CONTINUOUS REVIVAL OF THE PERIODIC SCHRÖDINGER EQUATION WITH PIECEWISE C^2 POTENTIAL

DINH-QUAN TRAN[1] AND PETER J. OLVER[2]

Abstract. In this paper, we investigate the revivals of the one-dimensional periodic Schrödinger equation with a piecewise C^2 potential function. As has been observed through numerical simulations of the equation with various initial data and potential functions, the solution, while remaining fractalized at irrational times, exhibits a form of revival at rational times. The goal is to prove that the solution at these rational times is given by a finite linear combination of translations and dilations of the initial datum, plus an additional continuous term, which we call "continuous revival". In pursuit of this result, we present a review of relevant properties of the periodic Schrödinger equation as an eigenvalue problem, including asymptotic results on both the eigenvalues and eigenfunctions.

Keywords: Schrödinger equation, revival, dispersive quantization, Talbot effect, Prüfer transformation.

1. Introduction

The Schrödinger equation, a partial differential equation that governs the wave function of non-relativistic quantum-mechanical systems, has been a cornerstone of mathematical physics. Named after the twentieth century physicist Erwin Schrödinger, the equation formed the basis of his Nobel Prize and continues to be of fundamental importance for applications, particularly in quantum mechanics. Within this paper, we are mainly concerned with the one-dimensional periodic Schrödinger equation.

Definition 1.1 (Periodic Schrödinger equation)**.** Let the potential $V(x)$ be a real-valued function. A solution $u(t,x)$ of the periodic Schrödinger equation satisfies the initial value problem

$$\begin{cases} \mathrm{i}\, u_t = -u_{xx} + Vu, \\ u(0,x) = f(x), \end{cases} \tag{1.2}$$

where $\mathrm{i} = \sqrt{-1}$, subject to periodic boundary conditions

$$\begin{cases} u(t,0) = u(t,2\pi), \\ u_x(t,0) = u_x(t,2\pi). \end{cases} \tag{1.3}$$

As a dispersive equation on a periodic domain, the Schrödinger equation displays two vastly different behaviors depending on whether the time step is rational relative to the period. While the solution remains fractalized at irrational time steps, it exhibits revival at rational times, being related to a finite linear combination of translations and dilations of the initial data. This quantization phenomenon was first observed by Henry Fox Talbot in 1836 when he noticed that the image of a diffraction grating is repeated at regular distances away from the grating plane. Since then, the phenomenon, called the "Talbot effect" [1, 2], or "dispersive quantization" [10], has been noted in various periodic dispersive equations, including the linearized Korteweg–deVries equation (also known as the Airy equation) [10], the free space Schrödinger equation without potential [1, 4], and related linear and nonlinear equations [4, 6, 9]. More recently, the continuous revival phenomenon has been investigated for the class of Schrödinger equations with H^2 potential [3], while the fractal dimension of the solution to the equation at irrational times has also been analyzed for a broader class of H^s potentials [5]. In order to extend our understanding to more general potential functions, it is important that we first characterize revival for potential functions with discontinuities. This paper,

Date: April 23, 2025.

therefore, aims to investigate the phenomenon in a simpler setting, where the potential $V(x)$ is piecewise C^2.

Assumption 1.4 (Piecewise C^2 potential function)**.** The potential function $V(x)$ is piecewise C^2 if there exists a finite number of points $0 = x_0 < x_1 < x_2 < ... < x_N < x_{N+1} = 2\pi$ such that

- $V(x)$ is bounded and 2π-periodic;
- $V(x) \in C^2(x_i, x_{i+1})$ for all $0 \leq i \leq N$;
- $V(x), V'(x)$, and $V''(x)$ have finite right and left limits at the discontinuities x_i.

The central focus of our project shall be the following theorem mirroring that of Boulton et al. [3]. Here, f^* is the 2π-periodic extension of the initial data f.

Theorem 1.5 (Continuous revival at rational times)**.** *Let $u(t,x)$ be the solution of the periodic Schrödinger equation (1.2). For $q, r \in \mathbb{N}$ co-prime numbers, define the revival function as*

$$\psi\left(2\pi\frac{q}{r}, x\right) = \frac{1}{r}\, e^{-2\pi \mathrm{i}\langle V\rangle q/r} \sum_{k,m=0}^{r-1} e^{2\pi \mathrm{i}\left(mk/r - m^2 q/r\right)} f^*\left(x - 2\pi\frac{k}{r}\right). \tag{1.6}$$

Then the value of u at rational times $t = 2\pi\frac{q}{r}$ is given by

$$u\left(2\pi\frac{q}{r}, x\right) = w\left(2\pi\frac{q}{r}, x\right) + \psi\left(2\pi\frac{q}{r}, x\right)$$

where $w(t,x)$ is a continuous function.

2. Sturm-Liouville theory for the Schrödinger equation

Inserting the ansatz $u(t,x) = e^{-\mathrm{i}\lambda t}\psi(x)$ into (1.2) produces the ordinary differential equation

$$\lambda\psi = -\psi'' + V\psi =: L[\psi] \tag{2.1}$$

along with the periodic boundary conditions

$$\begin{cases} \psi(0) = \psi(2\pi), \\ \psi'(0) = \psi'(2\pi). \end{cases} \tag{2.2}$$

Here $L[\psi]$ is a self-adjoint operator, and hence the eigenvalue equation (2.1) admits a countably infinite sequence of real eigenvalues

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \quad \text{with} \quad \lambda_n \to \infty \;\; \text{as} \;\; n \to \infty,$$

where double eigenvalues are counted twice. Using these eigenvalues, we can choose a basis of eigenfunctions to be real-valued and orthonormal on $[0, 2\pi]$; that is

$$\int_0^{2\pi} \psi_m(x)\psi_n(x)\, dx = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}.$$

These eigenfunctions can be trivially extended into continuously differentiable functions on the entirety of the real line thanks to the periodic boundary conditions (2.2). Our goal for this section is to quantify the asymptotic behavior of the eigenvalues and eigenfunctions of (2.1), which shall be important for our proof of Theorem 1.5. To prepare for those results, we shall, in the next subsections, look at some variational results and comparison theorems that shall give us insights into the properties of the eigenfunctions.

2.1. **Variational results.** Consider the Dirichlet integral given by

$$J(f,g) = \int_0^{2\pi} f'(x)g'(x) + V(x)f(x)g(x)\,dx. \tag{2.3}$$

Applying integration by parts and the boundary conditions (2.2), we obtain

$$\begin{aligned} J(f,g) &= -\int_0^{2\pi} f(x)\left[g''(x) - V(x)g(x)\right]dx + f(x)g'(x)\Big|_{x=0}^{x=2\pi} \\ &= -\int_0^{2\pi} f(x)\left[g''(x) - V(x)g(x)\right]dx \end{aligned}$$

where the boundary terms vanish due to the periodicity of f and g. Notably, we have

$$J(f,\psi_n) = -\int_0^{2\pi} f(x)\left[\psi_n''(x) - V(x)\psi_n(x)\right]dx = \int_0^{2\pi} f(x)\left[\lambda_n\psi_n(x)\right]dx = \lambda_n f_n \tag{2.4}$$

where f_n is the n-th Fourier coefficient of f. An immediate corollary is

$$J(\psi_m,\psi_n) = \begin{cases} \lambda_n, & \text{if } m = n. \\ 0, & \text{if } m \neq n. \end{cases}. \tag{2.5}$$

The following result gives an inequality involving J that elaborates on the familiar Parseval identity.

Lemma 2.6. *Let f be a real-valued function satisfying the boundary conditions (2.2) that is continuous with piecewise continuous derivative in $[0,2\pi]$. Then, with the Fourier coefficients f_n defined above,*

$$\sum_{n=0}^{\infty} \lambda_n f_n^2 \leq J(f,f)$$

Proof. We first suppose that $V(x) \geq 0$. Then, for any continuous $g(x)$ with piecewise continuous derivative, we have

$$J(g,g) = \int_0^{2\pi} [g'(x)]^2 + V(x)g^2(x)\,dx \geq 0.$$

In particular, for any positive integer N, we have

$$\begin{aligned} 0 \leq J\left(f - \sum_{n=0}^{N} f_n\psi_n, f - \sum_{n=0}^{N} f_n\psi_n\right) &= J(f,f) - 2\sum_{n=0}^{N} f_n J(f,\psi_n) + J\left(\sum_{n=0}^{N} f_n\psi_n, \sum_{n=0}^{N} f_n\psi_n\right) \\ &= J(f,f) - 2\sum_{n=0}^{N} \lambda_n f_n^2 + \sum_{n=0}^{N} \lambda_n f_n^2 \\ &= J(f,f) - \sum_{n=0}^{N} \lambda_n f_n^2 \end{aligned}$$

where, in the second-to-last inequality, we used (2.4) and (2.5). This gives the desired inequality for the case where $V(x)$ is positive, and we now aim for the general case. Let v_0 be a positive constant large enough such that

$$V(x) \geq -v_0$$

Now, the original relation (2.1) can be rewritten as

$$\psi''(x) + (\Lambda - \widetilde{V}(x))\psi(x) = 0$$

where $\Lambda = \lambda + v_0$ and $\widetilde{V}(x) = V(x) + v_0$ is a positive function. Applying the first part of the proof gives

$$\sum_{n=0}^{\infty} (\lambda_n + v_0) f_n^2 \leq \int_0^{2\pi} [f'(x)]^2 + (V(x) + v_0) f^2(x)\,dx = J(f,f) + v_0 \int_0^{2\pi} f^2(x)\,dx.$$

However, thanks to Parseval's formula,

$$\sum_{n=0}^{\infty} f_n^2 = \int_0^{2\pi} f^2(x)\,dx,$$

we can cancel the extra terms related to v_0. The general case follows as a result. □

Lemma 2.6 serves to establish the next variational result, which determines the changes in the eigenvalues with respect to that of the potential function.

