BÄCKLUND TRANSFORMATIONS FOR TRI-HAMILTONIAN DUAL STRUCTURES OF MULTI-COMPONENT INTEGRABLE SYSTEMS

JING KANG, XIAOCHUAN LIU, PETER J. OLVER, AND CHANGZHENG QU

Abstract. In this paper, the Bäcklund transformation based-approach is explored to obtain Hamiltonian operators of multi-component integrable systems which are governed by a compatible tri-Hamiltonian dual structures. The resulting Hamiltonian operators are used not only to derive multi-component bi-Hamiltonian integrable hierarchies and their dual integrable versions, but also to serve as a criterion to verify the compatibility for the corresponding (dual) bi-Hamiltonian operators. The approach is illustrated through the construction of two families of integrable bi-Hamiltonian hierarchies associated with certain two-component and three-component dispersive water wave systems and their modifications. The corresponding dual bi-Hamiltonian systems are also derived. Some of them are new integrable systems. Furthermore, we classify the analytic and nonanalytic traveling wave solutions to the two-component dual nonlinearly dispersive water wave system.

Key words: bi-Hamiltonian structure; Bäcklund transformation; tri-Hamiltonian duality; dispersive water wave system; nonanalytic traveling wave.
2000 *Mathematics Subject Classification. Primary:* 37K10, 37K35, 35Q58.

1. Introduction

In this paper, we develop the Bäcklund transformation based-method, in the multi-component setting, to derive Hamiltonian operators which admit compatible tri-Hamiltonian structures, and consequently lead to the associated bi-Hamiltonian integrable hierarchies and their dual counterparts endowed with nonlinear dispersion.

The algebraic theory of integrable bi-Hamiltonian systems of evolution equations was developed in detail over the last three decades. The innovative work due to Magri [37] establishes, for a bi-Hamiltonian system, the existence of an infinite hierarchy of mutually commuting conservation laws and bi-Hamiltonian flows. On the one hand, the recursion operator [38, 39] (see also [41]) can be derived from the bi-Hamiltonian structure of such integrable systems, and thus used to generate integrable hierarchies and higher-order symmetries. On the other hand, the bi-Hamiltonian property is closely related to the existence of a Lax pair representation. It has been found that a notable number of integrable systems are, in fact, bi-Hamiltonian. Those include a variety of well-known classical integrable systems such as the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV) equations, [41], as well as integrable systems endowed with nonlinear dispersion such as the Camassa-Holm (CH) equation, the modified Camassa-Holm (mCH) equation, etc., [5, 15, 42].

Several methods have been employed to obtain integrable bi-Hamiltonian systems endowed with nonlinear dispersion. In particular, a theory of tri-Hamiltonian duality was developed systematically in the references [18, 19, 42]. This approach starts from the basic observation that most of integrable soliton equations, which are known to possess a bi-Hamiltonian structure, actually support a compatible trio of Hamiltonian structures through an elementary scaling argument. Rearranging the three Hamiltonian operators provides an algorithmic method for systematically constructing dual integrable systems. Some typical dual integrable systems including the CH equation, the mCH equation, and certain two-component CH equations, etc., can be obtained in this manner. The resulting dual systems are endowed with nonlinear dispersion, and thus admit non-smooth solitons including compactons, cuspons, peakons, and more exotic species [35]. More precisely, applying tri-Hamiltonian duality to the bi-Hamiltonian representation of the KdV equation and the mKdV equation, the resulting dual integrable systems are the well-studied CH equation [5, 18, 19, 42] and the mCH equation [15, 18, 42]. As two prototypical models in the class of the bi-Hamiltonian integrable equations with quadratic and cubic nonlinearity respectively, the CH equation and the mCH equation have attracted enormous attention in recent years because of their remarkable properties: complete integrability [5, 6, 11, 20, 28, 45], physical relevance of the nonlinear shallow-water waves [5, 6, 13, 16, 27], non-smooth soliton structures of peakons and multi-peakons [3, 5, 6, 22], delicate geometric formulations [9, 22, 30] and the presence of breaking waves [10, 22, 34, 36].

In addition to the scalar setting, the approach of tri-Hamiltonian duality can also be applied to multi-component bi-Hamiltonian systems, with matrix-valued Hamiltonian operators. For instance, it was proved in [42] that the bi-Hamiltonian structure of the integrable *Ito system* [26]

$$u_t = u_{xxx} + 3uu_x + vv_x, \qquad v_t = (uv)_x, \tag{1.1}$$

supports the required tri-Hamiltonian dual structure. The corresponding dual bi-Hamiltonian system takes the form

$$m_t + 2u_x m + u m_x + \rho\rho_x = 0, \qquad \rho_t + (u\,\rho)_x = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

which is the so-called *two-component CH system* [7, 12]. It, together with several generalizatons, has recently been extensively studied from a variety of perspectives [14, 21, 23, 25, 43].

It is well-known that Bäcklund transformations play an important role in soliton theory and integrable systems, and are a useful tool to obtain new integrable systems from some known integrable ones, and to construct new solutions from known ones [44]. A Bäcklund transformation is a system of first-order partial differential equations relating solutions of two equations under consideration. The approach we adopt here generalizes the Bäcklund transformation method, and is based on the tri-Hamiltonian dual structure.

We shall focus our attention on the two-component and three-component systems. Our analysis is based on generalized Miura-type Bäcklund transformations depending on several arbitrary parameters, which, on the one hand, can be utilized to give rise to the Hamiltonian operators supporting tri-Hamiltonian structures, and on the other hand, can serve as an alternative method to verify the compatibility of the desired Hamiltonian pair. In such situations, the rearrangement of the Hamiltonian triple will produce two pairs of compatible Hamiltonian operators which lead to the multi-component integrable systems and their corresponding dual counterparts, respectively. In particular, we exploit this approach to derive 2×2 Hamiltonian operators with appropriate parameters which, in view of the intrinsic tri-Hamiltonian structure, produce two classes of integrable hierarchies involving the so-called *dispersive water wave* (*DWW*) *system* as well as the corresponding dual integrable version. The DWW system takes the following form [4, 31]

$$q_t = \big(-q_x + 2qr\big)_x, \qquad r_t = \big(r_x + r^2 + 2q\big)_x, \tag{1.3}$$

which is an integrable bi-Hamiltonian system, and is recognized to be in the form of a bi-directional system of Boussinesq type, modeling the propagation of shallow water waves [40]. In light of the tri-Hamiltonian characterization of the bi-Hamiltonian structure of system

(1.3), we derive the corresponding dual integrable system in the form

$$\begin{cases} g_t - \nu g_{xt} = \big(- g_x + 2fg - \nu\, f g_x \big)_x, \\ f_t + \nu f_{xt} = \big(f_x + 2g + f^2 + \nu\, f f_x \big)_x, \end{cases} \tag{1.4}$$

where $\nu = \pm 1$. The bi-Hamiltonian structure of system (1.4) naturally follows from the tri-Hamiltonian structure of system (1.3). In addition, Lax formulations admitted by systems (1.3) and (1.4) are also obtained.

In [31], it was shown that, via the Miura-type Bäcklund transformation

$$q = -u_x - u^2 + uv, \qquad r = v, \tag{1.5}$$

the DWW system (1.3) is mapped to the *modified DWW (mDWW) system*

$$\begin{cases} u_t = \big(- u_x + 2uv - u^2 \big)_x, \\ v_t = \big(v_x - 2u_x - 2u^2 + 2uv + v^2 \big)_x. \end{cases} \tag{1.6}$$

Making use of the recursion operator of the DWW system and the Bäcklund transformation (1.5), we establish a bi-Hamiltonian representation of the mDWW system (1.6) and its dual counterpart.

Furthermore, the following three-component integrable system

$$s_t = (sr)_x, \qquad q_t = \left(-q_x + 2qr + \tfrac{1}{2} s^2 \right)_x, \qquad r_t = \left(r_x + r^2 + 2q \right)_x, \tag{1.7}$$

was proposed in [1] as a model of a quadri-Hamiltonian system. It is noted that (1.7) reduces to the DWW system (1.3) when $s = 0$, and hence (1.7) can be regarded as an integrable three-component generalization. In this paper, we introduce a general Bäcklund transformation that enables us to derive a 3×3 Hamiltonian operator that induces a compatible tri-Hamiltonian structure; this allows us to construct two families of three-component integrable hierarchies involving the system (1.7) and its dual. In addition, again using corresponding Bäcklund transformations, several modified versions of these three-component integrable systems and their related dual systems are also derived.

The existence of peaked solitons is one of the non-trivial properties of nonlinearly dispersive wave equations of CH type [5], which helps explain why these systems have attracted so much attention in the last thirty years. Recently, it was found that the mCH equation, as the dual equation of the mKdV equation, also admits peaked solitons [22]. However, the two-component CH system (1.2), which is the dual system of the Ito system, does not admit non-trivial peaked solitons [7, 12, 25]. A natural question remains: what types of dual integrable systems obtained through the tri-Hamiltonian duality approach admit peaked solitons. To this end, it is of interest to study and classify the traveling wave solutions of the dual system of the DDW system and its modified versions, leading to some new types of nonanalytic solitary wave solutions. We refer the reader to [29] for further analysis of the existence, non-existence, stability, and other properties of solitary wave solutions to higher order wave models.

The remainder of this paper is organized as follows. In Section 2, we first formalize some notations and definitions in the multi-component setting of Hamiltonian operators, bi-Hamiltonian structures, as well as Bäcklund transformations, and recall some basic results required throughout this paper. Next, we present two theorems—Theorem 2.2 and Theorem 2.3—which demonstrate the relationship not only between the matrix Hamiltonian operators but also between the matrix recursion operators subject to multi-component Bäcklund transformations. In Section 3, we combine the Bäcklund transformations with tri-Hamiltonian duality to study two-component bi-Hamiltonian integrable hierarchies involving the DWW and mDWW systems, as well as their dual systems. We use the same approach to propose several three-component bi-Hamiltonian integrable systems and the corresponding dual versions in Section 4. Finally, in Section 5, the analytic and nonanalytic traveling wave solutions to the two-component dual DWW system are fully classified.

2. Preliminaries

Throughout this paper, we consider evolution equations involving a single spacial variable $x \in \mathbb{R}$ and time $t \in \mathbb{R}$. We let $\mathcal{A}$ denote the space of differential functions, and understand the functions in $\mathcal{A}$ depending on the indicated dependent variables and their spatial derivatives only. We further define $\mathcal{A}^n$ to be the space of n-component differential functions.

Consider an n-component system of evolution equations

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}), \qquad \mathbf{u} = (u_1(t,\, x)\,, \ldots, u_n(t,\, x))^T, \tag{2.1}$$

where $\mathbf{K}(\mathbf{u}) = (K_1(\mathbf{u})\,, \ldots,\, K_n(\mathbf{u}))^T \in \mathcal{A}^n$ is an n-component differential function depending on the components of $\mathbf{u}$ and their x-derivatives up to a given order. The system (2.1) is called *Hamiltonian* if it can be written in the form

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}) = \mathcal{J}\delta\mathcal{H}(\mathbf{u}),$$

where $\mathcal{H}(\mathbf{u})$ is the *Hamiltonian functional*, $\delta\mathcal{H}(\mathbf{u}) = (\delta\mathcal{H}/\delta u_1\,, \ldots,\, \delta\mathcal{H}/\delta u_n)^T$ is the *variational derivative* of $\mathcal{H}$, and the $n \times n$ matrix operator $\mathcal{J}$ is a *Hamiltonian operator*, [41]. For a candidate Hamiltonian operator $\mathcal{J}$, its corresponding *Poisson bracket* is defined by L^2 inner product for the functions in the Schwartz space

$$\big\{\mathcal{P},\ \mathcal{L}\big\}_{\mathcal{J}} = \langle\, \delta\mathcal{P},\ \mathcal{J}\delta\mathcal{L}\,\rangle = \int \delta\mathcal{P} \cdot \mathcal{J}\delta\mathcal{L}\, \mathrm{d}x,$$

which is required to be both *skew-symmetric*:

$$\big\{\mathcal{P},\ \mathcal{Q}\big\}_{\mathcal{J}} = -\big\{\mathcal{Q},\ \mathcal{P}\big\}_{\mathcal{J}}$$

and satisfy the *Jacobi identity*:

$$\big\{\big\{\mathcal{P}\,, \mathcal{Q}\big\}_{\mathcal{J}},\ \mathcal{R}\big\}_{\mathcal{J}} + \big\{\big\{\mathcal{Q},\ \mathcal{R}\big\}_{\mathcal{J}},\ \mathcal{P}\big\}_{\mathcal{J}} + \big\{\big\{\mathcal{R},\ \mathcal{P}\big\}_{\mathcal{J}},\ \mathcal{Q}\big\}_{\mathcal{J}} = 0, \tag{2.2}$$

for all functionals $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, cf. [41].

The system (2.1) is said to be *bi-Hamiltonian*, if it can be written in the form

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}) = \mathcal{J}_1\delta\mathcal{H}_1(\mathbf{u}) = \mathcal{J}_2\delta\mathcal{H}_0(\mathbf{u}), \tag{2.3}$$

where $\mathcal{H}_0(\mathbf{u})$ and $\mathcal{H}_1(\mathbf{u})$ are the Hamiltonian functionals, $\mathcal{J}_1$ and $\mathcal{J}_2$ are independent $n \times n$ Hamiltonian operators, satisfying the *compatibility condition* that every linear combination $c_1\mathcal{J}_1 + c_2\mathcal{J}_2$ is Hamiltonian, i.e., satisfies the Jacobi identity (2.2). Assume that the operator $\mathcal{J}_1$ of the Hamiltonian pair is nondegenerate, then the operator $\mathcal{R} = \mathcal{J}_2\mathcal{J}_1^{-1}$ is a recursion operator of the bi-Hamiltonian system (2.3) [37, 41]. The following theorem, due to Magri [37] (see also [41]), summarizes the basic properties of bi-Hamiltonian systems.

Theorem 2.1. *Consider a bi-Hamiltonian system of evolution equations* (2.3). *Assume that the Hamiltonian operator $\mathcal{J}_1$ is nondegenerate. Let $\mathcal{R} = \mathcal{J}_2\mathcal{J}_1^{-1}$ and*

$$\mathbf{K}_0(\mathbf{u}) = \big(K_0^1(\mathbf{u})\,, \ldots,\, K_0^n(\mathbf{u})\big)^T = \mathcal{J}_1\delta\mathcal{H}_0.$$

For each $m = 1, 2, \ldots$, define $\mathbf{K}_m(\mathbf{u}) = \mathcal{R}\,\mathbf{K}_{m-1}(\mathbf{u})$. Then there exists a sequence of functionals $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$, such that

(i) *for each $m \geq 1$, the evolution system*

$$\mathbf{u}_t = \mathbf{K}_m(\mathbf{u}) = \mathcal{J}_1\delta\mathcal{H}_m(\mathbf{u}) = \mathcal{J}_2\delta\mathcal{H}_{m-1}(\mathbf{u}) \tag{2.4}$$

is a bi-Hamiltonian system;

(ii) *the Hamiltonian functionals $\mathcal{H}_m$ are all in involution with respect to either Poisson bracket:*

$$\{\mathcal{H}_l, \mathcal{H}_m\}_{\mathcal{J}_1} = 0 = \{\mathcal{H}_l, \mathcal{H}_m\}_{\mathcal{J}_2}, \qquad l, m \geq 0,$$

and hence provide an infinite collection of conservation laws for each of the bi-Hamiltonian systems (2.4).

The method of tri-Hamiltonian duality [19, 42] can be applied to the bi-Hamiltonian systems (2.4) whose associated Hamiltonian operator $\mathcal{J}_2$ can be written as the sum of two distinct Hamiltonian operators $\mathcal{K}_2$ and $\mathcal{K}_3$ that scale independently under $x \mapsto \lambda x$ and/or $u \mapsto \mu u$. The fact that, under the scaling transformations, $\mathcal{J}_2 = \mathcal{K}_2 + \mathcal{K}_3$ is mapped into $\bar{\mathcal{J}}_2 = \lambda^\alpha \mathcal{K}_2 + \lambda^\beta \mathcal{K}_3$ with $\alpha \neq \beta$ immediately implies that $\mathcal{K}_2$ and $\mathcal{K}_3$ form a compatible Hamiltonian pair. We thus conclude that each linear combination of $\mathcal{J}_1 = \mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ is a Hamiltonian operator. In particular, each possible pair of $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ is compatible. As a consequence, by introducing an alternative Hamiltonian pair $\widehat{\mathcal{J}}_1 = \mathcal{K}_1 \pm \mathcal{K}_2$ and $\widehat{\mathcal{J}}_2 = \mathcal{K}_3$, one can produce a hierarchy of integrable equations, which can be viewed as the dual counterpart (called the tri-Hamiltonian dual in [42]) of the original hierarchy (2.4) that is generated by $\mathcal{J}_1$ and $\mathcal{J}_2$.

To illustrate the method more precisely, let us consider the scalar setting, so $n = 1$. We assume that $\mathcal{K}_1 = \partial_x$ and $\mathcal{K}_2$ are constant coefficient skew-adjoint differential operators, and, further, that $\widehat{\mathcal{J}}_1$ factorizes into a product of ∂_x with a self-adjoint constant coefficient differential operator A, i.e., $\widehat{\mathcal{J}}_1 = \partial_x A$. One usually introduce the new variable $\rho = Au$ to replace u, while $\widehat{J}_2$ is obtained from K_3 by replacing u by ρ. Then the dual integrable system is found by applying the recursion operator $\widehat{\mathcal{R}} = \widehat{\mathcal{J}}_2 \widehat{\mathcal{J}}_1^{-1} = \widehat{\mathcal{J}}_2 A^{-1} \partial_x^{-1}$ to the seed equation $\rho_t = \rho_x$, the resulting dual counterpart will thus take the form $\rho_t = \widehat{\mathcal{J}}_2 A^{-1} \rho$. Hence the corresponding dual integrable system can be written in local form in terms of the dependent variable u. Furthermore, applying the recursion operator $\widehat{\mathcal{R}}$ succesively to $\rho_t = \rho_x$ produces the dual hierarchy of integrable systems, and each flow in the hierarchy will be a bi-Hamiltonian system

$$\rho_t = \widehat{\mathbf{K}}_m(\rho) = \widehat{\mathcal{J}}_1 \delta \widehat{\mathcal{H}}_m = \widehat{\mathcal{J}}_2 \delta \widehat{\mathcal{H}}_{m-1},$$

which is governed by the dual Hamiltonian pair $\widehat{\mathcal{J}}_1$ and $\widehat{\mathcal{J}}_2$.

We now consider the effect of a Bäcklund transformation relating n-component systems (2.1) involving $\mathbf{u}$ and a similar system involving the transformed dependent variables $\tilde{\mathbf{u}}$:

$$\tilde{\mathbf{u}}_t = \mathbf{G}(\tilde{\mathbf{u}}), \quad \tilde{\mathbf{u}} = \big(\tilde{u}_1(t,\, x)\,, \ldots, \tilde{u}_n(t,\, x)\big)^T, \tag{2.5}$$

where $\mathbf{G}(\tilde{\mathbf{u}}) = (G_1(\tilde{\mathbf{u}})\,, \ldots,\, G_n(\tilde{\mathbf{u}}))^T \in \mathcal{A}^n$.

Definition 2.1. *An n-component implicit equation of the form*

$$\mathbf{B}(\mathbf{u}, \tilde{\mathbf{u}}) = (B_1(\mathbf{u}, \tilde{\mathbf{u}})\,, \ldots,\, B_n(\mathbf{u}, \tilde{\mathbf{u}}))^T = \mathbf{0}, \tag{2.6}$$

where each $B_i(\mathbf{u}, \tilde{\mathbf{u}}), i = 1, \ldots, n$, is a differential function depending on $\mathbf{u}$, $\tilde{\mathbf{u}}$ and their first-order x-derivatives, is called a Bäcklund transformation *between systems* (2.1) *and* (2.5)*, if, whenever $\mathbf{u}(t,x), \tilde{\mathbf{u}}(t,x)$ are any two solutions of* (2.1) *and* (2.5)*, respectively, such that* (2.6) *holds at one time $t = t_0$, then* (2.6) *holds identically for all (t,x) with $t > t_0$.*

Assume that systems (2.1) and (2.5) are both Hamiltonian systems. Given a Bäcklund transformation (2.6) relating (2.1) and (2.5), the relationship between their respective Hamiltonian operators can be established.

Theorem 2.2. *Given a Bäcklund transformation* (2.6) *between the n-component systems* (2.1) *and* (2.5)*, let $\mathbf{B}_{\mathbf{u}}$ and $\mathbf{B}_{\tilde{\mathbf{u}}}$ denote the $n \times n$ matrix differential operators given by their Fréchet derivatives with respect to $\mathbf{u}, \tilde{\mathbf{u}}$, respectively, and assume $\mathbf{B}_{\mathbf{u}}$ and $\mathbf{B}_{\tilde{\mathbf{u}}}$ are invertible. Set*

$$\mathbf{T} = \mathbf{B}_{\tilde{\mathbf{u}}}^{-1} \mathbf{B}_{\mathbf{u}}, \tag{2.7}$$

and let $\mathbf{T}^*$ *denote its formal adjoint. If* (2.1) *is Hamiltonian with Hamiltonian operator* $\mathcal{J}$*, then* (2.5) *is also Hamiltonian with Hamiltonian operator*

$$\widetilde{\mathcal{J}}(\tilde{\mathbf{u}}) = \mathbf{T}\,\mathcal{J}(\mathbf{u})\,\mathbf{T}^*. \tag{2.8}$$

The effect of a Bäcklund transformation on recursion operators also involves the operator (2.7).

Theorem 2.3. *Consider a Bäcklund transformation* (2.6) *between the n-component systems* (2.1) *and* (2.5)*, and let* $\mathbf{T}$ *be the* $n \times n$ *matrix differential operator given in* (2.7)*. If* $\mathcal{R}(\mathbf{u})$ *is a recursion operator for* (2.1)*, then*

$$\widetilde{\mathcal{R}}(\tilde{\mathbf{u}}) = \mathbf{T}\,\mathcal{R}(\mathbf{u})\,\mathbf{T}^{-1}$$

is a recursion operator admitted by (2.5)*.*

The proofs for Theorems 2.2 and 2.3 in the scalar case ($n = 1$) were given respectively in [20] and [17]. We now prove Theorems 2.2 and 2.3 in multi-component case. Let us begin with some notations and preliminary remarks. Taking the t-derivative of each equation in system (2.6) and using systems (2.1) and (2.5), we arrive at the following identities

$$\sum_{j=1}^{n} \left(B_{i,u_j}\, K_j + B_{i,\tilde{u}_j}\, G_j \right) = 0, \quad i = 1\,, \ldots,\, n.$$

For convenience, we write the above expression using vectorial notation:

$$\mathbf{B}_{\mathbf{u}}\,\mathbf{K} + \mathbf{B}_{\tilde{\mathbf{u}}}\,\mathbf{G} = \mathbf{0}. \tag{2.9}$$

Since $\mathbf{B}_{\tilde{\mathbf{u}}}$ is invertible, (2.9) together with (2.7) immediately leads to

$$\mathbf{G} = -\mathbf{T}\mathbf{K}. \tag{2.10}$$

Next, the nondegeneracy assumption on $\mathbf{B}_{\tilde{\mathbf{u}}}$ ensures that the Bäcklund transformation (2.6) uniquely determines, for each $i = 1, \ldots, n$,

$$u_i = U_i\left(\tilde{u}_1\,, \ldots,\, \tilde{u}_n\right), \tag{2.11}$$

which implies that

$$B_i\left(U_1\left(\tilde{u}_1\,, \ldots,\, \tilde{u}_n\right)\,, \ldots,\, U_n\left(\tilde{u}_1\,, \ldots,\, \tilde{u}_n\right),\, \tilde{u}_1\,, \ldots,\, \tilde{u}_n\right) = 0, \quad i = 1, \ldots, n, \tag{2.12}$$

hold identically, for $\tilde{\mathbf{u}} = \left(\tilde{u}_1\,, \ldots,\, \tilde{u}_n\right)^T$. Then, we denote the total derivative of the differential function $\mathbf{B} = \left(B_1\,, \ldots,\, B_n\right)^T \in \mathcal{A}^n$ with respect to $\tilde{\mathbf{u}}$ by $\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{B}$, which is a $n \times n$ matrix differential operator with entries

$$\left(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{B}\right)_{ij} = \mathrm{d}_{\tilde{u}_j} B_i, \quad i, j = 1, \ldots, n,$$

defined for $\alpha = \left(\alpha_1\,, \ldots,\, \alpha_n\right)^T$ by

$$\left(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{B}\right)\alpha = \left(\sum_{j=1}^{n} \left(\mathrm{d}_{\tilde{u}_j} B_1\right)\alpha_j\,, \ldots,\, \sum_{j=1}^{n} \left(\mathrm{d}_{\tilde{u}_j} B_n\right)\alpha_j \right)^T,$$

where, in view of (2.11) and (2.12), for each $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} \left(\mathrm{d}_{\tilde{u}_j} B_i\right)\alpha_j = \sum_{j=1}^{n} \left(B_{i,\tilde{u}_j} + \sum_{k=1}^{n} B_{i,u_k} U_{k,\tilde{u}_j} \right)\alpha_j,$$

with $U_{k,\tilde{u}_j}$ being the Fréchet derivative of differential function U_k with respect to $\tilde{u}_j$ for $j, k = 1, \ldots, n$. Therefore, we arrive at

$$\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{B} = \mathbf{B}_{\tilde{\mathbf{u}}} + \mathbf{B}_{\mathbf{u}} \frac{\partial\left(U_1\,, \ldots,\, U_n\right)}{\partial\left(\tilde{u}_1\,, \ldots,\, \tilde{u}_n\right)},$$

where the last term denotes the $n \times n$ matrix differential operator with entries

$$\left(\frac{\partial \left(U_1 \,, \ldots \,, \, U_n \right)}{\partial \left(\tilde{u}_1 \,, \ldots \,, \, \tilde{u}_n \right)} \right)_{ij} = U_{i, \tilde{u}_j} \,, \quad i, \, j = 1 \,, \ldots \,, \, n.$$

Since the relation (2.12) implies $\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{B} \equiv \mathbf{0}$, it then follows that

$$\frac{\partial \left(U_1 \,, \ldots \,, \, U_n \right)}{\partial \left(\tilde{u}_1 \,, \ldots \,, \, \tilde{u}_n \right)} = -\mathbf{T}^{-1}. \tag{2.13}$$

Finally, for the $n \times n$ matrix differential operator $\mathbf{T}$, denote its partial and total derivatives with respect to $\tilde{\mathbf{u}}$ along a direction α by $\big(\partial_{\tilde{\mathbf{u}}} \mathbf{T}\big)[\alpha]$ and $\big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big)[\alpha]$, respectively. To be precise, using entries $T_{ij}, \ i, j = 1, \ldots, n$, the partial derivative $\big(\partial_{\tilde{\mathbf{u}}} \mathbf{T}\big)[\alpha]$ along the direction $\alpha = \left(\alpha_1 \,, \ldots \,, \, \alpha_n\right)^T$ is a $n \times n$ matrix differential operator with the corresponding entries

$$\Big(\left(\partial_{\tilde{\mathbf{u}}} \mathbf{T}\right) [\alpha] \Big)_{ij} = \sum_{k=1}^{n} T_{ij, \tilde{u}_k} \left[\alpha_k\right], \quad i, \, j = 1 \,, \ldots \,, \, n,$$

where $T_{ij,\tilde{u}_k}[\cdot]$ is the partial derivative of T_{ij} with respect to $\tilde{u}_k$ along some direction. While, the total derivative $\big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big)[\alpha]$ is also a $n \times n$ matrix differential operator, which has entries

$$\Big(\left(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\right) [\alpha] \Big)_{ij} = \sum_{k=1}^{n} \left(\mathrm{d}_{\tilde{u}_k} T_{ij}\right) \left[\alpha_k\right], \quad i, \, j = 1 \,, \ldots \,, \, n,$$

where

$$\left(\mathrm{d}_{\tilde{u}_k} T_{ij}\right) \left[\alpha_k\right] = \left(T_{ij, \tilde{u}_k}\right) \left[\alpha_k\right] + \sum_{l=1}^{n} \left(T_{ij, u_l}\right) \left[U_{l, \tilde{u}_k} \alpha_k\right],$$

and $T_{ij,u_l}[\cdot]$ is the partial derivative of T_{ij} with respect to u_l along some direction. Consequently, using (2.13), we deduce that

$$\left(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\right) [\alpha] = \left(\partial_{\tilde{\mathbf{u}}} \mathbf{T}\right) [\alpha] - \left(\partial_{\mathbf{u}} \mathbf{T}\right) \left[\mathbf{T}^{-1} \alpha\right]. \tag{2.14}$$