Theorem 2.7. *Let*

$$V_1(x) \geq V(x) \quad \text{for all} \quad x \in \mathbb{R}$$

and let $\lambda_{1,n}$ *denote the eigenvalues of the periodic problem* (2.1) *with potential* $V_1(x)$. *Then*

$$\lambda_{1,n} \geq \lambda_n \quad \text{for all} \quad n \geq 0.$$

Proof. Let $\psi_{1,n}(x)$ denote the eigenfunction in the orthonormal basis of the modified problem that corresponds with $\lambda_{1,n}$, and let $J_1(f,g)$ denote the Dirichlet integral (2.3) but with $V(x)$ replaced by $V_1(x)$. We obtain the following relation between the Dirichlet integrals:

$$J_1(f,f) = \int_0^{2\pi} [f'(x)]^2 + V_1(x) f^2(x)\,dx \geq \int_0^{2\pi} [f'(x)]^2 + V(x) f^2(x)\,dx = J(f,f)$$

First, we seek to prove the theorem for $n = 0$. Since λ_0 is the smallest of the eigenvalues, applying Lemma 2.6 to any function f gives us

$$J(f,f) \geq \lambda_0 \sum_{n=0}^{\infty} f_n^2 = \lambda_0 \int_0^{2\pi} f^2(x)\,dx.$$

Consequently,

$$\lambda_{1,0} = J_1(\psi_{1,0}, \psi_{1,0}) \geq J(\psi_{1,0}, \psi_{1,0}) \geq \lambda_0 \int_0^{2\pi} \psi_{1,0}^2\,dx = \lambda_0$$

which completes the case of $n = 0$. For $n = 1$, we consider a function

$$f(x) = c_0 \psi_{1,0}(x) + c_1 \psi_{1,1}(x)$$

where the real constants c_0 and c_1 are such that

$$\begin{cases} c_0^2 + c_1^2 = 1 \\ c_0 A_0 + c_1 A_1 = 0 \end{cases}$$

with the constants A_k for $k \in \{0,1\}$ defined by

$$A_k = \int_0^{2\pi} \psi_{1,k} \psi_0\,dx.$$

Here, the first condition means that

$$\int_0^{2\pi} f^2(x)\,dx = c_0^2 \int_0^{2\pi} \psi_{1,0}^2(x)\,dx + c_1^2 \int_0^{2\pi} \psi_{1,1}^2(x)\,dx = c_0^2 + c_1^2 = 1$$

while the second condition implies

$$f_0 = \int_0^{2\pi} f(x)\psi_0(x)\,dx = c_0 A_0 + c_1 A_1 = 0.$$

Applying (2.5) to J_1, we have

$$J_1(f,f) = c_0^2 \lambda_{1,0} + c_1^2 \lambda_{1,1} \leq \lambda_{1,1}$$

while applying Lemma 2.6 along with the fact that $f_0 = 0$ gives

$$J(f,f) \geq \sum_{n=1}^{\infty} \lambda_n f_n^2 \geq \lambda_1 \sum_{n=1}^{\infty} f_n^2 = \lambda_1 \int_0^{2\pi} f^2(x)\,dx = \lambda_1.$$

Combining these two statements gives

$$\lambda_{1,1} \geq J_1(f,f) \geq J(f,f) \geq \lambda_1$$

which completes the case for $n = 1$. The same approach applies for a general natural number n - we consider

$$f(x) = \sum_{j=0}^{n} c_j \psi_{1,j}$$

along with the n homogenous linear algebraic equations given by

$$f_j = 0$$

for all $0 \leq j \leq n-1$. Following the same method as in the proof for $n = 1$ shall be sufficient to finish the result. □

2.2. **Roots of eigenfunctions.** In this section, we aim to identify the roots of the eigenfunctions of (2.1), subject to the boundary conditions (2.2). To do this, we first review the well-known Sturm Comparison Theorem, which illustrates the oscillatory nature of the solutions as well as allowing us to estimate the number of roots. We shall use the following formulation of Simons [12].

Theorem 2.8 (Sturm Comparison Theorem)**.** *Let V be a bounded function and let λ_1 and λ_2 be real numbers. For $i \in \{1,2\}$, let ψ_i be a non-trivial solution to the corresponding version of* (2.1)*,*

$$\lambda_i \psi = -\psi'' + V\psi.$$

Furthermore, suppose that ψ_1 has two roots $a < b$. Then ψ_2 has a root in (a,b) if one of the following two conditions hold:

- $\lambda_1 < \lambda_2$*, or*
- $\lambda_1 = \lambda_2$ *and* $\psi_2(a) \neq 0$*.*

We start by denoting by μ_n and ξ_n the eigenvalues and eigenfunctions to (2.1) with the semi-periodic boundary condition

$$\begin{cases} \xi(0) = -\xi(2\pi), \\ \xi'(0) = -\xi'(2\pi). \end{cases} \tag{2.9}$$

and Λ_n and Ψ_n the eigenvalues and eigenfunctions to (2.1) with the Dirichlet boundary condition

$$\Psi(0) = \Psi(2\pi) = 0. \tag{2.10}$$

Here, the number of roots of the eigenfunctions Ψ_n is well-established in the literature on Sturm-Liouville operators; see the following theorem in Eastham [7].

Theorem 2.11. *The eigenfunction Ψ_n has exactly n roots in the open interval $(0, 2\pi)$.*

We shall also utilize the following theorem in Eastham [8] to compare the eigenvalues of the associated boundary value problems.

Theorem 2.12. *The eigenvalues λ_n are interlaced with the eigenvalues μ_n of* (2.1) *with boundary conditions* (2.9) *according to*

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \ldots \tag{2.13}$$

Theorem 2.14. *The eigenvalues λ_n of* (2.1) *with boundary conditions* (2.2) *and the eigenvalues Λ_n of* (2.1) *with boundary conditions* (2.10) *are related by*

$$\lambda_{2m+1} \leq \Lambda_{2m+1} \leq \lambda_{2m+2}. \tag{2.15}$$

for any $m \geq 0$. On the other hand, the eigenvalues Λ_n are related to the eigenvalues μ_n of (2.1) *with boundary conditions* (2.9) *by*

$$\mu_{2m} \leq \Lambda_{2m} \leq \mu_{2m+1}. \tag{2.16}$$

An immediate corollary of (2.13), (2.15), and (2.16) is the following interlacing of λ_n and Λ_n.

$$\lambda_0 < \Lambda_0 < \lambda_1 \leq \Lambda_1 \leq \lambda_2 < \Lambda_2 < \lambda_3 \leq \Lambda_3 \leq \lambda_4 < \dots \tag{2.17}$$

These results, along with the variational result in Theorem 2.7, shall be utilized in proving the following key theorem regarding the roots of the eigenfunctions to our original problem.

Theorem 2.18. *Let ψ_n be the eigenfunctions of (2.1) with boundary conditions (2.2). Then*

(i) ψ_0 has no roots in $[0, 2\pi]$;
(ii) For all $m \geq 0$, ψ_{2m+1} and ψ_{2m+2} have precisely $2m+2$ roots in $[0, 2\pi)$.

Proof. For part *(i)*, by Theorem 2.11, the eigenfunction Ψ_0 has no roots in $(0, 2\pi)$. Furthermore, by (2.17), $\lambda_0 < \Lambda_0$, so by the Sturm Comparison Theorem, ψ_0 cannot have two roots in $[0, 2\pi]$. However, because of its periodic boundary conditions (2.2), ψ_0 must have an even number of roots in $[0, 2\pi)$. Hence, ψ_0 has no roots in $[0, 2\pi]$.

For part *(ii)*, consider ψ_{2m+1} for any $m \geq 0$. From (2.17), we find

$$\Lambda_{2m} < \lambda_{2m+1} \leq \Lambda_{2m+1}.$$

Here, note that by Theorem 2.11, the eigenfunctions Ψ_{2m} and Ψ_{2m+1} have $2m$ and $2m+1$ roots, respectively. Thus, the Sturm Comparison Theorem suggests that ψ_{2m+1} can have at least $2m+1$ and at most $2m+2$ roots. However, again because of its periodic boundary conditions (2.2), ψ_{2m+1} needs to have an even number of roots in $[0, 2\pi)$, which means that its number of roots in this interval is $2m+2$. The same argument applies to the eigenfunction ψ_{2m+2}, but with the relation

$$\Lambda_{2m+1} \leq \lambda_{2m+2} < \Lambda_{2m+2}.$$

This completes the proof. □

3. Eigenvalue and eigenfunction asymptotics

3.1. **The Prüfer transformation.** In his 1926 paper, Prüfer [11] introduced a transformation that greatly simplifies the analysis of Sturm-Liouville operators, as it allows the utilization of polar coordinates to obtain existence of eigenvalues and oscillation of eigenfunctions. Consider a differential equation with positive real-valued coefficients $C(x)$ and $D(x)$ with piecewise continuous derivatives

$$[C(x)u'(x)]' + D(x)u(x) = 0 \qquad \text{on } x \in [x_1, x_2]. \tag{3.1}$$

Set

$$R(x) = [C(x)D(x)]^{1/2}.$$

For every non-trivial, real-valued solution $u(x)$ of (3.1), we can transfer to polar coordinates by writing

$$R(x)u(x) = \rho(x)\sin\theta(x), \qquad C(x)u'(x) = \rho(x)\cos\theta(x) \tag{3.2}$$

so that

$$\begin{cases} \rho = [R^2u^2 + C^2(u')^2]^{1/2}, \\ \tan\theta = Ru/Cu'. \end{cases} \tag{3.3}$$

We complete the definition by selecting a point $a_0 \in [x_1, x_2]$ to specify that

$$-\pi < \theta(a_0) < \pi$$

and, in particular, if $u(a_0) \geq 0$, then

$$0 \leq \theta(a_0) < \pi. \tag{3.4}$$

With these definitions, the following two properties — the formula for the derivative of the phase function $\theta(x)$, and the estimate for the value of the phase function depending on the number of zeros — will be extremely important for our asymptotics.