Proof of Theorem 2.2. It is obvious that the skew-symmetric property of $\widetilde{\mathcal{J}}$ is preserved subject to the transformation (2.8). So it remains to verify the Jacobi identity for $\widetilde{\mathcal{J}}$, which, in the framework of n-component systems, is equivalent to

$$\Big\langle \mathbf{P} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \widetilde{\mathcal{J}}\big) \big[\widetilde{\mathcal{J}} \mathbf{L}\big] \mathbf{Q} \Big\rangle + \Big\langle \mathbf{Q} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \widetilde{\mathcal{J}}\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L} \Big\rangle + \Big\langle \mathbf{L} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \widetilde{\mathcal{J}}\big) \big[\widetilde{\mathcal{J}} \mathbf{Q}\big] \mathbf{P} \Big\rangle = 0, \tag{2.15}$$

where $\mathbf{P}, \ \mathbf{Q}, \ \mathbf{L} \in \mathcal{A}^n$ are the variational derivatives of the indicated Hamiltonian functionals $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{L}$, respectively. In view of the expressions (2.7) and (2.8), we find

$$\big(\mathrm{d}_{\tilde{\mathbf{u}}} \widetilde{\mathcal{J}}\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L} = \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathcal{J} \mathbf{T}^* \mathbf{L} + \mathbf{T} \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathcal{J}\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{T}^* \mathbf{L} + \mathbf{T} \mathcal{J} \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L}. \tag{2.16}$$

Using the fact that $\mathcal{J}$ is skew-symmetric and the symmetry property of the operator $\mathbf{T}^*$, say

$$\big\langle \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) [\alpha] \mathbf{P} \,; \, \mathbf{f} \, \big\rangle = \big\langle \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) \big[\mathbf{T} \, \mathbf{f}\big] \mathbf{P} \,; \, \mathbf{T}^{-1} \alpha \, \big\rangle = \big\langle \, \mathbf{P} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big) \big[\mathbf{T} \, \mathbf{f}\big] \mathbf{T}^{-1} \alpha \, \big\rangle, \tag{2.17}$$

for $\mathbf{f} = \left(f_1 \,, \ldots \,, \, f_n\right)^T$ and $\alpha = \left(\alpha_1 \,, \ldots \,, \, \alpha_n\right)^T$, we compute the inner product of $\mathbf{Q}$ and the third term in the expression (2.16) as

$$\begin{aligned} \Big\langle \mathbf{Q} \,; \, \mathbf{T} \mathcal{J} \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L} \Big\rangle &= \Big\langle \mathbf{T}^* \mathbf{Q} \,; \, \mathcal{J} \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L} \Big\rangle \\ &= -\Big\langle \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}^*\big) \big[\widetilde{\mathcal{J}} \mathbf{P}\big] \mathbf{L} \,; \, \mathcal{J} \mathbf{T}^* \mathbf{Q} \Big\rangle = -\Big\langle \mathbf{L} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big) \big[\widetilde{\mathcal{J}} \mathbf{Q}\big] \mathcal{J} \mathbf{T}^* \mathbf{P} \Big\rangle. \end{aligned}$$

Now, denote

$$\Big\langle \mathbf{L} \,; \, \big(\mathrm{d}_{\tilde{\mathbf{u}}} \mathbf{T}\big) \big[\widetilde{\mathcal{J}} \mathbf{Q}\big] \mathcal{J} \mathbf{T}^* \mathbf{P} \Big\rangle = \big| \, \mathbf{L}, \ \mathbf{Q}, \ \mathbf{P} \, \big|_{\widetilde{\mathcal{J}}} \,.$$

Then we have

$$\Big\langle\, \mathbf{Q}\,;\, \mathbf{T}\mathcal{J}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}^*\big)\big[\widetilde{\mathcal{J}}\mathbf{P}\big]\mathbf{L}\,\Big\rangle = -\mid \mathbf{L},\ \mathbf{Q},\ \mathbf{P}\mid_{\widetilde{\mathcal{J}}}$$

and

$$\Big\langle\, \mathbf{Q}\,;\, \big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\widetilde{\mathcal{J}}\mathbf{P}\big]\mathcal{J}\mathbf{T}^*\mathbf{L}\,\Big\rangle = \mid \mathbf{Q},\ \mathbf{P},\ \mathbf{L}\mid_{\widetilde{\mathcal{J}}}.$$

Set for simplicity $\widetilde{\mathbf{P}} = \mathbf{T}^*\mathbf{P}$, $\widetilde{\mathbf{Q}} = \mathbf{T}^*\mathbf{Q}$ and $\widetilde{\mathbf{L}} = \mathbf{T}^*\mathbf{L}$. A direct computation shows that

$$\Big\langle\, \mathbf{Q}\,;\, \mathbf{T}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathcal{J}\big)\big[\widetilde{\mathcal{J}}\mathbf{P}\big]\mathbf{T}^*\mathbf{L}\,\Big\rangle = -\Big\langle\, \mathbf{Q}\,;\, \mathbf{T}\big(\partial_{\mathbf{u}}\mathcal{J}\big)\Big[\mathbf{T}^{-1}\widetilde{\mathcal{J}}\mathbf{P}\Big]\widetilde{\mathbf{L}}\,\Big\rangle = -\Big\langle\, \widetilde{\mathbf{Q}}\,;\, \big(\partial_{\mathbf{u}}\mathcal{J}\big)\big[\mathcal{J}\widetilde{\mathbf{P}}\big]\widetilde{\mathbf{L}}\,\Big\rangle.$$

Hence, we deduce that the second term in the Jacobi identity (2.15) becomes

$$\Big\langle\, \mathbf{Q}\,;\, \big(\mathrm{d}_{\tilde{\mathbf{u}}}\widetilde{\mathcal{J}}\big)\big[\widetilde{\mathcal{J}}\mathbf{P}\big]\mathbf{L}\,\Big\rangle = \mid \mathbf{Q},\ \mathbf{P},\ \mathbf{L}\mid_{\widetilde{\mathcal{J}}} - \mid \mathbf{L},\ \mathbf{Q},\ \mathbf{P}\mid_{\widetilde{\mathcal{J}}} - \Big\langle\, \widetilde{\mathbf{Q}}\,;\, \big(\partial_{\mathbf{u}}\mathcal{J}\big)\big[\mathcal{J}\widetilde{\mathbf{P}}\big]\widetilde{\mathbf{L}}\,\Big\rangle,$$

which implies that the Jacobi identity associated to $\widetilde{\mathcal{J}}$ is equivalent to that corresponding to $\mathcal{J}$. We thus complete the proof of Theorem 2.2. □

Turning to the proof of Theorem 2.3, the following lemma is useful.

Lemma 2.1. *Let* $\mathbf{T}$ *be the matrix differential operator given by* (2.7). *Then subject to the Bäcklund transformation* (2.6), *there holds*

$$\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{f} = \big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{T}\mathbf{f}\big]\mathbf{T}^{-1}\alpha, \tag{2.18}$$

for $\mathbf{f} = (f_1, \ldots, f_n)^T$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n)^T$.

Proof. We start with some operator identities required. The first two formulae present the symmetry property of the corresponding second-order derivatives. More precisely, for $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $\beta = (\beta_1, \ldots, \beta_n)^T$,

$$\big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\mathbf{f} = \big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{f}\big]\alpha \quad \text{and} \quad \big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\beta = \big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\beta\big]\alpha.$$

The third formula is associated with the partial derivative of the inverse operator:

$$\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big)\big[\alpha\big] = -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}. \tag{2.19}$$

We are now in a position to prove the lemma. By (2.14),

$$\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{f} = \big(\partial_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{f} - \big(\partial_{\mathbf{u}}\mathbf{T}\big)\big[\alpha\big]\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f}.$$

In view of (2.19), we obtain

$$\begin{aligned}
&\big(\partial_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{f} = \big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big)\big[\alpha\big]\mathbf{B}_{\mathbf{u}}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\alpha\big]\mathbf{f}\\
&\quad = -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\mathbf{B}_{\mathbf{u}}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\alpha\big]\mathbf{f} = -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\mathbf{T}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\alpha\big]\mathbf{f}
\end{aligned}$$

and

$$\begin{aligned}
\big(\partial_{\mathbf{u}}\mathbf{T}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f} &= \big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{B}_{\mathbf{u}}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f}\\
&= -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\mathbf{B}_{\mathbf{u}}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f}\\
&= -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{T}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f}.
\end{aligned}$$

On the other hand, using the symmetry formulae of the second-order derivatives, (2.13) and (2.19), the right hand side of (2.18) becomes

$$\begin{aligned}
\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{T}\mathbf{f}\big]\mathbf{T}^{-1}\alpha =& \big(\partial_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{T}\mathbf{f}\big]\mathbf{T}^{-1}\alpha - \big(\partial_{\mathbf{u}}\mathbf{T}\big)\big[\mathbf{f}\big]\mathbf{T}^{-1}\alpha\\
=& \big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big)\big[\mathbf{T}\mathbf{f}\big]\mathbf{B}_{\mathbf{u}}\mathbf{T}^{-1}\alpha + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{T}\mathbf{f}\big]\mathbf{T}^{-1}\alpha\\
&\quad - \big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big)\big[\mathbf{f}\big]\mathbf{B}_{\mathbf{u}}\mathbf{T}^{-1}\alpha - \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{f}\big]\mathbf{T}^{-1}\alpha\\
=& -\mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\alpha\big]\mathbf{T}\mathbf{f} + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\tilde{\mathbf{u}}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{T}\mathbf{f}\\
&\qquad + \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\tilde{\mathbf{u}}}\mathbf{B}_{\mathbf{u}}\big)\big[\alpha\big]\mathbf{f} - \mathbf{B}_{\tilde{\mathbf{u}}}^{-1}\big(\partial_{\mathbf{u}}\mathbf{B}_{\mathbf{u}}\big)\big[\mathbf{T}^{-1}\alpha\big]\mathbf{f},
\end{aligned}$$

which suffices to prove the lemma. □

Proof of Theorem 2.3. First, since $\mathcal{R}(\mathbf{u})$ is a recursion operator of system (2.1), we deduce that the differential function $\mathbf{K}=(K_1(\mathbf{u}),\ldots,K_n(\mathbf{u}))^T$ together with $\mathcal{R}(\mathbf{u})$ satisfies the following operator identity

$$\big(\partial_{\mathbf{u}}\mathcal{R}\big)\big[\mathbf{K}\big]=\big[\mathbf{K}_{\mathbf{u}},\,\mathcal{R}\big]=\mathbf{K}_{\mathbf{u}}\mathcal{R}-\mathcal{R}\mathbf{K}_{\mathbf{u}},$$

where the $n\times n$ matrix differential operator $\mathbf{K}_{\mathbf{u}}$ with entries

$$\left(\mathbf{K}_{\mathbf{u}}\right)_{ij}=K_{i,u_j},\quad i,j=1,\ldots,n,$$

is the Fréchet derivative of $\mathbf{K}$, and $\big(\partial_{\mathbf{u}}\mathcal{R}\big)(\mathbf{u})\big[\mathbf{K}\big]$, or $\big(\partial_{\mathbf{u}}\mathcal{R}\big)\big[\mathbf{K}\big]$ for brevity, is the partial derivative of operator $\mathcal{R}$ along the direction $\mathbf{K}$.

Next, according to Lemma 2.1, and using the formulae (2.10) and (2.13), we have, for $\alpha=(\alpha_1,\ldots,\alpha_n)^T$,

$$\begin{aligned}\big(\partial_{\tilde{\mathbf{u}}}\tilde{\mathcal{R}}\big)(\tilde{\mathbf{u}})\big[\mathbf{G}\big]\alpha&=\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\mathcal{R}\mathbf{T}^{-1}\big)\big[\mathbf{G}\big]\alpha\\ &=\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{G}\big]\mathcal{R}\mathbf{T}^{-1}\alpha+\mathbf{T}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathcal{R}(u,v)\big)\big[\mathbf{G}\big]\mathbf{T}^{-1}\alpha+\mathbf{T}\mathcal{R}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}^{-1}\big)\big[\mathbf{G}\big]\alpha\\ &=\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{T}\mathcal{R}\mathbf{T}^{-1}\alpha\big]\mathbf{T}^{-1}\mathbf{G}-\mathbf{T}\big(\partial_{\mathbf{u}}\mathcal{R}\big)\big[\mathbf{T}^{-1}\mathbf{G}\big]\mathbf{T}^{-1}\alpha-\mathbf{T}\mathcal{R}\mathbf{T}^{-1}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\mathbf{G}\big]\mathbf{T}^{-1}\alpha\\ &=-\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\tilde{\mathcal{R}}\alpha\big]\mathbf{K}+\mathbf{T}\big(\partial_{\mathbf{u}}\mathcal{R}\big)\big[\mathbf{K}\big]\mathbf{T}^{-1}\alpha+\tilde{\mathcal{R}}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{K}.\end{aligned}$$

Since $\mathcal{R}(\mathbf{u})$ is a recursion operator of system (2.1), we have

$$\mathbf{T}\big(\partial_{\mathbf{u}}\mathcal{R}\big)\big[\mathbf{K}\big]\mathbf{T}^{-1}\alpha=\mathbf{T}\mathbf{K}_{\mathbf{u}}\mathbf{T}^{-1}\tilde{\mathcal{R}}\alpha-\tilde{\mathcal{R}}\mathbf{T}\mathbf{K}_{\mathbf{u}}\mathbf{T}^{-1}\alpha.$$

On the other hand,

$$\begin{aligned}\mathbf{G}_{\tilde{\mathbf{u}}}\tilde{\mathcal{R}}\alpha-\tilde{\mathcal{R}}\mathbf{G}_{\tilde{\mathbf{u}}}\alpha&=-\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\mathbf{K}\big)\big[\tilde{\mathcal{R}}\alpha\big]+\tilde{\mathcal{R}}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\mathbf{K}\big)\big[\alpha\big]\\ &=-\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\tilde{\mathcal{R}}\alpha\big]\mathbf{K}-\mathbf{T}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{K}\big)\tilde{\mathcal{R}}\alpha+\tilde{\mathcal{R}}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{K}+\tilde{\mathcal{R}}\mathbf{T}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{K}\big)\alpha\\ &=-\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\tilde{\mathcal{R}}\alpha\big]\mathbf{K}+\mathbf{T}\mathbf{K}_{\mathbf{u}}\mathbf{T}^{-1}\tilde{\mathcal{R}}\alpha+\tilde{\mathcal{R}}\big(\mathrm{d}_{\tilde{\mathbf{u}}}\mathbf{T}\big)\big[\alpha\big]\mathbf{K}-\tilde{\mathcal{R}}\mathbf{T}\mathbf{K}_{\mathbf{u}}\mathbf{T}^{-1}\alpha.\end{aligned}$$

Hence the identity

$$\big(\partial_{\tilde{\mathbf{u}}}\tilde{\mathcal{R}}\big)\big[\mathbf{G}\big]=\mathbf{G}_{\tilde{\mathbf{u}}}\tilde{\mathcal{R}}-\tilde{\mathcal{R}}\mathbf{G}_{\tilde{\mathbf{u}}}=[\mathbf{G}_{\tilde{\mathbf{u}}},\,\tilde{\mathcal{R}}]$$

holds, completing the proof of the theorem. □

In view of Theorem 2.2, if we introduce a generalized Bäcklund transformation that involves certain arbitrary parameters, and apply it to a given Hamiltonian operator, then (2.8) implies that the transformed Hamiltonian operator will depend on the associated parameters. This motivates us to analyze families of n-component Bäcklund transformations depending on, for example, three constant parameters α, β, γ, to construct a new $n\times n$ matrix Hamiltonian operator $\widetilde{\mathcal{J}}$ from a given $n\times n$ matrix Hamiltonian operator $\mathcal{J}$, hoping that the resulting operator takes the form

$$\widetilde{\mathcal{J}}=\tilde{\alpha}\widetilde{\mathcal{K}}_1+\tilde{\beta}\widetilde{\mathcal{K}}_2+\tilde{\gamma}\widetilde{\mathcal{K}}_3, \tag{2.20}$$

where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ depend on α, β and γ. Assuming the three operators $\widetilde{\mathcal{K}}_1$, $\widetilde{\mathcal{K}}_2$ and $\widetilde{\mathcal{K}}_3$ appearing in (2.20) scale independently under scaling transformations of the dependent variables, they then form a compatible Hamiltonian triple, meaning that each linear combination (2.20) is a Hamiltonian operator. As a consequence, two pairs of compatible Hamiltonian operators are readily constructed by taking distinct recombinations of the Hamiltonian triple, in which, one pair will typically generate a classical soliton system, while, the other dual counterpart is an integrable bi-Hamiltonian system, which, usually is endowed with nonlinear dispersion, [42]. Moreover, it is worth mentioning that, since such linear combinations of the recombined pairs are members of the 3-parameter family (2.20), this argument also serves to automatically verify the compatibility of two Hamiltonian operators.

In the next two sections, we shall demonstrate the efficacy of this approach through examples of two-component and three-component systems which arise from the models in the shallow water wave propagation.

3. The two-component DWW and mDWW systems

Consider the following *dispersive water wave* (*DWW*) *system* [31]

$$\begin{cases} q_t = (-q_x + 2qr)_x\,, \\ r_t = \left(r_x + 2q + r^2\right)_x\,, \end{cases} \tag{3.1}$$

which is one of the dispersive generalizations of the classical dispersiveless long wave equation and belongs to the family of the bi-directional Boussinesq-type systems modeling the propagation of shallow water waves [4, 40]. The DWW system is related to the *modified dispersive water wave* (*mDWW*) *system*

$$\begin{cases} u_t = \left(-u_x + 2uv - u^2\right)_x\,, \\ v_t = \left(v_x - 2u_x - 2u^2 + 2uv + v^2\right)_x\,, \end{cases} \tag{3.2}$$

via the Miura-type transformation [2, 31]

$$q = -u_x - u^2 + uv, \qquad r = v. \tag{3.3}$$

Moreover, the mDWW system admits the following constant coefficient Hamiltonian operator

$$\mathcal{J}(u,v) = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 2\,\partial_x \end{pmatrix}. \tag{3.4}$$

We introduce a family of Miura-type transformations

$$q = -\gamma\left(u_x + u^2 - uv\right) + \frac{\alpha}{\bar\beta}u, \qquad r = \left(\frac{\gamma}{\bar\beta} - \bar\beta\right)u + \bar\beta v, \tag{3.5}$$

where $\bar\beta = \beta/\gamma$ and α, β and γ ($\beta\gamma \neq 0$) are arbitrary constants. Applying (3.5) and using (3.4) as the original Hamiltonian operator, we can obtain a 3-parameter family of Hamiltonian operators

$$\widetilde{\mathcal{J}}(q,r) = \begin{pmatrix} \gamma(q\partial_x + \partial_x q) & -\beta\partial_x^2 + \alpha\partial_x + \gamma r\partial_x \\ \beta\partial_x^2 + \alpha\partial_x + \gamma\partial_x r & 2\gamma\partial_x \end{pmatrix}. \tag{3.6}$$

Obviously, $\widetilde{\mathcal{J}}$ is a linear combination of the following triple of Hamiltonian operators:

$$\widetilde{\mathcal{K}}_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \qquad \widetilde{\mathcal{K}}_2 = \begin{pmatrix} 0 & -\partial_x^2 \\ \partial_x^2 & 0 \end{pmatrix}, \qquad \widetilde{\mathcal{K}}_3 = \begin{pmatrix} q\partial_x + \partial_x q & r\partial_x \\ \partial_x r & 2\partial_x \end{pmatrix}.$$

We now define the operators

$$\widetilde{\mathcal{J}}_1 = \widetilde{\mathcal{K}}_1 + \nu\widetilde{\mathcal{K}}_2, \qquad \widetilde{\mathcal{J}}_2 = \widetilde{\mathcal{J}},$$

where ν is a constant. Since γ in $\widetilde{\mathcal{J}}$ should be nonzero, we set $\gamma = 1$ without loss of generality. The fact that the linear combinations $c_1\widetilde{\mathcal{J}}_1 + c_2\widetilde{\mathcal{J}}_2$ are members of the 3-parameter family of Hamiltonian operators (3.6) justifies their compatibility, and thus Magri's Theorem [37] establishes the formal existence of a hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \mathbf{G}_n = \widetilde{\mathcal{J}}_1\delta\widetilde{\mathcal{H}}_n = \widetilde{\mathcal{J}}_2\delta\widetilde{\mathcal{H}}_{n-1}, \quad n = 1, 2, \ldots, \qquad \delta\widetilde{\mathcal{H}}_n = \left(\frac{\delta\widetilde{\mathcal{H}}_n}{\delta q}, \frac{\delta\widetilde{\mathcal{H}}_n}{\delta r}\right)^T, \tag{3.7}$$

of higher-order commuting bi-Hamiltonian systems, with associated higher-order Hamiltonian functionals $\widetilde{\mathcal{H}}_n$, $n = 0, 1, 2, \ldots$, which are conservation laws common to all members of the hierarchy. The members in the hierarchy (3.7) are obtained by applying successively the recursion operator $\widetilde{\mathcal{R}}=\widetilde{\mathcal{J}}_2\widetilde{\mathcal{J}}_1^{-1}$ to the seed system

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \mathbf{G}_1 = \begin{pmatrix} q \\ r \end{pmatrix}_x = \widetilde{\mathcal{J}}_1 \delta \widetilde{\mathcal{H}}_1 .$$

For the second flow in this hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \mathbf{G}_2 = \widetilde{\mathcal{R}}\mathbf{G}_1 = \widetilde{\mathcal{J}}_1 \delta \widetilde{\mathcal{H}}_2 = \widetilde{\mathcal{J}}_2 \delta \widetilde{\mathcal{H}}_1 , \tag{3.8}$$

we consider two cases:

Case 1. When $\nu=0$, (3.8) becomes

$$\begin{cases} q_t = (-\beta q_x + \alpha q + 2qr)_x \, , \\ r_t = \left(\beta r_x + \alpha r + 2q + r^2\right)_x \end{cases} \tag{3.9}$$

with the associated Hamiltonian functionals

$$\widetilde{\mathcal{H}}_1 = \int q\, r \; \mathrm{d}x, \qquad \widetilde{\mathcal{H}}_2 = \int \left(\beta r_x q + r^2 q + q^2 + \alpha q r\right) \, \mathrm{d}x .$$

The system (3.9) admits the following Lax pair with spectral parameter λ:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.10}$$

where

$$\begin{aligned} U &= \frac{1}{2}\begin{pmatrix} \lambda + \dfrac{1}{\beta}\left(r+\dfrac{\alpha}{2}\right) & -\dfrac{2}{\beta^2}\, q \\ 2 & -\left(\lambda + \dfrac{1}{\beta}\left(r+\dfrac{\alpha}{2}\right)\right) \end{pmatrix}, \\ V &= \frac{1}{2}\begin{pmatrix} -\beta\,\lambda^2 + r_x + \dfrac{1}{\beta}\left(r+\dfrac{\alpha}{2}\right)^2 & \dfrac{2\lambda}{\beta}\, q + \dfrac{2}{\beta}\, q_x - \dfrac{2}{\beta^2}\left(r+\dfrac{\alpha}{2}\right) q \\ 2\left(-\beta\,\lambda + r + \dfrac{\alpha}{2}\right) & \beta\,\lambda^2 - r_x - \dfrac{1}{\beta}\left(r+\dfrac{\alpha}{2}\right)^2 \end{pmatrix}. \end{aligned} \tag{3.11}$$

Case 2. When $\nu \neq 0$, without loss of generality, we set $\nu = \pm 1$. Setting

$$q = g - \nu g_x, \qquad r = f + \nu f_x,$$

allows us to write the resulting bi-Hamiltonian system (3.8) in the local form

$$\begin{cases} q_t = (-\beta g_x + \alpha g + (q+g) f)_x \, , \\ r_t = (\beta f_x + \alpha f + 2g + r f)_x \, , \end{cases} \tag{3.12}$$

or, in full detail,

$$\begin{cases} g_t - \nu g_{xt} = \big(-\beta g_x + \alpha g + (2g - \nu g_x) f\big)_x, \\ f_t + \nu f_{xt} = \big(\beta f_x + \alpha f + 2g + (f + \nu f_x) f\big)_x, \end{cases}$$

where the Hamiltonian functionals are

$$\widetilde{\mathcal{H}}_1 = \int (f - \nu f_x) g \; \mathrm{d}x, \qquad \widetilde{\mathcal{H}}_2 = \int \big(-\beta g_x + \alpha g + r g\big) f \; \mathrm{d}x.$$

The system (3.12) admits the Lax pair (3.10) with

$$U = \frac{1}{2}\begin{pmatrix} \dfrac{\alpha}{\beta} + \dfrac{1}{\beta}(f + \nu\, f_x) & -\dfrac{2}{\lambda\,\beta^2}(g - \nu\, g_x) \\ 2\lambda & -\dfrac{\alpha}{\beta} - \dfrac{1}{\beta}(f + \nu\, f_x) \end{pmatrix},$$

$$V = \begin{pmatrix} \dfrac{1}{2\beta}\left(\beta\, f_x + \alpha\, f + f^2 + \nu\, f f_x\right) & \dfrac{1}{\lambda\,\beta^2}\left(\nu\, f g_x - g f + \beta\, g_x\right) \\ \lambda\, f & -\dfrac{1}{2\beta}\left(\beta\, f_x + \alpha\, f + f^2 + \nu\, f f_x\right) \end{pmatrix}.$$

The tri-Hamiltonian duality argument implies that the dual integrable bi-Hamiltonian system is obtained by rearranging the Hamiltonian triple to form the dual Hamiltonian pair

$$\widehat{\mathcal{J}}_1 = \begin{pmatrix} 0 & \partial_x - \partial_x^2 \\ \partial_x + \partial_x^2 & 0 \end{pmatrix}, \qquad \widehat{\mathcal{J}}_2 = \begin{pmatrix} \widehat{q}\partial_x + \partial_x \widehat{q} & \widehat{r}\partial_x \\ \partial_x \widehat{r} & 2\partial_x \end{pmatrix}. \tag{3.13}$$

The dual counterpart of the DWW system (3.1) is thus

$$\begin{cases} \widehat{q}_t = ((\widehat{q} + g) f)_x\,, & \widehat{q} = g - g_x, \\ \widehat{r}_t = (2g + \widehat{r} f)_x\,, & \widehat{r} = f + f_x, \end{cases} \tag{3.14}$$

which, in fact, belongs to the integrable family (3.12) derived in Case 2.

Observe that the second Hamiltonian operator $\widehat{\mathcal{J}}_2$ in (3.13) admits the Casimir functional

$$\widehat{\mathcal{H}}_C = \int \sqrt{4\widehat{q} - \widehat{r}^2}\, \mathrm{d}x,$$

with variational derivative

$$\delta \widehat{\mathcal{H}}_C = \left(\frac{\delta \widehat{\mathcal{H}}_C}{\delta \widehat{q}},\ \frac{\delta \widehat{\mathcal{H}}_C}{\delta \widehat{r}} \right)^T = \left(\frac{2}{\sqrt{4\widehat{q} - \widehat{r}^2}},\ -\frac{\widehat{r}}{\sqrt{4\widehat{q} - \widehat{r}^2}} \right)^T.$$

On the one hand, the functional $\widehat{\mathcal{H}}_C$ is an additional conservation law admitted by (3.14); on the other hand, it leads to the associated *Casimir system*

$$\begin{pmatrix} \widehat{q} \\ \widehat{r} \end{pmatrix}_t = \widehat{\mathbf{G}}_{-1} = \widehat{\mathcal{J}}_1 \delta \widehat{\mathcal{H}}_C,$$

which takes the explicit form

$$\widehat{q}_t = (-\partial_x + \partial_x^2)\left(\frac{\widehat{r}}{\sqrt{4\widehat{q} - \widehat{r}^2}} \right), \qquad \widehat{r}_t = (\partial_x + \partial_x^2)\left(\frac{2}{\sqrt{4\widehat{q} - \widehat{r}^2}} \right). \tag{3.15}$$

Starting from the Casimir system (3.15), one (formally) constructs an infinite hierarchy of higher-order commuting bi-Hamiltonian systems and corresponding Hamiltonian functionals $\{\widehat{\mathcal{H}}_{-n}\}$ in the negative direction:

$$\begin{pmatrix} \widehat{q} \\ \widehat{r} \end{pmatrix}_t = \widehat{\mathbf{G}}_{-n} = \widehat{\mathcal{J}}_1 \delta \widehat{\mathcal{H}}_{-n} = \widehat{\mathcal{J}}_2 \delta \widehat{\mathcal{H}}_{-(n+1)}, \quad n = 1, 2, \ldots,$$

with $\widehat{\mathcal{H}}_{-1} = \widehat{\mathcal{H}}_C$ and $\delta \widehat{\mathcal{H}}_{-n} = \left(\delta \widehat{\mathcal{H}}_{-n}/\delta \widehat{q},\ \delta \widehat{\mathcal{H}}_{-n}/\delta \widehat{r} \right)^T$.