Property 3.5. The derivative of the phase function $\theta'(x)$ is given by the formula

$$\theta'(x) = \left(\frac{D(x)}{C(x)}\right)^{1/2} + \frac{(C(x)D(x))'}{4C(x)D(x)}\sin(2\theta(x)). \tag{3.6}$$

Proof. Differentiating the first equation in (3.2) gives

$$R'(x)u(x) + R(x)u'(x) = \rho'(x)\sin(\theta(x))) + \rho(x)\cos(\theta(x))\theta'(x),$$

which, after rearragement, can be evaluated as

$$\begin{aligned}\cos(\theta(x))\theta'(x) &= -\frac{\rho'(x)}{\rho(x)}\sin(\theta(x)) + \frac{R'(x)}{\rho(x)}u(x) + \frac{R(x)}{\rho(x)}u'(x)\\ &= -\frac{\rho'(x)}{\rho(x)}\sin(\theta(x)) + \frac{R'(x)}{R(x)}\sin(\theta(x)) + \frac{R(x)}{C(x)}\cos(\theta(x))\end{aligned} \tag{3.7}$$

where on the last line we have substituted $u(x)$ and $u'(x)$ using the appropriate equation from (3.3). On the other hand, by substituting $u(x)$ and $u'(x)$ using the same equations into (3.1), we obtain

$$0 = D(x)\frac{\rho(x)\sin(\theta(x))}{R(x)} + \rho'(x)\cos(\theta(x)) - \rho(x)\sin(\theta(x))\theta'(x)$$

which translates to

$$\sin(\theta(x))\theta'(x) = \frac{\rho'(x)}{\rho(x)}\cos(\theta(x)) + \frac{D(x)}{R(x)}\sin(\theta(x)). \tag{3.8}$$

Finally, from equation (3.7) and (3.8), we eliminate the term containing $\rho'(x)/\rho(x)$ to obtain

$$\begin{aligned}\theta'(x) = [\sin^2(\theta(x)) + \cos^2(\theta(x))]\theta'(x) &= \frac{R'(x)}{R(x)}\sin(\theta(x))\cos(\theta(x)) + \frac{R(x)}{C(x)}\cos^2(\theta(x)) + \frac{D(x)}{R(x)}\sin^2(\theta(x))\\ &= \frac{(C(x)D(x))'}{4C(x)D(x)}\sin(2\theta(x)) + \left(\frac{D(x)}{C(x)}\right)^{1/2}\end{aligned}$$

where in the last equality we substituted $R(x) = (C(x)D(x))^{1/2}$. This proves the identity (3.6). □

Property 3.9. If a solution $u(x)$ of (3.1) has N roots in $(a_0, y]$ with $u(a_0) \geq 0$ and $a_0 < y \leq x_2$, then

$$N\pi \leq \theta(y) < (N+1)\pi \tag{3.10}$$

Proof. First, note that by the first equation of (3.2), $u(x)$ is zero only when $\theta(x)$ is a multiple of π. Applying this to the derivative formula (3.6) shows that $\theta'(x)$ is positive at any root of $u(x)$. Since $u(x)$ and $u'(x)$ are never simultaneously zero, we know that $\rho(x)$ is always positive. We can then refine the previous statement to say that $u(x)$ is zero if and only if $\theta(x)$ is a multiple of π.

Let the roots of $u(x)$ in $(a_0, y]$ be

$$a_0 < \alpha_1 < ... < \alpha_N \leq y$$

and consider first the interval $(a_0, \alpha_1]$. From the assumptions, we know that

- $0 \leq \theta(a_0) < \pi$,
- $\theta(x)$ is not a multiple of π anywhere in (a_0, α_1) since $u(x)$ has no roots there, and
- $\theta(\alpha_1) > 0$.

These facts allow us to conclude that $\theta(\alpha_1) = \pi$. We shall proceed accordingly — for the interval $(\alpha_1, \alpha_2]$, note that

- $\theta(x)$ is not a multiple of π anywhere in (α_1, α_2) since $u(x)$ has no roots here, and
- $\theta(\alpha_2) > 0$.

We conclude from these that $\theta(\alpha_2) = 2\pi$. Continuing this process shall give us the desired inequality (3.10), where the inequality on the left occurs if $\alpha_N = y$. □

Our goal is to apply this transformation to the Schrödinger eigenvalue equation (2.1). Because of the conditions imposed on $C(x)$ and $D(x)$, we consider differential equations of the form

$$u'' + (\lambda - V_1)u = 0$$

where V_1 is a continuously differentiable function of period 2π. Since we are concerned with the estimates as $\lambda \to 0$, the second coefficient $\lambda - V_1$ can also be considered positive for λ sufficiently large. The phase function, under these, becomes a two-parameter function that, since $u(x)$ has period 2π, satisfy

$$\theta(2\pi, \lambda) - \theta(0, \lambda) = 2k\pi \tag{3.11}$$

for any integer k. Furthermore, following (3.6), it has the derivative

$$\theta'(x, \lambda) = (\lambda - V_1(x))^{1/2} - \frac{V_1'(x)}{4(\lambda - V_1(x))} \sin(2\theta(x, \lambda)) \tag{3.12}$$

with asymptotic value

$$\lim_{\lambda \to \infty} \theta'(x, \lambda) = \lambda^{1/2} + \mathrm{O}(1). \tag{3.13}$$

This asymptotic estimate is crucial for the next helpful lemma following Eastham [8].

Lemma 3.14. *Let $f(x)$ be integrable over $[0, 2\pi]$ and let c be a constant. Then*

$$\int_0^{2\pi} f(x) \sin(c\,\theta(x, \lambda))\, dx \;\longrightarrow\; 0 \tag{3.15}$$

as $\lambda \to 0$. The same result holds with $\sin(c\,\theta(x, \lambda))$ replaced by $\cos(c\,\theta(x, \lambda))$.

Proof. For any fixed $\varepsilon > 0$, let $g(x)$ be a continuously differentiable function such that

$$\int_0^{2\pi} |f(x) - g(x)|\, dx < \varepsilon.$$

Then, we have

$$\left| \int_0^{2\pi} f(x) \sin(c\,\theta(x, \lambda))\, dx \right| < \varepsilon + \left| \int_0^{2\pi} g(x) \sin(c\,\theta(x, \lambda))\, dx \right|. \tag{3.16}$$

On the other hand, by utilizing (3.13), as $\lambda \to \infty$,

$$\begin{aligned} \int_0^{2\pi} g(x) \sin(c\,\theta(x, \lambda))\, dx &= \lambda^{-1/2} \int_0^{2\pi} g(x) \sin(c\,\theta(x, \lambda))\theta'(x, \lambda)\, dx + \mathrm{O}(\lambda^{-1/2}) \\ &= \frac{\lambda^{-1/2}}{c} \left[g(x) \cos(c\,\theta(x, \lambda)) \Big|_0^{2\pi} - \int_0^{2\pi} g'(x) \cos(c\,\theta(x, \lambda))\, dx \right] + \mathrm{O}(\lambda^{-1/2}). \end{aligned}$$

This means that the entire expression is $\mathrm{O}(\lambda^{-1/2})$, and for a sufficiently large λ, we have

$$\left| \int_0^{2\pi} g(x) \sin(c\,\theta(x, \lambda))\, dx \right| < \varepsilon,$$

which means, by (3.16),

$$\left| \int_0^{2\pi} f(x) \sin(c\,\theta(x, \lambda))\, dx \right| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily chosen, we deduce the desired result (3.15). The case where $\sin(c\,\theta(x, \lambda))$ is replaced by $\cos(c\,\theta(x, \lambda))$ can then be arrived at using a similar argument. □

3.2. **Eigenvalue asymptotics.** The proof of Theorem 1.5 relies on asymptotic estimates of eigenvalues and eigenfunctions — in particular, whether the eigenvalues are sufficiently close to an integer and the eigenfunctions are sufficiently close to the trigonometric basis of a Fourier series. Both of these properties have been observed in numerical simulations, even in the presence of discontinuities in the potential function. In particular, for the asymptotic behavior of eigenvalues, our goal is the following theorem, courtesy of Eastham [8].

Theorem 3.17. *Consider the differential equation (2.1) with a real-valued potential $V_1(x)$ with continuous derivative up to the second order. Let $\lambda_{1,n}$ be the ascending eigenvalues of the problem, along with the corresponding eigenfunctions $\psi_{1,n}(x)$. Then, as $m \to \infty$, the eigenvalues $\lambda_{1,2m+1}$ and $\lambda_{1,2m+2}$ satisfy the asymptotic estimate*

$$\lambda^{1/2} = (m+1) + \frac{A_1}{(m+1)} + \mathrm{O}(m^{-3}) \tag{3.18}$$

where the coefficient

$$A_1 = \frac{1}{4\pi}\int_0^{2\pi} V_1(x)\,dx \tag{3.19}$$

is independent of m.