In the case of $\alpha = 0$, $\beta = \gamma = 1$, the Bäcklund transformation (3.5) reduces to (3.3), which connects the DWW system (3.1) and the mDWW system (3.2). Now, let us focus our attention on the mDWW system (3.2). First of all, in view of Theorem 2.3 and transformation (3.3), we deduce that the recursion operator $\mathcal{R}(u, v)$ of mDWW system satisfies

$$\begin{pmatrix} \partial_x + 2u - v & -u \\ 0 & -1 \end{pmatrix} \mathcal{R}(u, v) = \widetilde{\mathcal{R}}(q, r) \begin{pmatrix} \partial_x + 2u - v & -u \\ 0 & -1 \end{pmatrix},$$

where $\widetilde{\mathcal{R}}(q,r)=\widetilde{\mathcal{J}}_2\widetilde{\mathcal{J}}_1^{-1}$ is the recursion operator for the DWW system. It follows that

$$\mathcal{R}(u,v)=\begin{pmatrix} v-2u-\partial_x & u+\partial_x u\partial_x^{-1} \\ 2(v-2u-\partial_x) & 2u+\partial_x+\partial_x v\partial_x^{-1}\end{pmatrix}.$$

The second Hamiltonian operator admitted by the mDWW system (3.2) is thereby obtained,

$$\mathcal{J}_2(u,v)=\mathcal{R}\mathcal{J}_1=\begin{pmatrix} u\partial_x+\partial_x u & -\partial_x^2+2\partial_x u+v\partial_x \\ \partial_x^2+2u\partial_x+\partial_x v & 2(v\partial_x+\partial_x v)\end{pmatrix},$$

where $\mathcal{J}_1=\mathcal{J}(u,v)$ given by (3.4) is the first Hamiltonian operator of system (3.2). Therefore, the mDWW system (3.2) possesses the bi-Hamiltonian structure

$$\begin{pmatrix} u \\ v\end{pmatrix}_t=\mathcal{J}_1\delta\mathcal{H}_2=\mathcal{J}_2\delta\mathcal{H}_1,$$

with Hamiltonian functionals

$$\mathcal{H}_1=\int u(v-u)\,\mathrm{d}x, \qquad \mathcal{H}_2=\int u(v_x+v^2-uv)\,\mathrm{d}x.$$

In addition, the Lax formulation (3.10) for the mDWW system (3.2) is based on

$$U=\frac{1}{2}\begin{pmatrix}\lambda+v & 2\left(u_x+u^2-uv\right) \\ 2 & -\lambda-v\end{pmatrix},$$
$$V=\frac{1}{2}\begin{pmatrix}-\lambda^2+v_x+v^2 & -2\big(\partial_x-v+\lambda\big)(u_x+u^2-uv) \\ 2(v-\lambda) & \lambda^2-v_x-v^2\end{pmatrix}.$$

Next, note that the mDWW system (3.2) also admits the tri-Hamiltonian structure, and its dual version relies on the recombined dual Hamiltonian pair

$$\overline{\mathcal{J}}_1=\begin{pmatrix} 0 & \partial_x-\partial_x^2 \\ \partial_x+\partial_x^2 & 2\partial_x\end{pmatrix}, \qquad \overline{\mathcal{J}}_2=\begin{pmatrix}\bar{u}\partial_x+\partial_x\bar{u} & 2\partial_x\bar{u}+\bar{v}\partial_x \\ 2\bar{u}\partial_x+\partial_x\bar{v} & 2(\bar{v}\partial_x+\partial_x\bar{v})\end{pmatrix}. \tag{3.16}$$

Applying the dual bi-Hamiltonian structure (3.16) to the seed system

$$\begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix}_t=\overline{\mathbf{K}}_1=\begin{pmatrix}\bar{u} \\ \bar{v}\end{pmatrix}_x,$$

we obtain the integrable bi-Hamiltonian hierarchy

$$\begin{pmatrix}\bar{u}_t \\ \bar{v}_t\end{pmatrix}=\overline{\mathbf{K}}_n=\overline{\mathcal{J}}_1\delta\overline{\mathcal{H}}_n=\overline{\mathcal{J}}_2\delta\overline{\mathcal{H}}_{n-1}, \qquad n\in\mathbb{Z}, \tag{3.17}$$

with $\delta\overline{\mathcal{H}}_n=\big(\delta\overline{\mathcal{H}}_n/\delta\bar{u},\,\delta\overline{\mathcal{H}}_n/\delta\bar{v}\big)^T$. The initial members of the sequence of Hamiltonian functionals are

$$\overline{\mathcal{H}}_0=\frac{1}{4}\int(2\bar{u}+\bar{v})\,\mathrm{d}x, \qquad \overline{\mathcal{H}}_1=\int\Big((g+g_x)\bar{v}-(g^2+g_x^2)\Big)\,\mathrm{d}x,$$
$$\overline{\mathcal{H}}_2=\frac{1}{2}\int\Big(2(g_x^2-g^2)(f-f_{xx})+(2g+3g_x-g_{xxx})f^2+(2g+g_x-g_{xx})f_x^2\Big)\,\mathrm{d}x,$$

with $\bar{u}=(1-\partial_x^2)g$ and $\bar{v}=(1-\partial_x^2)f$. In particular, the case $n=2$ in (3.17) corresponds to the dual mDWW system

$$\begin{cases}\bar{u}_t=\big((g+g_x)(g_x-g-f_x+f)+(2g_x-f_x+f)\bar{u}\big)_x, \\ \bar{v}_t=\big(2(g+g_x)(g_x-g-f_x+f)+(2g_x-f_x+f)\bar{v}\big)_x.\end{cases}$$

Again, the second Hamiltonian operator $\overline{\mathcal{J}}_2$ admits a Casimir functional

$$\overline{\mathcal{H}}_C=\int\frac{\bar{u}}{2\bar{u}-\bar{v}}\,\mathrm{d}x,$$

leading to the following Casimir system

$$\bar{u}_t = (\partial_x - \partial_x^2)\left(\frac{\bar{u}}{(2\bar{u}-\bar{v})^2}\right), \qquad \bar{v}_t = \partial_x\left(\frac{1}{2\bar{u}-\bar{v}}\right) - \partial_x^2\left(\frac{\bar{v}}{(2\bar{u}-\bar{v})^2}\right), \tag{3.18}$$

for the dual mDWW system, which is the first member of the dual hierarchy (3.17) in the negative direction. Interestingly, when $\bar{u}=0$, (3.18) reduces to

$$\bar{v}_t = -(\partial_x + \partial_x^2)\left(\frac{1}{\bar{v}}\right). \tag{3.19}$$

Using the reciprocal transformation

$$y = \int^x \bar{v}(\xi,\, t)\,\mathrm{d}\xi, \qquad \tau = t, \qquad Q(y,\tau) = \frac{1}{\bar{v}(x,t)},$$

equation (3.19) is mapped into Burgers' equation

$$Q_\tau = Q_{yy} + 2QQ_y. \tag{3.20}$$

On the other hand, for the mDWW system (3.2), if we take

$$u = 0, \qquad v(x,t) = Q(y,\tau),$$

the resulting equation is also Burgers' equation (3.20). Motivated by this argument, it is anticipated that there exists such an analogous reciprocal correspondence between the mDWW system (3.2) and the associated Casimir system (3.18) for the dual mDWW system, and thereby leading to a generalization for their entire hierarchies.

4. Three-component integrable generalizations

In this section, we shift our attention to three-component systems. Motivated by a system studied in [1] — see (4.7a) therein — we consider the following pair of three-component systems:

$$\begin{cases} w_t = \left(-\lambda u_x + w + bc\lambda u + \frac{3}{2}\lambda w^2 + wv + \frac{1}{2}ab\lambda u^2 + b\lambda uv\right)_x, \\ u_t = \left(-\frac{1}{b}u_x + cu + \frac{1}{2b}w^2 + \lambda wu + \frac{1}{2}au^2 + 2uv\right)_x, \\ v_t = \left(\frac{\lambda}{b}w_x + \frac{a}{b}u_x + \frac{1}{b}v_x + c\lambda w - ac\,u + c\,v - \frac{a}{2b}w^2 + \lambda wv - \frac{1}{2}a^2u^2 - auv + v^2\right)_x, \end{cases} \tag{4.1}$$

and

$$\begin{cases} s_t = \left(s + \lambda q - ds^2 + sr\right)_x, \\ q_t = \left(-\frac{1}{b}q_x + cq + 2qr - 2dsq + \frac{1}{2}s^2\right)_x, \\ r_t = \left(-\frac{d}{b}s_x + \frac{1}{b}r_x + (d+\lambda-cd)\,s + \left(\lambda^2 + d\lambda - \frac{a}{b}\right)q + c\,r + r^2 - dsr\right)_x, \end{cases} \tag{4.2}$$

where $a \neq 0$, $b \neq 0$, c, d, λ are constants. Systems (4.1) and (4.2) are related by the following Miura-type transformation

$$s = w, \qquad q = -u_x + \frac{1}{2}w^2 + \frac{ab}{2}u^2 + b\,uv + bc\,u, \qquad r = (d+\lambda)w + v.$$

Note that system (4.1) can be written in a Hamiltonian form

$$\begin{pmatrix} w \\ u \\ v \end{pmatrix}_t = \mathcal{J}(w,u,v)\begin{pmatrix} \delta\mathcal{H}/\delta w \\ \delta\mathcal{H}/\delta u \\ \delta\mathcal{H}/\delta v \end{pmatrix},$$

with the Hamiltonian operator

$$\mathcal{J}(w,u,v) = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & 0 & \frac{1}{b}\partial_x \\ 0 & \frac{1}{b}\partial_x & -\frac{a}{b}\partial_x \end{pmatrix} \tag{4.3}$$

and the associated Hamiltonian functional

$$\mathcal{H}(w,u,v) = \frac{1}{2}\int \Big(\lambda\, w^3 + w^2 v + ab\lambda\, wu^2 + 2b\lambda\, wuv + w^2 + 2bc\lambda\, wu - 2\lambda w u_x\Big)\,\mathrm{d}x + \frac{1}{2}\int \Big(u^2 v + 2b\, uv^2 + 2bc\, uv - 2\, u_x v\Big)\,\mathrm{d}x.$$

Consider the following Miura-type Bäcklund transformations

$$\begin{aligned} &B_1(w,u,v,s,q,r) = s - c_1 w = 0,\\ &B_2(w,u,v,s,q,r) = q - \big(c_2 w^2 + c_3 u_x + c_4 u^2 + c_5 uv + c_6 u\big) = 0,\\ &B_3(w,u,v,s,q,r) = r - (c_7 w + c_8 u + c_9 v) = 0, \end{aligned} \tag{4.4}$$

where $c_1, \ldots, c_9$ are constant parameters. Plugging the original Hamiltonian operator (4.3) into formula (2.8) under the transformations (4.4) yields the Hamiltonian operator

$$\widetilde{\mathcal{J}}(s,q,r) = \begin{pmatrix} \alpha\, \partial_x & \gamma\, \partial_x s & \kappa\, \partial_x \\ \gamma\, s\, \partial_x & \gamma(q\, \partial_x + \partial_x q) & \epsilon\, \partial_x - \beta \partial_x^2 + \gamma\, r \partial_x \\ \kappa\, \partial_x & \epsilon\, \partial_x + \beta \partial_x^2 + \gamma\, \partial_x r & \xi \partial_x \end{pmatrix}, \tag{4.5}$$

where

$$\alpha = c_1^2,\ \ \beta = -\frac{1}{b} c_3 c_9,\ \ \gamma = 2c_2 = \frac{2}{ab} c_4 = \frac{1}{b} c_5,\ \ \epsilon = \frac{1}{b} c_6 c_9,\ \ \xi = c_7^2 + \frac{2}{b} c_8 c_9 - \frac{a}{b} c_9^2,\ \ \kappa = c_1 c_7.$$

Using a scaling argument, we extract a compatible Hamiltonian triple (2.20) from the operator (4.5). Consider the compatible Hamiltonian pair

$$\widetilde{\mathcal{J}}_1 = \widetilde{\mathcal{K}}_1 + \nu \widetilde{\mathcal{K}}_2, \qquad \widetilde{\mathcal{J}}_2 = \widetilde{\mathcal{J}}, \tag{4.6}$$

where

$$\widetilde{\mathcal{K}}_1 = \begin{pmatrix} \alpha_1 & 0 & \kappa_1 \\ 0 & 0 & 1 \\ \kappa_1 & 1 & \xi_1 \end{pmatrix} \partial_x, \qquad \widetilde{\mathcal{K}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x^2 \tag{4.7}$$

depending upon the parameters $\alpha_1 \neq 0$, κ_1, ξ_1. Applying successively the resulting recursion operator $\widetilde{\mathcal{R}} = \widetilde{\mathcal{J}}_2 \widetilde{\mathcal{J}}_1^{-1}$ to the seed system

$$\begin{pmatrix} s \\ q \\ r \end{pmatrix}_t = \mathbf{G}_1(s,q,r) = \begin{pmatrix} s \\ q \\ r \end{pmatrix}_x$$

produces a hierarchy of three-component higher-order commuting bi-Hamiltonian systems, namely

$$\begin{pmatrix} s \\ q \\ r \end{pmatrix}_t = \mathbf{G}_n(s,q,r) = \widetilde{\mathcal{J}}_1 \delta \widetilde{\mathcal{H}}_n = \widetilde{\mathcal{J}}_2 \delta \widetilde{\mathcal{H}}_{n-1}, \qquad n = 1, 2, \ldots,$$

where the variational derivative of the Hamiltonian functionals is given by

$$\delta \widetilde{\mathcal{H}}_n = \left(\frac{\delta \widetilde{\mathcal{H}}_n}{\delta s}, \frac{\delta \widetilde{\mathcal{H}}_n}{\delta q}, \frac{\delta \widetilde{\mathcal{H}}_n}{\delta r} \right)^T.$$

Focussing our attention on the second flow of this hierarchy

$$\begin{pmatrix} s \\ q \\ r \end{pmatrix}_t = \mathbf{G}_2(s,q,r) = \widetilde{\mathcal{R}} \mathbf{G}_1(s,q,r) = \widetilde{\mathcal{R}} \begin{pmatrix} s \\ q \\ r \end{pmatrix}_x, \tag{4.8}$$

we set the parameters $\gamma = \alpha_1 = 1$ for simplicity, and distinguish two cases.

Case 1. When $\nu = 0$,

$$\widetilde{\mathcal{J}}_1^{-1} = \begin{pmatrix} 1 & -\kappa_1 & 0 \\ -\kappa_1 & \xi_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x^{-1}, \qquad \text{where} \qquad \xi_2 = \kappa_1^2 - \xi_1.$$

The flow (4.8) takes the bi-Hamiltonian form

$$\begin{pmatrix} s \\ q \\ r \end{pmatrix}_t = \widetilde{\mathcal{J}}_2\, \delta\widetilde{\mathcal{H}}_1 = \widetilde{\mathcal{J}}_1\, \delta\widetilde{\mathcal{H}}_2 = \widetilde{\mathcal{J}} \begin{pmatrix} s - \kappa_1 q \\ -\kappa_1 s + \xi_2 q + r \\ q \end{pmatrix},$$

where

$$\widetilde{\mathcal{H}}_1 = \frac{1}{2} \int \Big(s^2 + \xi_2\, q^2 + 2(r - \kappa_1 s)\, q \Big)\, \mathrm{d}x,$$

$$\begin{aligned} \widetilde{\mathcal{H}}_2 = \frac{1}{2} \int \Big(&\xi_2^2 q^3 - \kappa_1 s^3 + \Big((\xi_2 + 2\kappa_1^2) s - 3\kappa_1 \xi_2\, q \Big)\, sq + (s - 4\kappa_1 q)\, sr + (3\xi_2 q + 2r)\, qr \\ &+ \alpha(s - \kappa_1 q)^2 + (2\epsilon\xi_2 - 2\kappa\kappa_1 + \xi)\, q^2 + 2(\kappa - \epsilon\kappa_1) sq + 2\epsilon\, qr + 2\beta\, q_x(\kappa_1 s - r) \Big)\, \mathrm{d}x. \end{aligned}$$

Explicitly,

$$\begin{cases} s_t = \Big(\alpha s + (\kappa - \alpha\kappa_1)\, q + (\xi_2 q - \kappa_1 s + r) s \Big)_x, \\ q_t = \Big(-\beta\, q_x + \epsilon\, q + 2 \left(\frac{3\xi_2}{4} q - \kappa_1 s + r \right) q + \frac{1}{2} s^2 \Big)_x, \\ r_t = \Big(-\beta\kappa_1 s_x + \beta\xi_2 q_x + \beta\, r_x + (\kappa - \epsilon\kappa_1) s + (\epsilon\, \xi_2 + \xi - \kappa\kappa_1)\, q + \epsilon\, r + (\xi_2 q - \kappa_1 s + r)\, r \Big)_x. \end{cases} \tag{4.9}$$

Case 2. When $\nu \neq 0$, without loss of generality, we set $\nu = \pm 1$, and so

$$\widetilde{\mathcal{J}}_1^{-1} = \begin{pmatrix} 1 & -\kappa_1 (1 - \nu\partial_x)^{-1} & 0 \\ -\kappa_1 (1 + \nu\partial_x)^{-1} & \xi_2 (1 - \partial_x^2)^{-1} & (1 + \nu\partial_x)^{-1} \\ 0 & (1 - \nu\partial_x)^{-1} & 0 \end{pmatrix} \partial_x^{-1}, \qquad \xi_2 = \kappa_1^2 - \xi_1.$$

Thus system (4.8) becomes

$$\begin{pmatrix} s \\ q \\ r \end{pmatrix}_t = \widetilde{\mathcal{J}} \begin{pmatrix} s - \kappa_1 (1 - \nu\partial_x)^{-1} q \\ (1 + \nu\partial_x)^{-1} (r - \kappa_1 s) + \xi_2 (1 - \partial_x^2)^{-1} q \\ (1 - \nu\partial_x)^{-1} q \end{pmatrix}. \tag{4.10}$$

Two subcases are considered:

Case 2.1. Setting $\kappa_1 = \xi_2 = 0$, so that $\xi_1 = 0$ in (4.7), we introduce new variables

$$g = (1 - \nu\partial_x)^{-1} q, \qquad f = (1 + \nu\partial_x)^{-1} r.$$

Then (4.10) becomes

$$\begin{cases} s_t = (\alpha s + \kappa g + s f)_x, \\ g_t - \nu g_{xt} = \big(-\beta g_x + \epsilon g + (g + q) f + \frac{1}{2} s^2 \big)_x, \\ f_t + \nu f_{xt} = (\beta f_x + \kappa s + \xi g + \epsilon f + r\, f)_x, \end{cases}$$

and the associated Hamiltonian functionals are

$$\widetilde{\mathcal{H}}_1 = \frac{1}{2} \int \Big(s^2 + 2(g + \nu g_x) f \Big)\, \mathrm{d}x,$$

$$\widetilde{\mathcal{H}}_2 = \frac{1}{2} \int \Big(s^2 f + 2r f g + \alpha s^2 + (\xi g + 2\kappa s g + 2\epsilon\, f + 2\beta f_x) g \Big)\, \mathrm{d}x.$$

Case 2.2. When $\kappa_1^2 + \xi_2^2 \neq 0$, we define the variables h, f and g by

$$s = (1 - \partial_x^2) h, \qquad q = (1 - \partial_x^2) g, \qquad r = (1 - \partial_x^2) f,$$

respectively. We then arrive at the following bi-Hamiltonian system

$$\begin{cases} h_t - h_{xxt} = \Big(\alpha\, s + (\kappa - \alpha\kappa_1)(\nu g_x + g) + \big(f - \nu f_x + \xi_2 g - \kappa_1(h - \nu h_x)\big)s\Big)_x, \\ g_t - g_{xxt} = \Big(- \beta\nu g_{xx} + (\epsilon\nu - \beta)g_x + \epsilon g + \frac{\xi_2}{2}\left(3g^2 - g_x^2 - 2gg_{xx}\right) \\ \qquad\qquad + \left(f - \nu f_x - \kappa_1(h - \nu h_x)\right)\left(2g + \nu g_x - g_{xx}\right) + \frac{1}{2}\, s^2\Big)_x, \\ f_t - f_{xxt} = \Big(\big(\beta\kappa_1\nu + \kappa\big)h_{xx} - \beta\nu f_{xx} - \kappa_1(\beta - \epsilon\nu)h_x + (\beta\xi_2 + \xi\nu - \kappa\kappa_1\nu)g_x + (\beta - \epsilon\nu)f_x \\ \qquad\qquad +\big(\kappa - \epsilon\kappa_1\big)h + (\epsilon\xi_2 + \xi - \kappa\kappa_1)g + \epsilon f + \big(f - \nu f_x - \kappa_1(h - \nu h_x) + \xi_2 g\big)\, r\Big)_x \end{cases} \tag{4.11}$$

and the associated Hamiltonian functionals are

$$\widetilde{\mathcal{H}}_1 = \frac{1}{2}\int \Big(s^2 + \xi_2(g^2 + g_x^2) + 2(q + \nu q_x)(f - \kappa_1 h)\Big)\,\mathrm{d}x,$$

$$\begin{aligned} \widetilde{\mathcal{H}}_2 = \frac{1}{2}\int \Big(& s^2\big(f - \nu f_x - \kappa_1(h - \nu h_x) + \xi_2 g\big) + \big(\kappa_1(h - \nu h_x) - f + \nu f_x\big)^2(q + g + \nu g_x) \\ & + \xi_2\left(-\kappa_1(h - \nu h_x) + f - \nu f_x\right)\left(3g^2 - g_x^2 - 2gg_{xx}\right) + \xi_2^2(g^3 + gg_x^2) + \alpha s^2 \\ & + 2(\kappa - \alpha\kappa_1)\left(g + \nu g_x\right)s + 2\big(- \kappa_1(h - \nu h_x) + f - \nu f_x\big)\big(\epsilon g + (\epsilon\nu - \beta)g_x - \beta\nu g_{xx}\big) \\ & + \left(\kappa_1(\alpha\kappa_1 - \kappa) + \epsilon\xi_2\right)g^2 + \left(\kappa_1(\alpha\kappa_1 - \kappa) + \beta\nu\xi_2\right)g_x^2\Big)\,\mathrm{d}x. \end{aligned}$$

Next, we focus our attention on the three-component DWW system

$$\begin{cases} s_t = (sr)_x, \\ q_t = \left(-q_x + 2qr + \frac{1}{2}s^2\right)_x, \\ r_t = \left(r_x + 2q + r^2\right)_x, \end{cases} \tag{4.12}$$

which is a special case of system (4.9) corresponding to the choice of $\alpha = \kappa = \kappa_1 = \epsilon = \xi_2 = 0$, $\beta = 1$ and $\xi = 2$ in (4.9). This system was proposed in [1] as an example of a three-component quadri-Hamiltonian system, and it appears as a three-component generalization of the DWW system (3.1).

According to the preceding analysis, the dual integrable counterpart of system (4.12) is the system arising from Case 2.1, i.e., the bi-Hamiltonian system admitting the Hamiltonian operators $\widetilde{\mathcal{J}}_1$ and $\widetilde{\mathcal{J}}_2$ defined in (4.6) and (4.7) with $\alpha_1 = \gamma = 1,\ \nu = \pm 1$ and $\kappa_1 = \xi_1 = 0$. More precisely, the following dual Hamiltonian pair

$$\widehat{\mathcal{J}}_1 = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & 0 & \partial_x - \nu\partial_x^2 \\ 0 & \partial_x + \nu\partial_x^2 & 0 \end{pmatrix}, \qquad \widehat{\mathcal{J}}_2 = \begin{pmatrix} 0 & \partial_x\widehat{s} & 0 \\ \widehat{s}\partial_x & \widehat{q}\partial_x + \partial_x\widehat{q} & \widehat{r}\partial_x \\ 0 & \partial_x\widehat{r} & 2\partial_x \end{pmatrix}, \tag{4.13}$$

with $\nu = \pm 1$, gives rise to the three-component bi-Hamiltonian system

$$\begin{cases} \widehat{s}_t = (\widehat{s}f)_x, \\ \widehat{q}_t = \left((\widehat{q} + g)\, f + \frac{1}{2}\widehat{s}^2\right)_x, & \widehat{q} = g - \nu g_x, \\ \widehat{r}_t = (2g + \widehat{r}f)_x, & \widehat{r} = f + \nu f_x, \end{cases} \tag{4.14}$$

which turns out to be the dual version of (4.12). In particular, when $\widehat{s} = 0$, (4.14) reduces to the dual counterpart (3.14) of the DWW system (3.1). The Hamiltonian operators $\widehat{\mathcal{J}}_1$ and $\widehat{\mathcal{J}}_2$ given in (4.13), when projected to the $(\widehat{q},\ \widehat{r})$ subspace, yield the Hamiltonian pair (3.13) admitted by the two-component dual system (3.14).

Finally, it is easy to see that setting $b = 1$ and $a = -2$ in (4.2) produces the system

$$\begin{cases} s_t = \left(s + \lambda q - ds^2 + sr\right)_x, \\ q_t = \left(-q_x + c\, q + 2qr - 2dsq + \frac{1}{2}s^2\right)_x, \\ r_t = \left(-ds_x + r_x + (d + \lambda - cd)s + (\lambda^2 + d\lambda + 2)q + c\, r + r^2 - dsr\right)_x, \end{cases} \tag{4.15}$$

which belongs to the integrable family (4.9) corresponding to

$$\alpha=\beta=1,\quad \epsilon=c,\quad \kappa=d+\lambda,\quad \xi=(d+\lambda)^2+2,\quad \kappa_1=d,\quad \xi_2=0\ (\text{i.e.}, \xi_1=d^2).$$

Therefore, following from the Hamiltonian operators $\widetilde{\mathcal{J}}_1$ and $\widetilde{\mathcal{J}}_2$ of system (4.9) defined in (4.5), (4.6), and (4.7) with $\nu=0$, (4.15) admits the Hamiltonian pair

$$\widetilde{\mathcal{J}}_1'=\begin{pmatrix}1&0&d\\0&0&1\\d&1&d^2\end{pmatrix}\partial_x,\qquad \widetilde{\mathcal{J}}_2'=\begin{pmatrix}\partial_x&\partial_x s&(d+\lambda)\partial_x\\ s\partial_x&q\partial_x+\partial_x q&-\partial_x^2+c\,\partial_x+r\partial_x\\ (d+\lambda)\partial_x&\partial_x^2+c\,\partial_x+\partial_x r&\big((d+\lambda)^2+2\big)\,\partial_x\end{pmatrix}.$$

In addition, (4.15) admits the following 2×2 Lax pair with spectral parameter μ:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}\mu&1\\ L&-\mu\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\qquad \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t=\begin{pmatrix}A&B\\ C&-A\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where

$$\begin{aligned}
L&=-\frac{d}{4}\left(\lambda+d+\frac{2}{\lambda}\right)s^2+\frac14(r+1)^2-\frac12(\lambda^2+d\lambda+2)q+\frac12(cd-\lambda)\,r-\frac14 r_x\\
&\quad+\frac14\left(\frac{\lambda}{d}+2-c\right)\left(\frac{\lambda}{d}-c\right)-\mu^2,\\
A&=-2B\mu+B_x,\qquad\qquad B=-ds+r+\frac{\lambda}{d}+1,\\
C&=2B\mu^2-2B_x\mu+\frac{d}{\lambda}(\lambda+d+1)\,Bs^2-\frac12 Br^2+(\lambda^2+d\lambda+2)Bq\\
&\quad+\left(\frac{\lambda}{d}+1-c\right)Br+Br_x+\frac12\left(B_{xx}-B-\left(\frac{\lambda}{d}+2-c\right)\left(\frac{\lambda}{d}-c\right)B\right).
\end{aligned}$$

On the other hand, (4.15) is related to

$$\begin{cases}
w_t=\left(-\lambda u_x+w+c\lambda u+wv-\lambda u^2+\lambda uv+\frac32\lambda w^2\right)_x,\\
u_t=\left(-u_x+c\,u-u^2+\lambda wu+2uv+\frac12 w^2\right)_x,\\
v_t=\left(\lambda w_x-2u_x+v_x+c\lambda w+2c\,u+c\,v+v^2+\lambda wv+2uv-2u^2+w^2\right)_x
\end{cases}\tag{4.16}$$

via the Miura-type transformation

$$s=w,\qquad q=-u_x-u^2+\frac12 w^2+uv+cu,\qquad r=(d+\lambda)w+v.$$

According to Theorem 2.3, the recursion operator $\mathcal{R}(w,u,v)$ of (4.16) satisfies

$$\mathbf{T}\,\mathcal{R}(w,u,v)=\widetilde{\mathcal{R}}(s,q,r)\,\mathbf{T},$$

where

$$\mathbf{T}=\begin{pmatrix}1&0&0\\ w&\Phi&u\\ d+\lambda&0&1\end{pmatrix},\qquad \Phi=-\partial_x-2u+v+c,\tag{4.17}$$

and

$$\begin{aligned}
\widetilde{\mathcal{R}}(s,q,r)&=\widetilde{\mathcal{J}}_2'\widetilde{\mathcal{J}}_1'^{-1}\\
&=\begin{pmatrix}1-d\partial_x s\partial_x^{-1}&\lambda&\partial_x s\partial_x^{-1}\\ s-dq-d\partial_x q\partial_x^{-1}&-\partial_x-ds+r+c&q+\partial_x q\partial_x^{-1}\\ -d\partial_x-d\partial_x r\partial_x^{-1}+\lambda+d-cd&\lambda^2+\lambda d+2&\partial_x+\partial_x r\partial_x^{-1}+c\end{pmatrix}
\end{aligned}$$

is the recursion operator of system (4.15). Consequently, the recursion operator of (4.16) is

$$\mathcal{R}(w,u,v)=\begin{pmatrix}1+\lambda w+\lambda\partial_x w\partial_x^{-1}&\lambda\Phi&\lambda u+\partial_x w\partial_x^{-1}\\ w+\lambda\partial_x u\partial_x^{-1}&\Phi&u+\partial_x u\partial_x^{-1}\\ \lambda c+2w+\lambda\partial_x+\lambda\partial_x v\partial_x^{-1}&2\Phi&c+2u+\partial_x+\partial_x v\partial_x^{-1}\end{pmatrix},$$

where the operator Φ is defined in (4.17).