Proof. Without loss of generality, assume that $\psi_{1,n}(0) \geq 0$ for all $n \geq 0$. Here, we apply the Prüfer transform (3.3) to $u(x) = \psi_{1,2m+1}(x)$ with $a_0 = 0$, which satisfies (3.4). Equation (3.11) then gives us the range of the value of the phase function as

$$2k\pi \leq \theta(2\pi, \lambda_{1,2m+1}) < (2k+1)\pi$$

for some integer k. However, since Theorem 2.18 dictates that $\psi_{1,2m+1}$ has $2m+2$ zeros in $(0, 2\pi]$, then Property 3.9 allows us to deduce that

$$k = m+1$$

which transforms (3.11) into

$$\begin{aligned} 2(m+1)\pi = \theta(2\pi, \lambda_{1,2m+1}) - \theta(0, \lambda_{1,2m+1}) &= \int_0^{2\pi} \theta'(x, \lambda_{1,2m+1})\,dx \\ &= \int_0^{2\pi} (\lambda_{1,2m+1} - V_1(x))^{1/2}\,dx - \frac{1}{4}\int_0^{2\pi} \frac{V_1'(x)}{\lambda_{1,2m+1} - V_1(x)} \sin(2\theta(x, \lambda_{1,2m+1}))\,dx \end{aligned} \tag{3.20}$$

where the last equality is derived using (3.12). We shall focus on the latter term first, abbreviating $\lambda_{1,2m+1}$ as λ and further substituting using (3.12), which yields

$$\begin{aligned} &\frac{1}{4}\int_0^{2\pi} \frac{V_1'(x)}{(\lambda - V_1(x))^{3/2}} \left[\theta'(x,\lambda) + \frac{V_1'(x)}{4(\lambda - V_1(x))}\sin(2\theta(x,\lambda))\right] \sin(2\theta(x,\lambda))\,dx \\ &\quad = \frac{1}{2}\int_0^{2\pi} \left[\frac{d}{dx}\frac{V_1'(x)}{(\lambda - V_1(x))^{3/2}}\right] \cos(2\theta(x,\lambda))\,dx + \frac{1}{8}\int_0^{2\pi} \frac{V_1'(x)}{(\lambda - V_1(x))^{5/2}}(1 - \cos(4\theta(x,\lambda)))\,dx \end{aligned}$$

where we used an integration by parts on the term involving $\theta'(x, \lambda)$. Utilizing Lemma 3.14, we see that the first term is $\mathrm{O}(\lambda^{-3/2})$ while the second term is $\mathrm{O}(\lambda^{-5/2})$, giving the overall estimate of $\mathrm{O}(\lambda^{-3/2})$. Applying the binomial expansion to the other term in (3.20) yields

$$2(m+1)\pi = 2\pi\lambda^{1/2} - \frac{\lambda^{-1/2}}{2}\int_0^{2\pi} V_1(x)\,dx + \mathrm{O}(\lambda^{-3/2}).$$

The most significant contribution comes from the first term, which means $m \sim \lambda^{1/2}$. We rewrite the expression as

$$\lambda^{1/2} = (m+1) + \frac{\lambda^{-1/2}}{4\pi}\int_0^{2\pi} V_1(x)\,dx + \mathrm{O}(m^{-3}) = (m+1) + \lambda^{-1/2}A_1 + \mathrm{O}(m^{-3}) \tag{3.21}$$

where the coefficient A_1 is given by (3.19). Taking the inverse of this gives

$$\lambda^{-1/2} = \frac{1}{m+1}\left[1 - \frac{\lambda^{-1/2}A_1}{m+1} + \mathrm{O}(m^{-4})\right] = \frac{1}{m+1} + \mathrm{O}(m^{-3}). \tag{3.22}$$

Substituting both (3.22) back to (3.21) yields the estimate

$$\lambda^{1/2} = (m+1) + \left[\frac{1}{m+1} + \mathrm{O}(m^{-3})\right] A_1 + \mathrm{O}(m^{-3}) = (m+1) + \frac{A_1}{m+1} + \mathrm{O}(m^{-3}),$$

which establishes (3.18). □

Here, notice that our potential given by Assumption 1.4 is only piecewise continuously differentiable up to the second order, so Theorem 3.17 does not apply directly. However, thanks to Theorem 2.7, we can still use this asymptotic result with the following corollary.

Corollary 3.23. *The estimate* (3.18) *can be applied for the differential equation* (2.1) *when the real-valued potential $V(x)$ is piecewise continuously differentiable up to the second order, as defined by Assumption 1.4.*

Proof. Consider an arbitrarily small $\varepsilon > 0$, and choose a function $V_1(x)$ that is continuously differentiable up to the second order such that

$$\begin{cases} V_1(x) \geq V(x) & \text{for all } \ x \in [0, 2\pi] \\ \|V_1(x)\|_{L^1[0,2\pi]} \leq \|V(x)\|_{L^1[0,2\pi]} + \varepsilon. \end{cases} \tag{3.24}$$

Let $\lambda_{1,n}$ and $\psi_{1,n}$ be the eigenvalue and eigenfunction with the potential $V_1(x)$. Furthermore, let $A_{1,1}$ be the coefficient in the asymptotic expansion (3.18) associated with the potential $V_1(x)$, and define A_1 using the same formulas but with $V(x)$. Our goal is to prove that A_1 is the correct coefficient in the asymptotic estimate of λ_n.

Using Theorem 2.7, the first condition of (3.24) means that for all $n \geq 0$, we have

$$\lambda_{1,n} \geq \lambda_n. \tag{3.25}$$

On the other hand, regarding the coefficient A_1, (3.24) means that

$$A_1 \leq A_{1,1} \leq A_1 + \frac{\varepsilon}{2}. \tag{3.26}$$

Combining (3.25) and (3.26), along with utilizing the asymptotic expansion (3.18) gives

$$\lambda_n^{1/2} \leq \lambda_{1,n}^{1/2} = (m+1) + \frac{A_{1,1}}{m+1} + \mathrm{O}(m^{-3}) \leq (m+1) + \frac{A_1}{m+1} + \mathrm{O}(m^{-3}) + \frac{\varepsilon}{2(m+1)}. \tag{3.27}$$

We can also do the same process but with a real-valued, continuously twice differentiable function $V_2(x)$ satisfying

$$\begin{cases} V_2(x) \leq V(x) & \text{for all } x \in [0, 2\pi] \\ \|V_2(x)\|_{L^1[0,2\pi]} \geq \|V(x)\|_{L^1[0,2\pi]} - \varepsilon. \end{cases}$$

to obtain

$$\lambda_n^{1/2} \geq (m+1) + \frac{A_1}{m+1} + \mathrm{O}(m^{-3}) - \frac{\varepsilon}{2(m+1)}. \tag{3.28}$$

Since ε can be taken arbitrarily small, the two-sided bound formed by (3.27) and (3.28) gives the estimate

$$\lambda_n^{1/2} = (m+1) + \frac{A_1}{m+1} + \mathrm{O}(m^{-3})$$

as $\varepsilon \to 0$. This concludes the proof. □

3.3. **Eigenfunction asymptotics.** Our second goal for this section is to obtain asymptotic estimates for eigenfunctions, partially relying on the asymptotic expansion of eigenvalues (3.18). To do that, we first acknowledge that any solution ψ of (2.1) can be written as a linear combination

$$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x) \tag{3.29}$$

where $\{\phi_1, \phi_2\}$ is a fundamental set of solutions for (2.1), satisfying

$$\begin{cases}\phi_1(0,\lambda) = 1\\ \phi_1'(0,\lambda) = 0\end{cases} \qquad \text{and} \qquad \begin{cases}\phi_2(0,\lambda) = 0\\ \phi_2'(0,\lambda) = 1\end{cases}. \tag{3.30}$$

Using variation of constants, we can obtain solution representations for ϕ_1 and ϕ_2 as

$$\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) + \lambda^{-1/2}\int_0^x \sin((x-y)\sqrt{\lambda})V(y)\phi_1(y,\lambda)\,dy \tag{3.31}$$

and

$$\phi_2(x,\lambda) = \lambda^{-1/2}\sin(x\sqrt{\lambda}) + \lambda^{-1/2}\int_0^x \sin((x-y)\sqrt{\lambda})V(y)\phi_2(y,\lambda)\,dy. \tag{3.32}$$

Both of these formulas can be regarded as recursive definitions of the fundamental solutions, and therefore we can iterate them as many times as we wish to yield terms of higher order in λ. Let us present the resulting asymptotic estimates for the two fundamental solutions.

Theorem 3.33. *Assume that $V(x)$ is a real-valued, piecewise continuous potential function with mean zero:*

$$\langle V\rangle = \frac{1}{2\pi}\int_0^{2\pi} V(x)\,dx = 0$$

Then, the fundamental solutions (3.30) satisfy the asymptotics

$$\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) + \lambda^{-1/2}\int_0^x \sin((x-y)\sqrt{\lambda})\cos(y\sqrt{\lambda})V(y)\,dy + \mathrm{O}(\lambda^{-1})$$

and

$$\phi_2(x,\lambda) = \lambda^{-1/2}\sin(x\sqrt{\lambda}) + \lambda^{-1}\int_0^x \sin((x-y)\sqrt{\lambda})\sin(y\sqrt{\lambda})V(y)\,dy + \mathrm{O}(\lambda^{-3/2}).$$

Proof. First, we make some remarks about the boundedness of the fundamental solutions. Let $M_1(\lambda)$ be the maximum value of ϕ_1 over $[0,2\pi]$. Then, (3.31) gives

$$M_1(\lambda) \le 1 + \lambda^{-1/2}M_1(\lambda)\int_0^{2\pi}|V(y)|\,dy$$

which means

$$M_1(\lambda) \le \left(1 - \lambda^{-1/2}\int_0^{2\pi}|V(y)|\,dy\right)^{-1}$$

provided that λ is large enough for the right-hand side to be positive. Since $V(x)$ is bounded,

$$M_1(\lambda) = \mathrm{O}(1) \quad \text{as} \quad \lambda \longrightarrow \infty.$$