Furthermore, in the case $b = 1$ and $a = -2$, the first Hamiltonian operator $\mathcal{J}_1(w,u,v)$ of system (4.16) is $\mathcal{J}(w,u,v)$ given by (4.3), which, along with $\mathcal{R}(w,u,v)$, gives rise to the second Hamiltonian operator

$$
\begin{aligned}
&\mathcal{J}_2(w,u,v) = \mathcal{R}(w,u,v)\mathcal{J}_1(w,u,v) \\
&= \begin{pmatrix} \partial_x + \lambda\,(w\partial_x + \partial_x w) & \lambda u\partial_x + \partial_x w & -\lambda\partial_x^2 + \lambda(c+v)\partial_x + 2\partial_x w \\ w\partial_x + \lambda\partial_x u & \partial_x u + u\partial_x & -\partial_x^2 + (c+v)\partial_x + 2\partial_x u \\ \lambda\partial_x^2 + (c\lambda + 2w)\partial_x + \lambda\partial_x v & \partial_x^2 + (c+2u)\partial_x + \partial_x v & 4c\,\partial_x + 2(v\partial_x + \partial_x v) \end{pmatrix}.
\end{aligned}
$$

With the Hamiltonian pair $\mathcal{J}_1$ and $\mathcal{J}_2$ in hand, the dual integrable bi-Hamiltonian hierarchy for system (4.16) can also be readily constructed. For instance, if we set $c = \lambda = 0$, the system (4.16) reduces to

$$
\begin{cases} w_t = (w + wv)_x\,, \\ u_t = \left(-u_x - u^2 + 2uv + \frac{1}{2}w^2\right)_x, \\ v_t = \left(-2u_x + v_x + v^2 + 2uv - 2u^2 + w^2\right)_x, \end{cases} \tag{4.18}
$$

which can be viewed as a three-component generalization of the mDWW system (3.2). The dual version of system (4.18) is obtained by recombining the Hamiltonian pair

$$
\mathcal{J}_1' = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & 0 & \partial_x \\ 0 & \partial_x & 2\partial_x \end{pmatrix}, \qquad \mathcal{J}_2' = \begin{pmatrix} \partial_x & \partial_x w & 2\partial_x w \\ w\partial_x & \partial_x u + u\partial_x & -\partial_x^2 + v\partial_x + 2\partial_x u \\ 2w\partial_x & \partial_x^2 + \partial_x v + 2u\partial_x & 2(v\partial_x + \partial_x v) \end{pmatrix},
$$

admitted by (4.18) to define the dual Hamiltonian pair

$$
\overline{\mathcal{J}}_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1-\partial_x \\ 0 & 1+\partial_x & 2 \end{pmatrix}\partial_x, \quad \overline{\mathcal{J}}_2 = \begin{pmatrix} 0 & \partial_x\bar{w} & 2\partial_x\bar{w} \\ \bar{w}\partial_x & \partial_x\bar{u} + \bar{u}\partial_x & \bar{v}\partial_x + 2\partial_x\bar{u} \\ 2\bar{w}\partial_x & 2\bar{u}\partial_x + \partial_x\bar{v} & 2\,(\bar{v}\partial_x + \partial_x\bar{v}) \end{pmatrix}.
$$

The dual of the original integrable system (4.18) will thus take the form

$$
\begin{cases} \bar{w}_t = ((2g_x - f_x + f)\bar{w})_x\,, \\ \bar{u}_t = \left((g+g_x)(g_x - g - f_x + f) + (2g_x - f_x + f)\bar{u} + \bar{w}^2\right)_x, \\ \bar{v}_t = \left(2(g+g_x)(g_x - g - f_x + f) + (2g_x - f_x + f)\bar{v} + 2\bar{w}^2\right)_x, \end{cases}
$$

where $f = (1-\partial_x^2)^{-1}\bar{v}$ and $g = (1-\partial_x^2)^{-1}\bar{u}$, and

$$
\begin{aligned}
\overline{\mathcal{H}}_1 &= \int \Big(\bar{w}^2 - (g^2 + g_x^2) + (g + g_x)\bar{v}\Big)\,\mathrm{d}x, \\
\overline{\mathcal{H}}_2 &= \frac{1}{2}\int \Big(2\bar{w}^2(2g_x + f - f_x) + 2(g_x^2 - g^2)(f - f_{xx}) \\
&\qquad + \Big(2g + 3g_x - g_{xxx}\Big)f^2 + \Big(2g + g_x - g_{xx}\Big)f_x^2\Big)\,\mathrm{d}x
\end{aligned}
$$

are the required Hamiltonian functionals.

5. Solitary wave solutions of the dual DWW system (3.12)

In this section, we analyze the solitary wave solutions of the following dual DWW system

$$
\begin{cases} g_t - \nu g_{xt} = \big(-\beta g_x + 2gf - \nu g_x f\big)_x, \\ f_t + \nu f_{xt} = \big(\beta f_x + 2g + f^2 + \nu f_x f\big)_x, \quad \nu = \pm 1, \end{cases} \tag{5.1}
$$

which is equivalent to system (3.12) under the Galilean transformation $x \mapsto x - \alpha t$. We consider the solitary wave solutions which take the form

$$
(g(t,x),\, f(t,x)) = (\psi(x - c\,t),\, \phi(x - c\,t))\,, \quad c \in \mathbb{R},
$$

for some $\psi, \phi : \mathbb{R} \to \mathbb{R}$ such that $\psi \to a$ ($a \geq 0$), $\phi \to 0$ as $|x| \to \infty$. In view of the asymptotic behavior of ψ, we set $g = a + h$ with $h \to 0$ as $|x| \to \infty$ and rewrite system (5.1) as

$$\begin{cases} h_t - \nu h_{xt} = \big(- \beta h_x + 2af + 2hf - \nu h_x f\big)_x, \\ f_t + \nu f_{xt} = \big(\beta f_x + 2h + f^2 + \nu f_x f\big)_x, \qquad \nu = \pm 1. \end{cases} \tag{5.2}$$

Definition 5.1. *Let $0 < T \leq \infty$. A function $(h,\, f) \in C\left([0,\, T), X\right)$ is called a* solution *of* (5.2) *on* $[0,\, T)$ *if it satisfies* (5.2) *in the distribution sense on* $[0,\, T)$.

Definition 5.2. *A* solitary wave *of* (5.2) *is a nontrivial traveling wave solution of* (5.2) *of the form $\Phi_c(t,x) = (\eta(x - c\,t),\, \phi(x - c\,t))$ with the constant wave speed $c \in \mathbb{R}$, where $(\eta,\, \phi) \in H^1_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ and $\eta,\, \phi$ vanish at infinity.*

Definition 5.3. *We say that a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ has a* peak *at $\bar{x}$ if ϕ is smooth locally on either side of $\bar{x}$ and*

$$0 \neq \lim_{x \uparrow \bar{x}} \phi_x(x) = - \lim_{x \downarrow \bar{x}} \phi_x(x) \neq \pm\infty.$$

Wave profiles with peaks are called peaked waves or peakons.

Definition 5.4. *We say that a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ has a* cusp *at $\bar{x}$ if ϕ is smooth locally on either side of $\bar{x}$ and*

$$\lim_{x \uparrow \bar{x}} \phi_x(x) = - \lim_{x \downarrow \bar{x}} \phi_x(x) = \pm\infty.$$

Wave profiles with cusps are called cusped waves or cuspons.

Definition 5.5. *We say that a traveling wave solution $\phi : \mathbb{R} \to \mathbb{R}$ is a* kink solution *if ϕ satisfies the boundary conditions*

$$k_l = \lim_{x \to -\infty} \phi(x) \quad \text{and} \quad k_r = \lim_{x \to +\infty} \phi(x),$$

where $-\infty < k_l < k_r < +\infty$ or $k_l > k_r$. The function ϕ sometimes also satisfies the additional asymptotic condition

$$\lim_{|x| \to \infty} \phi_x^{(j)}(x) = 0, \quad j = 1, 2, \ldots.$$

Substituting the traveling wave ansatz $(h(t,x),\, f(t,x)) = (\eta(x - c\,t),\, \phi(x - c\,t))$, where the wave speed c is constant, into (5.2), one obtains the following system of ordinary differential equations for $(\eta,\, \phi)$

$$\begin{cases} -c\,\eta_x + c\,\nu\eta_{xx} = \big(- \beta\eta_x + 2a\phi + (2\eta - \nu\eta_x)\phi\big)_x, \\ -c\,\phi_x - c\,\nu\phi_{xx} = \big(\beta\phi_x + 2\eta + (\phi + \nu\phi_x)\phi\big)_x. \end{cases} \tag{5.3}$$

Integrating both equations produces

$$\begin{cases} (\nu\phi + c\nu + \beta)\eta_x - c\eta - 2(a + \eta)\phi = 0, \\ \eta = -\frac{1}{2}(\nu\phi + c\nu + \beta)\phi_x - \frac{1}{2}\phi^2 - \frac{1}{2}c\phi. \end{cases} \tag{5.4}$$

Then we substitute the second equation in system (5.4) into the first to derive a single second order equation for $\phi(x)$:

$$(\phi - \tilde{c})^2\phi_{xx} + (\phi - \tilde{c})\phi_x^2 - 2\phi^3 - 3c\phi^2 - (c^2 - 4a)\phi = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}), \tag{5.5}$$

where $\tilde{c} = -(c + \nu\beta)$. Further calculation gives

$$\left[(\phi - \tilde{c})^3\right]_{xx} = 3(\phi - \tilde{c})\phi_x^2 + 3\left[2\phi^3 + 3c\phi^2 + (c^2 - 4a)\phi\right], \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{5.6}$$

The following lemma deals with the regularity of the solitary waves. The idea is inspired by the study of the traveling waves of the CH equation [32]; see also [8] and [24].

Lemma 5.1. *Let* $(\eta,\, \phi) \in H^1_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ *be a solitary wave of* (5.2). *Then*

$$(\phi - \tilde{c})^k \in C^j \left(\mathbb{R} \setminus \phi^{-1}(\tilde{c})\right), \qquad \text{for } k \geq 2^j. \tag{5.7}$$

Therefore, $\phi \in C^\infty\left(\mathbb{R} \setminus \phi^{-1}(\tilde{c})\right)$.

Proof. We set $v = \phi - \tilde{c}$, and denote

$$G(v) = 2v^3 + 3(2\tilde{c} + c)v^2 + (6\tilde{c}^2 + c^2 + 6c\tilde{c} - 4a)v + [(2\tilde{c} + c)(\tilde{c} + c) - 4a]\tilde{c}.$$

Since $\phi \in H^1_{loc}(\mathbb{R})$ solves equation (5.6) in the distributional sense, v satisfies

$$(v^3)_{xx} = 3vv_x^2 + 3\,G(v), \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

We then deduce that $(v^3)_{xx} \in L^1_{loc}(\mathbb{R})$, which indicates that $(v^3)_x$ is absolutely continuous and v, v^2 as well as v^3 belong to $C^1\left(\mathbb{R} \setminus v^{-1}(0)\right)$. Moreover, for $k \geq 4$,

$$\begin{aligned}
(v^k)_{xx} &= \left(kv^{k-1}v_x\right)_x = \tfrac{1}{3}\,k\left[v^{k-3}(v^3)_x\right]_x = k(k-3)v^{k-2}v_x^2 + \tfrac{1}{3}\,k\,v^{k-3}(v^3)_{xx} \\
&= k(k-2)v^{k-2}v_x^2 + 3kv^{k-3}G(v) \\
&= \tfrac{1}{4}k(k-2)v^{k-4}\left[(v^2)_x\right]^2 + 3\,k\,v^{k-3}G(v) \in C\left(\mathbb{R} \setminus v^{-1}(0)\right).
\end{aligned} \tag{5.8}$$

It follows that $v^k \in C^2\left(\mathbb{R} \setminus v^{-1}(0)\right)$ for $k \geq 4$.

In the case of $k \geq 8$, based on the previous result, we have v^4 and v^{k-4} belong to $C^2\left(\mathbb{R} \setminus v^{-1}(0)\right)$, while $v^{k-3}G(v) \in C^1\left(\mathbb{R} \setminus v^{-1}(0)\right)$. On the other hand, since

$$v^{k-2}v_x^2 = \frac{1}{4(k-4)}\left(v^4\right)_x\left(v^{k-4}\right)_x \in C^1\left(\mathbb{R} \setminus v^{-1}(0)\right),$$

formula (5.8) allows us to deduce that

$$(v^k)_{xx} = \frac{k(k-2)}{k-4}\left(v^4\right)_x\left(v^{k-4}\right)_x + 3kv^{k-3}G(v) \in C^1\left(\mathbb{R} \setminus v^{-1}(0)\right),$$

which immediately leads to

$$v^k \in C^3\left(\mathbb{R} \setminus v^{-1}(0)\right), \qquad k \geq 8.$$

Extending these arguments to higher values of k concludes that $v^k \in C^j\left(\mathbb{R} \setminus v^{-1}(0)\right)$ for $k \geq 2^j$, and then (5.7) follows. This completes the proof. $\square$

Denote $\bar{x} = \min\{x\colon\ \phi(x) = \tilde{c}\}$, while if $\phi \neq \tilde{c}$ for all x we set $\bar{x} = +\infty$, so $\bar{x} \leq +\infty$. From Lemma 5.1, a solitary wave ϕ is smooth on $(-\infty,\, \bar{x})$ and hence (5.5) holds pointwise on $(-\infty,\, \bar{x})$. Therefore, we may multiply (5.5) by ϕ_x and integrate on $(-\infty,\, x]$ for $x < \bar{x}$ to obtain

$$\phi_x^2 = \frac{\phi^2(\phi - A_1)(\phi - A_2)}{(\phi - \tilde{c})^2} := F(\phi), \tag{5.9}$$

where $A_1 = -c - 2\sqrt{a}$ and $A_2 = -c + 2\sqrt{a}$.