A similar evaluation can be conducted for ϕ_2 to get that it grows to $\mathrm{O}(\lambda^{-1/2})$. Here, from (3.31), we iterate one more time by replacing the right-hand side in place of $\phi_1(x,\lambda)$ in the integral to get

$$\begin{aligned}\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) &+ \lambda^{-1/2}\int_0^x \sin((x-y)\sqrt{\lambda})\cos(y\sqrt{\lambda})V(y)\,dy\\ &+ \lambda^{-1}\int_0^x \sin((x-y)\sqrt{\lambda})\,V(y)\left(\int_0^y \sin((y-z)\sqrt{\lambda})V(z)\phi_1(z,\lambda)\,dz\right)dy\end{aligned}$$

where, with upper bound as above, the asymptotic estimate

$$\phi_1(x,\lambda) = \cos(x\sqrt{\lambda}) + \lambda^{-1/2}\int_0^x \sin((x-y)\sqrt{\lambda})\cos(y\sqrt{\lambda})V(y)\,dy + \mathrm{O}(\lambda^{-1}) \tag{3.34}$$

holds. On the other hand, iterating one more time for (3.32) gives

$$\phi_2(x,\lambda) = \lambda^{-1/2}\sin(x\sqrt{\lambda}) + \lambda^{-1}\int_0^x \sin((x-y)\sqrt{\lambda})\sin(y\sqrt{\lambda})V(y)\,dy \\ + \lambda^{-1}\int_0^x \sin((x-y)\sqrt{\lambda})\,V(y)\left(\int_0^y \sin((y-z)\sqrt{\lambda})V(z)\phi_2(z,\lambda)\,dz\right)dy$$

and, by substituting the upper bound,

$$\phi_2(x,\lambda) = \lambda^{-1/2}\sin(x\sqrt{\lambda}) + \lambda^{-1}\int_0^x \sin((x-y)\sqrt{\lambda})\sin(y\sqrt{\lambda})V(y)\,dy + \mathrm{O}(\lambda^{-3/2}). \tag{3.35}$$

This completes the proof. □

Here, note that with the mean zero condition on the potential, (3.18) implies that the eigenvalues satisfy

$$\lambda_{2m-1}^{1/2} \simeq \lambda_{2m}^{1/2} = m + \mathrm{O}(m^{-3}) \quad \text{as} \quad m \longrightarrow \infty..$$

Applying this to (3.34) yields

$$\phi_1(x,\lambda_{2m-1}) \simeq \phi_1(x,\lambda_{2m}) = \cos(mx) + \frac{1}{m}\int_0^x \sin(m(x-y))\cos(my)V(y)\,dy + \mathrm{O}(m^{-2}). \tag{3.36}$$

This is because the $\mathrm{O}(m^{-3})$ term within the cosine function translates to $\mathrm{O}(m^{-6})$ maximum error, which is absorbed in the $\mathrm{O}(\lambda^{-1}) = \mathrm{O}(m^{-2})$ term. On the other hand, the $\mathrm{O}(m^{-3})$ term within the sine function translates to $\mathrm{O}(m^{-3})$ maximum error, so (3.35) gives

$$\phi_2(x,\lambda_{2m-1}) \simeq \phi_2(x,\lambda_{2m}) = \frac{1}{m}\sin(mx) + \frac{1}{m^2}\int_0^x \sin(m(x-y))\sin(my)V(y)\,dy + \mathrm{O}(m^{-3}). \tag{3.37}$$

Finally, knowing that the eigenfunctions ψ_n are linear combinations of the fundamental solutions as in (3.29), we have the following lemma on the relative size of the constants.

Lemma 3.38. *For $i \in \{2m-1, 2m\}$, the eigenfunctions ψ_i can be written as*

$$\psi_i(x) = \alpha_i\phi_1(x,\lambda_i) + \beta_i m\phi_2(x,\lambda_i)$$

where both coefficients are $\mathrm{O}(1)$ as $m \to \infty$.

Proof. Note that using (3.36) and (3.37), the eigenfunction can be expressed as

$$\psi_i(x) = \alpha_i\cos(mx) + \beta_i\sin(mx) + (\alpha_i+\beta_i)\mathrm{O}(m^{-1})$$

for $i \in \{2m-1, 2m\}$. If either constant is $\mathrm{O}(m^k)$ for $k \geq 1$, then

$$||\psi_i||_{L^2[0,2\pi]} = \mathrm{O}(m) \longrightarrow \infty \quad \text{as} \quad m \longrightarrow \infty,$$

which contradicts the assumption that ψ_i is part of an orthonormal basis. On the other hand, if both constants are $\mathrm{O}(m^k)$ for $k \leq -1$, then

$$||\psi_i||_{L^2[0,2\pi]} = \mathrm{O}(m^{-1}) \ \longrightarrow \ 0$$

which is also contradictory. This means at least one of the constants is $\mathrm{O}(1)$ - without loss of generality, assume that it is α_i. If $\beta_i = \mathrm{O}(m^k)$ for $k \leq -1$, since ψ_{2m-1} and ψ_{2m} are mutually orthogonal, we have

$$0 = \langle\psi_{2m-1}, \psi_{2m}\rangle = \alpha_{2m-1}\alpha_{2m}||\cos(mx)||^2 + \mathrm{O}(m^{-1})$$

which means that

$$\alpha_{2m-1}\alpha_{2m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty,$$

which contradicts the fact that the coefficients are $\mathrm{O}(1)$ and non-zero. We conclude that both coefficients have to be $\mathrm{O}(1)$ as $m \to \infty$. □

From here on, we let

$$a_n = \sqrt{\frac{1}{\pi}}\cos(nx) \qquad \text{and} \qquad b_n = \sqrt{\frac{1}{\pi}}\sin(nx)$$

denote the standard orthonormal Fourier basis functions. With the lemma above, and by adjusting the coefficients α_i and β_i to the normalized bases, we can express the eigenfunction ψ_i for $i \in \{2m-1, 2m\}$ as

$$\psi_i(x) = \alpha_i a_m(x) + \beta_i b_m(x) + \frac{1}{m}\int_0^x \sin(m(x-y))[\alpha_i a_m(y) + \beta_i b_m(y)]V(y)\,dy + R_i(x) \tag{3.39}$$

where $||R_i||_{L^\infty} = \mathrm{O}(m^{-2})$. With this, the next lemma shall give further information about the constants α_i and β_i.

Lemma 3.40. *For all $m \geq 1$, there exists a y_m such that*

$$\alpha_{2m-1} = \cos(y_m) + \mathrm{O}(m^{-1}), \qquad \alpha_{2m} = \sin(y_m) + \mathrm{O}(m^{-1}),$$
$$\beta_{2m-1} = \sin(y_m) + \mathrm{O}(m^{-1}), \qquad \beta_{2m} = -\cos(y_m) + \mathrm{O}(m^{-1}).$$

Proof. Using (3.39) and the fact that ψ_{2m-1} is normalized, we have

$$1 = ||\psi_{2m-1}||^2_{L^2[0,2\pi]} = \alpha^2_{2m-1} + \beta^2_{2m-1} + \mathrm{O}(m^{-1}).$$

This means that there exists a y_m such that

$$\alpha_{2m-1} = \cos(y_m) + \mathrm{O}(m^{-1}), \qquad \beta_{2m-1} = \sin(y_m) + \mathrm{O}(m^{-1}).$$

Applying the same argument on ψ_{2m-1} means that there exists a $\widehat{y}_m$ such that

$$\alpha_{2m} = \cos(\widehat{y}_m) + \mathrm{O}(m^{-1}), \qquad \beta_{2m} = \sin(\widehat{y}_m) + \mathrm{O}(m^{-1}).$$

On the other hand, since ψ_{2m-1} and ψ_{2m} are orthogonal, we have

$$0 = \langle \psi_{2m-1}, \psi_{2m}\rangle = \alpha_{2m-1}\alpha_{2m} + \beta_{2m-1}\beta_{2m} + \mathrm{O}(m^{-1}) = \cos(y_m - \widehat{y}_m) + \mathrm{O}(m^{-1}).$$

This implies that

$$\widehat{y}_m = y_m + \left(k - \frac{1}{2}\right)\pi + \mathrm{O}(m^{-1})$$

which we can then apply to α_{2m} and β_{2m} to yield the desired estimates, up to flipping the sign of ψ_{2m}. □

Remark 3.41. It can be verified that the union of the two collections $\{\cos(y_m)a_m + \sin(y_m)b_m\}$ and $\{\sin(y_m)a_m - \cos(y_m)b_m\}$

- forms an orthonormal basis, and
- spans the same subspace of L^2 as the basis formed by the two collections $\{a_m\}$ and $\{b_m\}$.

In particular, we have the useful trigonometric identity

$$\begin{aligned}\langle f, a_m\rangle a_m(x) + \langle f, b_m\rangle b_m(x) = &(\cos(y_m)\langle f, a_m\rangle + \sin(y_m)\langle f, b_m\rangle)(\cos(y_m)a_m(x) + \sin(y_m)b_m(x)) \\ &+ (\sin(y_m)\langle f, a_m\rangle - \cos(y_m)\langle f, b_m\rangle)(\sin(y_m)a_m(x) - \cos(y_m)b_m(x)).\end{aligned} \tag{3.42}$$

4. Continuous revival of the Schrödinger equation with piecewise C^2 potential

With the asymptotics from the previous section, we are now equipped for a proof to Theorem 1.5. We present below a lemma that relates a solution of the periodic Schrödinger equation (1.2) to the Fourier cosine series of the initial condition. Theorem 1.5 shall be a corollary to this lemma.

Lemma 4.1. *Consider a real-valued potential function V that is piecewise continuously differentiable up to the second order with mean zero: $\langle V\rangle = 0$. Then a solution u to the periodic Schrödinger equation (1.2) has the form*

$$u(t,x) = w(t,x) + \sum_{n=1}^{\infty} e^{-\mathrm{i}n^2 t}\left[\langle f, a_n\rangle a_n(x) + \langle f, b_n\rangle b_n(x)\right] \tag{4.2}$$

where $w(t,x)$ is a continuous function for any $t > 0$.

Proof. First, consider the expansion

$$f(x)=\sum_{n=1}^{\infty}c_n\psi_n(x)\qquad\text{where}\qquad c_n=\langle f,\psi_n\rangle$$

of the initial condition f with respect to the orthonormal eigenfunction basis $\{\psi_n\}$. Using the eigenvalue asymptotics in (3.18),

$$\lambda=m^2+\frac{k_i}{m^2}\qquad\text{for}\qquad i\in\{2m-1,2m\},$$

where $\{k_i\}\in\ell^\infty$. Applying this to the eigenvalue expansion of the solution to (1.2) produces

$$\begin{aligned}u(t,x)-c_0\psi_0(x)&=\sum_{n=1}^{\infty}c_ne^{-\mathrm{i}\lambda_nt}\psi_n(x)\\&=\sum_{m=1}^{\infty}c_{2m-1}e^{-\mathrm{i}(m^2+k_{2m-1}/m^2)t}\psi_{2m-1}(x)+c_{2m}e^{-\mathrm{i}(m^2+k_{2m}/m^2)t}\psi_{2m}(x)=U_1(t,x)-U_2(t,x),\end{aligned}$$

where

$$\begin{aligned}U_1(t,x)&=\sum_{m=1}^{\infty}e^{-\mathrm{i}m^2t}\left(c_{2m-1}\psi_{2m-1}(x)+c_{2m}\psi_{2m}(x)\right),\\U_2(t,x)&=\sum_{m=1}^{\infty}\frac{\mathrm{i}e^{-\mathrm{i}m^2t}}{m^2}\left(\sum_{j\in\{0,1\}}c_{2m-j}k_{2m-j}\psi_{2m-j}(x)\int_0^te^{-\mathrm{i}k_{2m-j}s/m^2}ds\right).\end{aligned}$$

We shall prove that $U_2\in C^1[0,2\pi]$ first. Let

$$\zeta_{2m-j}(t,x)=\frac{\mathrm{i}\,e^{-\mathrm{i}m^2t}}{m^2}c_{2m-j}k_{2m-j}\psi_{2m-j}(x)\int_0^te^{-\mathrm{i}k_{2m-j}s/m^2}ds$$

Note that for all $m\geq 1$ and $j\in\{0,1\}$, since the sequence $\{k_{2m-j}\}$ is bounded, we have

$$|\zeta_{2m-j}(t,x)|\leq\frac{C\,||\psi_{2m-j}||_{L^\infty}|\langle f,\psi_{2m-j}\rangle|}{m^2}\,t\sup_{s\in[0,t]}\left|e^{-\mathrm{i}k_{2m-j}s/m^2}\right|\leq\frac{C\,||\psi_{2m-j}||_{L^\infty}||f||_{L^2[0,2\pi]}}{m^2}\,t.$$

Here, for a fixed t, we know that all the terms in the numerator are bounded by constants independent of m. Hence, by the Weierstrass M-test,

$$U_2(t,x)=\sum_{m=1}^{\infty}\left[\zeta_{2m-1}(t,x)+\zeta_{2m}(t,x)\right]$$

converges absolutely and uniformly to a C^1 function, as each component itself is C^1.

We now consider $U_1(t,x)$. Using (3.39), by letting

$$B_m(x)=\sum_{j\in\{0,1\}}c_{2m-j}\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right)$$

we can write

$$U_1(t,x)=U_3(t,x)+U_4(t,x)+U_5(t,x)$$

where the component functions are

$$\begin{aligned}U_3(t,x)&=\sum_{m=1}^{\infty}e^{-\mathrm{i}m^2t}B_m(x),\\U_4(t,x)&=\sum_{m=1}^{\infty}\frac{e^{-\mathrm{i}m^2t}}{m}\int_0^x\sin(m(x-y))B_m(y)V(y)\,dy,\\U_5(t,x)&=\sum_{m=1}^{\infty}e^{-\mathrm{i}m^2t}\sum_{j\in\{0,1\}}c_{2m-j}R_{2m-j}(x).\end{aligned}$$

Utilizing the same argument as we did for $U_2(t,x)$, for some constant $C>0$, each individual term of $U_5(t,x)$ has the bound

$$\left|e^{-\mathrm{i}m^2t}c_{2m-j}R_{2m-j}(x)\right| \le ||R_{2m-j}||_{L^\infty}|\langle f,\psi_{2m-j}\rangle| \le \frac{C||f||_{L^2[0,2\pi]}}{m^2}$$

which, when applied to the Weierstrass M-test, gives us that $U_5(t,x)$ is a C^1 function. We now turn to $U_4(t,x)$. The individual term of the series is given by

$$\eta_{2m-j}(t,x) = \frac{e^{-\mathrm{i}m^2t}}{m}c_{2m-j}\int_0^x \sin(m(x-y))[\alpha_{2m-j}a_m(y)+\beta_{2m-j}b_m(y)]V(y)\,dy$$

Note that, by using the Cauchy-Schwarz inequality and Parseval's identity, we get

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{|c_{2m-1}|+|c_{2m}|}{m} &\le \left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^{1/2}\left[\left(\sum_{m=1}^{\infty}|c_{2m-1}|^2\right)^{1/2}+\left(\sum_{m=1}^{\infty}|c_{2m}|^2\right)^{1/2}\right]\\ &\le \frac{\pi}{\sqrt{6}}\cdot 2\left(\sum_{m=1}^{\infty}|c_{2m-1}|^2+|c_{2m}|^2\right)^{1/2} \le \frac{\pi\sqrt{6}}{3}\,||f||_{L^2[0,2\pi]}<\infty\end{aligned}$$

Furthermore, with Lemma 3.38, for fixed t, there exists a constant C such that

$$|\eta_{2m-j}(t,x)| \le \left|\frac{c_{2m-j}}{m}\right|(|\alpha_{2m-j}|+|\beta_{2m-j}|)\int_0^x |V(y)|\,dy \le C\left|\frac{c_{2m-j}}{m}\right|$$

since both coefficients α_{2m-j} and β_{2m-j} are $\mathrm{O}(1)$ and the function $\int_0^x |V(y)|\,dy$ is bounded. Therefore,

$$\sum_{m=1}^{\infty}\sum_{j\in\{0,1\}}|\eta_{2m-j}(t,x)| \le C\sum_{m=1}^{\infty}\sum_{j\in\{0,1\}}\left|\frac{c_{2m-j}}{m}\right| \le \frac{C\pi\sqrt{6}}{3}\,||f||_{L^2[0,2\pi]}<\infty.$$

Using the Dominated Convergence Theorem, we deduce that $U_4(t,x)$ is a continuous function of x. Finally, we look at $U_3(t,x)$. From (3.39), we write

$$c_{2m-j} = \alpha_{2m-j}\langle f,a_m\rangle + \beta_{2m-j}\langle f,b_m\rangle + \frac{1}{m}\langle f,\widetilde{V}_{2m-j}\rangle + \langle f,R_{2m-j}\rangle$$

where the function $\widetilde{V}$ is given by

$$\widetilde{V}_{2m-j}(x) = \int_0^x \sin(m(x-y))\left(\alpha_{2m-j}a_m(y)+\beta_{2m-j}b_m(y)\right)V(y)\,dy$$

Then, we can write the function $U_3(t,x)$ as the sum

$$U_3(t,x) = U_6(t,x)+U_7(t,x)+U_8(t,x)$$

where the component functions are given by

$$\begin{aligned}U_6(t,x) &= \sum_{m=1}^{\infty}e^{-\mathrm{i}m^2t}\sum_{j\in\{0,1\}}\left(\alpha_{2m-j}\langle f,a_m\rangle+\beta_{2m-j}\langle f,b_m\rangle\right)\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right),\\ U_7(t,x) &= \sum_{m=1}^{\infty}\frac{e^{-\mathrm{i}m^2t}}{m}\sum_{j\in\{0,1\}}\langle f,\widetilde{V}_{2m-j}\rangle\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right),\\ U_8(t,x) &= \sum_{m=1}^{\infty}e^{-\mathrm{i}m^2t}\sum_{j\in\{0,1\}}\langle f,R_{2m-j}\rangle\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right).\end{aligned}$$

For $U_8(t,x)$, thanks to the Cauchy-Schwarz inequality, we have

$$\langle f,R_{2m-j}\rangle = \left|\int_0^{2\pi} f(x)R_{2m-j}(x)\,dx\right| \le \frac{2\pi C\,||f||_{L^2[0,2\pi]}}{m^2}$$

which means each individual term of $U_8(t,x)$ has the bound

$$\left| e^{-\mathrm{i}m^2 t} \langle f, R_{2m-j}\rangle \left(\alpha_{2m-j} a_m(x) + \beta_{2m-j} b_m(x)\right)\right| \leq \frac{C||f||_{L^2[0,2\pi]}}{m^2}$$

for some constant C, as the coefficients α_{2m-j} and β_{2m-j} are $\mathrm{O}(1)$. Applying the Weierstrass M-test here shows that $U_8(t,x)$ is a C^1 function. For $U_7(t,x)$, note that

$$\begin{aligned}\widetilde{V}_{2m-j}(x) = \sin(mx) &\int_0^x \cos(my)\left(\alpha_{2m-j} a_m(y) + \beta_{2m-j} b_m(y)\right) V(y)\, dy \\ &- \cos(mx) \int_0^x \sin(my)\left(\alpha_{2m-j} a_m(y) + \beta_{2m-j} b_m(y)\right) V(y)\, dy\end{aligned}$$

where the first term can be written as

$$\begin{aligned}&\sin(mx) \int_0^x \cos(my)\left(\alpha_{2m-j} a_m(y) + \beta_{2m-j} b_m(y)\right) V(y)\, dy \\ &\quad = \frac{\sin(mx)}{2\sqrt{\pi}} \int_0^x \left(\alpha_{2m-j}(1+\cos(2my)) + \beta_{2m-j}\sin(2my)\right) V(y)\, dy \\ &\quad = \frac{\alpha_{2m-j}}{2} b_m(x) \int_0^x V(y)\, dy + \frac{b_m(x)}{2}\int_0^x \left(\alpha_{2m-j}\cos(2my) + \beta_{2m-j}\sin(2my)\right) V(y)\, dy\end{aligned}$$

and the second term can be written as

$$\begin{aligned}&\cos(mx) \int_0^x \sin(my)\left(\alpha_{2m-j} a_m(y) + \beta_{2m-j} b_m(y)\right) V(y)\, dy \\ &\quad = \frac{\cos(mx)}{2\sqrt{\pi}} \int_0^x \left(\alpha_{2m-j}\sin(2my) + \beta_{2m-j}(1-\cos(2my))\right) V(y)\, dy \\ &\quad = \frac{\beta_{2m-j}}{2} a_m(x) \int_0^x V(y)\, dy + \frac{a_m(x)}{2}\int_0^x \left(\alpha_{2m-j}\sin(2my) - \beta_{2m-j}\cos(2my)\right) V(y)\, dy.\end{aligned}$$

Now, using integration by parts, with N_0 such that $x_{N_0} < x < x_{N_0+1}$ — that is, the $N_0{}^{\text{th}}$ discontinuous point is the greatest discontinuous point of $V(x)$ smaller than x — we have

$$\begin{aligned}\int_0^x \sin(2my)V(y)\, dy = &-\frac{1}{2m}\sum_{n=0}^{N_0-1}\left(\cos(2mx_{n+1}) \lim_{x\to x_{n+1}^-} V(x) - \cos(2mx_n)\lim_{x\to x_n^+} V(x)\right) \\ &- \frac{1}{2m}\left(\cos(2mx)V(x) - \cos(2mx_{N_0})\lim_{x\to x_{N_0}^+} V(x)\right) + \frac{1}{2m}\int_0^x \cos(2my)V(y)\, dy\end{aligned}$$

which grows like $\mathrm{O}(m^{-1})$. This also applies to

$$\begin{aligned}\int_0^x \cos(2my)V(y)\, dy = &\frac{1}{2m}\sum_{n=0}^{N_0-1}\left(\sin(2mx_{n+1}) \lim_{x\to x_{n+1}^-} V(x) - \sin(2mx_n)\lim_{x\to x_n^+} V(x)\right) \\ &+ \frac{1}{2m}\left(\sin(2mx)V(x) - \sin(2mx_{N_0})\lim_{x\to x_{N_0}^+} V(x)\right) - \frac{1}{2m}\int_0^x \sin(2my)V(y)\, dy.\end{aligned}$$

Then, each individual term of $U_7(t,x)$ can be split into three terms

$$\frac{e^{-\mathrm{i}m^2 t}}{m}\langle f, \widetilde{V}_{2m-j}\rangle\left(\alpha_{2m-j} a_m(x) + \beta_{2m-j} b_m(x)\right) = \rho_{1,2m-j}(t,x) + \rho_{2,2m-j}(t,x) + \rho_{3,2m-j}(t,x)$$

where

$$\begin{aligned}
\rho_{1,2m-j}(t,x) &= \frac{\alpha_{2m-j}e^{-\mathrm{i}m^2 t}}{2m}\left\langle fV_0, b_m\right\rangle\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right)\\
\rho_{2,2m-j}(t,x) &= \frac{\beta_{2m-j}e^{-\mathrm{i}m^2 t}}{2m}\left\langle fV_0, a_m\right\rangle\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right)\\
\rho_{3,2m-j}(t,x) &= e^{-\mathrm{i}m^2 t}\left\langle f, R^*_{2m-j}\right\rangle\left(\alpha_{2m-j}a_m(x)+\beta_{2m-j}b_m(x)\right)
\end{aligned}$$

with

$$V_0(x) = \int_0^x V(y)\,dy \quad \text{ and } \quad ||R^*_{2m-j}||_{L^\infty} = \mathrm{O}(m^{-2}).$$

As for the third sub-term, employing the same argument as with $U_8(t,x)$, we have the bound

$$|\rho_{3,2m-j}(t,x)| \le \frac{C\,||f||_{L^2[0,2\pi]}}{m^2}$$

which means that the Weierstrass M-test gives us that the sum across all of these sub-terms is a C^1 function. For the first sub-term, with α_{2m-j} and β_{2m-j} both being $\mathrm{O}(1)$, plus that $\{b_m\}$ is the orthonormal sine basis and $fV_0 \in L^2[0,2\pi]$, we have

$$\sum_{m=1}^{\infty}\sum_{j\in\{0,1\}}|\rho_{1,2m-j}(t,x)| \le \sum_{m=1}^{\infty} C\left|\frac{\langle fV_0, b_m\rangle}{m}\right| \le \frac{C\pi}{\sqrt{6}}\,||fV_0||_{L^2[0,2\pi]}.$$

This ensures that the sum across all of these sub-terms is continuous due to the Dominated Convergence Theorem. With the same method applied to the second sub-term, we conclude that

$$U_7(t,x) = \sum_{m=1}^{\infty}\sum_{j\in\{0,1\}}\left[\,\rho_{1,2m-j}(t,x)+\rho_{2,2m-j}(t,x)+\rho_{3,2m-j}(t,x)\,\right]$$

is a continuous function. This leaves us with $U_6(t,x)$. Using the asymptotics from Lemma 3.40, we can split it into two parts

$$U_6(t,x) = U_9(t,x) + U_{10}(t,x)$$

where the component functions are

$$\begin{aligned}
U_9(t,x) = \sum_{m=1}^{\infty} e^{-\mathrm{i}m^2 t}(&\langle f, \cos(y_m)a_m + \sin(y_m)b_m\rangle\left(\cos(y_m)a_m(x)+\sin(y_m)b_m(x)\right)\\
&+\langle f, \sin(y_m)a_m - \cos(y_m)b_m\rangle\left(\sin(y_m)a_m(x)-\cos(y_m)b_m(x)\right)),\\
U_{10}(t,x) = \sum_{m=1}^{\infty} e^{-\mathrm{i}m^2 t}\sum_{j\in\{0,1\}} &S_{1,2m-j}\langle f,a_m\rangle a_m(x) + S_{2,2m-j}\langle f,a_m\rangle b_m(x)\\
&+ S_{3,2m-j}\langle f,b_m\rangle a_m(x) + S_{4,2m-j}\langle f,b_m\rangle b_m(x)
\end{aligned}$$

with $||S_{i,2m-j}||_{L^\infty} = \mathrm{O}(m^{-1})$ for $1 \le i \le 4$. The identity (3.42) immediately identifies $U_9(t,x)$ as the revival component of the solution, while for $U_{10}(t,x)$, we can apply the same technique on the sum across each of the four sub-terms as we did for $U_7(t,x)$, which will lead to convergence to a continuous function. For example, the sum across the first sub-term has the bound

$$\sum_{m=1}^{\infty}\sum_{j\in\{0,1\}}|e^{-\mathrm{i}m^2 t}S_{1,2m-j}\langle f,a_m\rangle a_m| \le \sum_{m=1}^{\infty} C\left|\frac{\langle f,a_m\rangle}{m}\right| \le \frac{C\pi}{\sqrt{6}}||f||_{L^2[0,2\pi]},$$

which proves that $U_{10}(t,x)$ is continuous. Assembling all the components gives us that

$$u(t,x) = c_0\psi_0(x) + U_1(t,x) - U_9(t,x) - U_2(t,x) + \sum_{n=1}^{\infty} e^{-\mathrm{i}n^2 t}\left[\langle f,a_n\rangle a_n(x) + \langle f,b_n\rangle b_n(x)\right].$$

Setting $w(t,x) = c_0\psi_0(x) + U_1(t,x) - U_9(t,x) - U_2(t,x)$ then gives us the desired identity. □

The proof of Theorem 1.5 now follows from this lemma and a rearrangement with regard to the norm of the potential function.

Proof of Theorem 1.5. First, for $V=0$, the solution to (1.2) is given by

$$u_0(t,x)=\frac{\langle f^*,1\rangle}{2}+\sum_{n=1}^{\infty}e^{-\mathrm{i}n^2t}\left[\,\langle f^*,a_n\rangle a_n(x)+\langle f^*,b_n\rangle b_n(x)\,\right]=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{-\mathrm{i}n^2t}\langle f^*,e^{\mathrm{i}n(\cdot)}\rangle e^{\mathrm{i}nx}.$$

Then, at a rational time $t_0=2\pi q/r$, observe that

$$e^{\mathrm{i}n^2t_0}=e^{\mathrm{i}m^2t_0}$$

for any $n\equiv m \bmod r$. Thus, the solution has the form

$$u_0(t,x)=\frac{1}{2\pi}\sum_{m=0}^{r-1}e^{-2\pi\mathrm{i}m^2q/r}\sum_{n\equiv m \bmod r}\langle f^*,e^{\mathrm{i}n(\cdot)}\rangle e^{\mathrm{i}nx}.$$

We shall focus on the inner summation. Since

$$\sum_{k=0}^{r-1}e^{2\pi\mathrm{i}(m-n)k/r}=\begin{cases}r & \text{if } n\equiv m \bmod r\\ 0 & \text{otherwise}\end{cases}$$

we have

$$\begin{aligned}\sum_{n\equiv m \bmod r}\langle f,e^{\mathrm{i}n(\cdot)}\rangle e^{\mathrm{i}nx}&=\frac{1}{r}\sum_{k=0}^{r-1}e^{2\pi\mathrm{i}mk/r}\sum_{n\in\mathbb{Z}}e^{-2\pi\mathrm{i}nk/r}\langle f^*,e^{\mathrm{i}n(\cdot)}\rangle e^{\mathrm{i}nx}\\&=\frac{1}{r}\sum_{k=0}^{r-1}e^{2\pi\mathrm{i}mk/r}\sum_{n\in\mathbb{Z}}\left\langle f^*\left(\cdot-\frac{2\pi k}{r}\right),e^{\mathrm{i}n(\cdot)}\right\rangle e^{\mathrm{i}nx}=\frac{2\pi}{r}\sum_{k=0}^{r-1}e^{2\pi\mathrm{i}mk/r}f^*\left(x-\frac{2\pi k}{r}\right).\end{aligned}$$

Substituting this in the full expression above for the solution gives

$$u_0(t,x)=\frac{1}{r}\sum_{k,m=0}^{r-1}e^{2\pi\mathrm{i}\left(mk/r-m^2q/r\right)}f^*\left(x-2\pi\frac{k}{r}\right).$$

By (4.2), for any potential V satisfying $\langle V\rangle=0$,

$$u(t,x)=w(t,x)+\left[u_0(t,x)-\frac{\langle f^*,1\rangle}{2}\right]=\left[w(t,x)-\frac{\langle f^*,1\rangle}{2}\right]+\frac{1}{r}\sum_{k,m=0}^{r-1}e^{2\pi\mathrm{i}\left(mk/r-m^2q/r\right)}f^*\left(x-2\pi\frac{k}{r}\right).$$

Finally, notice that for a general potential V, the solution to (1.2) has the form

$$u(t,x)=e^{-\mathrm{i}\langle V\rangle t}u^*(t,x)$$

where u^* is the solution to

$$\begin{cases}\mathrm{i}\,u_t=-u_{xx}+[V-\langle V\rangle]\,u,\\ u(0,x)=f(x),\end{cases}$$

which has a potential that satisfies the mean-zero condition. Applying the derived result above, we have that

$$u(t,x)=e^{-2\pi\mathrm{i}\langle V\rangle q/r}\left(w(t,x)-\frac{\langle f^*,1\rangle}{2}\right)+\frac{1}{r}e^{-2\pi\mathrm{i}\langle V\rangle q/r}\sum_{k,m=0}^{r-1}e^{2\pi\mathrm{i}\left(mk/r-m^2q/r\right)}f^*\left(x-2\pi\frac{k}{r}\right).$$

Since the first term is a continuous function, the proof is complete. □

5. Numerical simulations

In this section, we wish to show various simulations of the solution to the Schrödinger equation. Consider the case where the potential is the following 2π-periodic piecewise constant function:

$$V(x) = \begin{cases} 0 & \text{for } x \in [0, \pi/2) \\ 9 & \text{for } x \in [\pi/2, 2\pi) \end{cases}$$

and the initial condition is the following 2π-periodic function:

$$f(x) = \begin{cases} -x/2\pi & \text{for } x \in [0, \pi) \\ 1 - x/2\pi & \text{for } x \in [\pi, 2\pi) \end{cases}.$$

Here, our plots were produced using Julia employing these assumptions:

- The number of eigenvalues used is 1000;
- The resolution of the plot is $\Delta x = .0005\pi$.

As a dispersive relation, the solution for the Schrödinger equation is piecewise smooth for times rational relative to the length of the interval. This can be seen in the plots of the real part of the solution below.

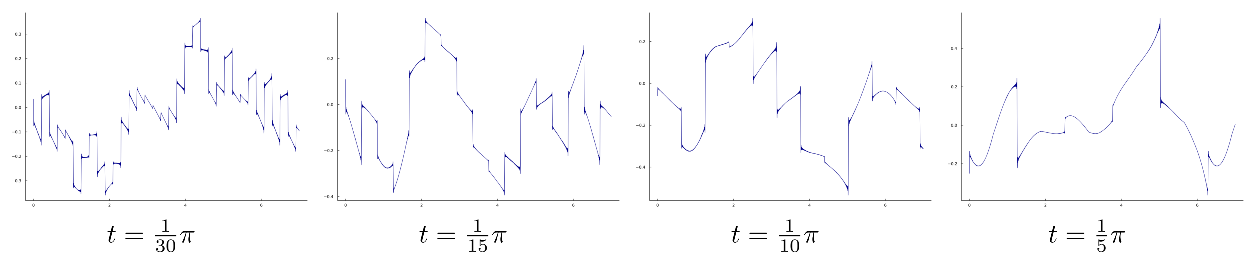

Figure 1. The real part of the solution.

We then consider the plots below of the same rational time steps, where we have the real part of the revival component and the remainder w on the top and bottom row, respectively.

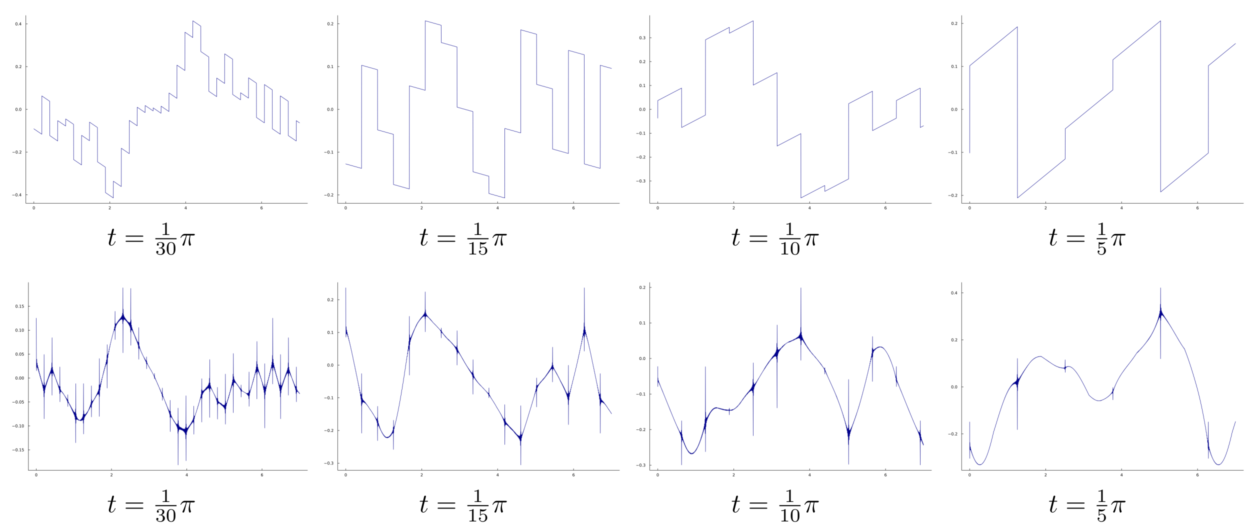

Figure 2. The real part of the revival component and the remainder.

Apart from the bars at the jump discontinuities caused by the Gibbs phenomenon, it can be observed that the real part of w is continuous across all time steps. Similarly, we demonstrate below the continuity for the imaginary part of w, which is plotted in the bottom row and is the difference between the imaginary part of the solution (top row) and the imaginary part of the revival component (middle row).

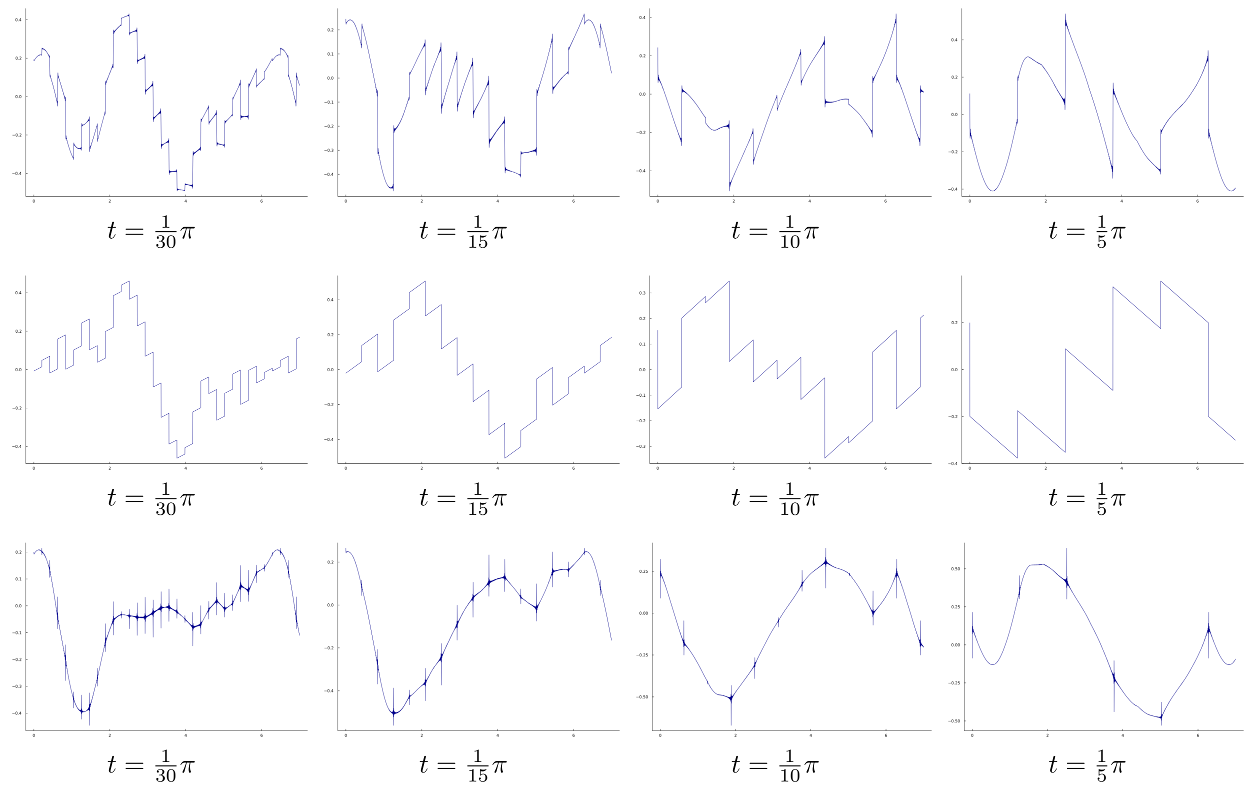

Figure 3. The imaginary part of the solution, the revival component, and the remainder.

This confirms the validity of Theorem 1.5 in this particular example.

Acknowledgements

The authors thank Jing Kang, Changzheng Qu, and Zihan Yin for their input on these results.

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[1]School of Mathematics, University of Minnesota, Minneapolis, MN 55455

Email address: dinhquan.tran23@gmail.com

[2]School of Mathematics, University of Minnesota, Minneapolis, MN 55455

Email address: olver@umn.edu

URL: http://www.math.umn.edu/∼olver