Next, we will explore the qualitative behavior of solutions to (5.9) near points where $F(\phi)$ has zero or a pole. Applying the similar arguments as introduced in [29, 32], we arrive at the following conclusions.

1. When $F(\phi)$ has a simple zero at $m = A_1$ or $m = A_2$, so that $F(m) = 0$ and $F'(m) \neq 0$, the solution ϕ of (5.9) satisfies

$$\phi_x^2 = (\phi - m)F'(m) + O\left((\phi - m)^2\right), \qquad \text{as} \quad \phi \to m,$$

which gives rise to

$$\phi(x) = m + \tfrac{1}{4}(x - x_0)^2F'(m) + O\left((x - x_0)^4\right), \qquad \text{as} \quad x \to x_0, \tag{5.10}$$

where $\phi(x_0) = m$.

2. When $F(\phi)$ has a double zero at $\phi = m$, so that $F(m) = F'(m) = 0$, $F''(m) > 0$, we have

$$\phi_x^2 = F''(m)(\phi - m)^2 + O\left((\phi - m)^3\right), \qquad \text{as} \quad \phi \to m,$$

and thus

$$\phi(x) - m \sim \gamma_1 \exp(-x\sqrt{F''(m)}), \qquad \text{as} \quad x \to \infty \tag{5.11}$$

for some constant γ_1. Thus $\phi \to m$ exponentially as $x \to \infty$.

Analogous computations allow us to draw the following conclusions for cases of the third and fourth order zeros:

3. When $F(\phi)$ has $k-$th order zero, $k = 3, 4$, at $\phi = m$, so that $F^{(i)}(m) = 0$, for $i = 0, \ldots, k-1$, while $F^{(k)}(m) \neq 0$, and

$$\phi_x^2 = F^{(k)}(m)(\phi - m)^k + O\left((\phi - m)^{k+1}\right), \qquad \text{as} \quad \phi \to m.$$

Hence we have

$$\phi(x) - m \sim x^{\frac{2}{2-k}}, \qquad \text{as} \quad x \to \infty.$$

Therefore, when $k = 3$ or $k = 4$, $\phi \to m$ algebraically, at the rate $O\left(1/x^2\right)$ and $O\left(1/x\right)$, respectively, as $x \to \infty$.

4. Peaked solitary waves occur when ϕ changes direction: $\phi_x \mapsto -\phi_x$ according to (5.9).

5. Suppose ϕ approaches a simple pole $\phi(\bar{x}) = \tilde{c}$, such that $1/F(\phi)$ has a simple zero. Without loss of generality, we assume that $\phi(x) < \tilde{c}$, and then

$$\phi(\bar{x}) - \tilde{c} = \gamma_2 |x - \bar{x}|^{2/3} + O\left((x - \bar{x})^{4/3}\right), \qquad \text{as} \quad x \to \bar{x}, \tag{5.12}$$

and

$$\phi_x = \begin{cases} \frac{2}{3}\gamma_2 |x - \bar{x}|^{-1/3} + O\left((x - \bar{x})^{1/3}\right), & x \downarrow \bar{x}, \\ -\frac{2}{3}\gamma_2 |x - \bar{x}|^{-1/3} + O\left((x - \bar{x})^{1/3}\right), & x \uparrow \bar{x}, \end{cases} \tag{5.13}$$

for some constant $\gamma_2 < 0$.

Furthermore, if ϕ approaches a pole of order 2 at $\phi(\bar{x}) = \tilde{c}$, such that $1/F(\phi)$ has a double zero, then

$$\phi(\bar{x}) - \tilde{c} = \gamma_3 |x - \bar{x}|^{1/2} + O(x - \bar{x}), \qquad \text{as} \quad x \to \bar{x}. \tag{5.14}$$

We thus have

$$\phi_x = \begin{cases} \frac{1}{2}\gamma_3 |x - \bar{x}|^{-1/2} + O\left(1\right), & x \downarrow \bar{x}, \\ -\frac{1}{2}\gamma_3 |x - \bar{x}|^{-1/2} + O\left(1\right), & x \uparrow \bar{x}, \end{cases} \tag{5.15}$$

for some constant $\gamma_3 < 0$. Altogether, it follows from (5.13) and (5.15) that

$$\lim_{x \uparrow \bar{x}} \phi_x = -\lim_{x \downarrow \bar{x}} \phi_x = \pm\infty,$$

which means that whenever F has a pole, the continuous solution ϕ will have a cusp at $\bar{x}$. We conclude that nonanalytic behavior will only arise at the singular points, which are precisely the points of genuine nonlinearity of the original systems.

Now, let us consider the classification of traveling waves of (5.2). As in [29], our approach is based on the configuration of the zeros and poles of $F(\phi)$ in formula (5.9). This allows to determine several precise parameter regimes in which (5.2) admits solitary wave solutions. In contrast to the dual counterpart of the KdV equation, whose qualitative properties change considerably as the coefficient ν in potential changes sign [33], the sign of ν involved in system (5.2) just affect the parameter regimes that govern the corresponding wave patterns. Thus, from here on, we only consider the case when $\nu = 1$. The qualitative behavior of the component $\phi(x)$ of the solitary wave solution to (5.2) in the case of $a > 0$ is summarized in the following theorem.

Theorem 5.1. *When $a > 0$, the necessary condition for the existence of the solitary wave solutions is $c \geq 2\sqrt{a}$ or $c \leq -2\sqrt{a}$. Moreover, we have*

Case 1. $c < -2\sqrt{a}$, *i.e.,* $0 < A_1 < A_2$.

1.1. *If $\beta < 2\sqrt{a}$, i.e., $\tilde{c} > A_1$, then there is a smooth solitary wave $\phi > 0$, with $\max_{x\in\mathbb{R}} \phi(x) = A_1$;*

1.2. *If $2\sqrt{a} \leq \beta < -c$, i.e., $0 < \tilde{c} \leq A_1$, then there is a cusped solitary wave $\phi > 0$, with $\max_{x\in\mathbb{R}} \phi(x) = \tilde{c}$;*

1.3. *If $\beta > -c$, i.e., $\tilde{c} < 0$, then there is a smooth solitary wave $\phi > 0$, with $\max_{x\in\mathbb{R}} \phi(x) = A_1$ and an anticusped solitary wave (the solution profile has a cusp pointing downward) $\phi < 0$, with $\min_{x\in\mathbb{R}} \phi(x) = \tilde{c}$.*

Case 2. *When $c = -2\sqrt{a}$, i.e., $A_1 = 0 < A_2$, a solitary wave exists if and only if $\beta > 2\sqrt{a}$, i.e., $\tilde{c} < 0$, and is anticusped with $\min_{x\in\mathbb{R}} \phi(x) = \tilde{c}$.*

Case 3. *When $c = 2\sqrt{a}$, i.e., $A_1 < 0 = A_2$, a solitary wave exists if and only if $\beta < -2\sqrt{a}$, i.e., $\tilde{c} > 0$, and is cusped with $\max_{x\in\mathbb{R}} \phi(x) = \tilde{c}$.*

Case 4. $c > 2\sqrt{a}$, *i.e.,* $A_1 < A_2 < 0$.

4.1. *If $\beta > -2\sqrt{a}$, i.e., $\tilde{c} < A_2$, then there is a smooth solitary wave $\phi < 0$, with $\min_{x\in\mathbb{R}} \phi(x) = A_2$;*

4.2. *If $-c < \beta \leq -2\sqrt{a}$, i.e., $A_2 \leq \tilde{c} < 0$, then there is an anticusped solitary wave $\phi < 0$, with $\min_{x\in\mathbb{R}} \phi(x) = \tilde{c}$;*

4.3. *If $\beta < -c$, i.e., $\tilde{c} > 0$, then there is a smooth solitary wave $\phi < 0$, with $\min_{x\in\mathbb{R}} \phi(x) = A_2$ and a cusped solitary wave $\phi > 0$, with $\max_{x\in\mathbb{R}} \phi(x) = \tilde{c}$.*

Moreover, each of the above solitary waves is unique and even up to translations. When $A_1 = 0$ or $A_2 = 0$, the corresponding anticusped or cusped waves decay algebraically at the rate of $O\left(1/x^2\right)$, while when $A_1 > 0$ or $A_2 < 0$, all solitary waves decay exponentially to zero at infinity. When $\tilde{c} = A_1 > 0$ or $\tilde{c} = A_2 < 0$, the corresponding cusped or anticusped solitary waves have the singular behavior in (5.12) *and* (5.13), *while the other cases satisfy* (5.14) *and* (5.15).

Proof. First of all, formula (5.9), together with the fact that $\phi(x)$ decays at infinity leads to the necessary condition for the existence of solitary wave: $A_1 \geq 0$ or $A_2 \leq 0$.

(1). If $A_1 = 0$, then (5.9) becomes

$$\phi_x^2 = \frac{\phi^3(\phi - A_2)}{(\phi - \tilde{c})^2} := F_1(\phi), \tag{5.16}$$

which indicates that $\phi(x) < 0$ near $-\infty$. Since $\phi(x) \to 0$, as $x \to -\infty$, there exists some x_0 sufficiently negative satisfying $\phi(x_0) = -\epsilon < 0$, with $\epsilon > 0$ sufficiently small, and $\phi_x(x_0) < 0$.

According to the standard ODE theory, one can generate a unique local solution $\phi(x)$ on $[x_0 - \delta,\, x_0 + \delta]$ for some $\delta > 0$. Furthermore, since $A_1 < A_2$, we deduce that

$$F_1'(\phi) = \frac{\phi^2\left(2\phi^2 - (A_2 + 4\tilde{c})\phi + 3A_2\tilde{c}\right)}{(\phi - \tilde{c})^3} = \frac{\phi^2(\phi - \mu_1)(\phi - \mu_2)}{(\phi - \tilde{c})^3},$$

$$\mu_1 = \frac{A_2 + 4\tilde{c} - \sqrt{A_2^2 - 16A_2\tilde{c} + 16\tilde{c}^2}}{4}, \quad \mu_2 = \frac{A_2 + 4\tilde{c} + \sqrt{A_2^2 - 16A_2\tilde{c} + 16\tilde{c}^2}}{4}.$$

If $\tilde{c} > 0$, then $0 < \mu_1 < \mu_2$, so $F_1'(\phi) < 0$ for $\phi < 0$. While, if $\tilde{c} < 0$, we have $\mu_1 < 0 < \mu_2$, and then $F_1'(\phi) < 0$ for $\mu_1 < \phi < 0$. In view of this, we can choose x_0, so that, in the case of $\tilde{c} < 0$, $\mu_1 < \phi < 0$ on $[x_0 - \delta,\, x_0 + \delta]$. Summarizing, one has $F_1'(\phi) < 0$, on $[x_0 - \delta,\, x_0 + \delta]$, which together with the fact that $\phi_{xx} = F_1'(\phi)/2$ implies $\phi_x(x)$ decreases on $[x_0 - \delta,\, x_0 + \delta]$. Hence $\phi_x(x) < 0$ on $[x_0,\, x_0 + \delta]$, which further implies that $\phi(x)$ decreases on $[x_0,\, x_0 + \delta]$.

If $\tilde{c} < 0$, since $\sqrt{F_1(\phi)}$ is locally Lipschitz in ϕ for $\tilde{c} < \phi \leq 0$, one can easily continue the local solution to $(-\infty,\, x_0 - \delta]$ with $\phi(x) \to 0$ as $x \to -\infty$. As for $x > x_0 + \delta$, we can solve the initial-value problem

$$\psi_x = -\sqrt{F_1(\psi)}, \qquad \psi(x_0+\delta) = \phi(x_0+\delta),$$

all the way until $\psi = \tilde{c}$, which is a double pole of $F_1(\psi)$. Therefore, according to (5.16), one can readily construct an anticusped solution with a cusp singularity at $\phi = \tilde{c}$. Moreover, since $\phi = 0$ is a third-order zero of $F_1(\phi)$, we deduce that $\phi(x)$ has the algebraic decay as $|x| \to \infty$ at the rate of $O\left(1/x^2\right)$.

If $\tilde{c} > 0$, the fact that $\sqrt{F_1(\phi)}$ is locally Lipschitz in ϕ allows us to extend the local solution to all of $\mathbb{R}$ and obtain that $\phi(x) \to -\infty$ as $x \to +\infty$, which fails to be in $H^1(\mathbb{R})$. Thus there is no solitary wave in this case.

If $\tilde{c} = 0$, then the equation (5.16) is reduced to $\phi^2\phi_x^2 = \phi^3(\phi - A_2)$. Since $A_2 > 0$, by the same argument as above, one may realize that there is no nontrivial bounded solution exists in this case.

In the case of $A_2 = 0$, an analogous analysis as used in the above cases leads to the conclusions that there is no solitary wave when $\tilde{c} \le 0$. When $\tilde{c} > 0$, there exists a cusped solution caused by the singularities at $\tilde{c}$ which decays algebrically at the rate $O\left(1/x^2\right)$.

(2). Now let us consider the cases $A_1 > 0$ or $A_2 < 0$, where we concentrate on the former because the latter can be analyzed similarly. From (5.9) we see that ϕ cannot oscillate around zero near infinity. Let us distinguish the following two subcases:

Case I. $\phi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\phi(x_0) = \epsilon > 0$, with ϵ sufficiently small and $\phi_x(x_0) > 0$.

(i). When $\tilde{c} > A_1 > 0$ or $\tilde{c} < 0$, $F(\phi)$ is locally Lipschitz in ϕ for $0 \le \phi \le A_1$. Hence, there is a local solution to

$$\phi_x = \sqrt{F(\phi)}, \qquad \phi(x_0) = \epsilon, \qquad \epsilon > 0,$$

on $[x_0 - \delta,\, x_0 + \delta]$ for some $\delta > 0$. As a consequence of (5.10) and (5.11), there is a smooth solitary wave with maximum height $\phi = A_1$, having exponential decay to zero at infinity

$$\phi(x) = O\left(\exp\left(-\sqrt{F''(0)}|x|\right)\right), \qquad \text{as} \quad |x| \to \infty,$$

where

$$\sqrt{F''(0)} = \frac{\sqrt{2(c^2-4a)}}{|\tilde{c}|}.$$

(ii). When $0 < \tilde{c} \le A_1$, the smooth solution can be constructed until $\phi = \tilde{c}$. However, at $\phi = \tilde{c}$, the slope $\phi_x \to \infty$, which gives rise to a cusp. Moreover, since $\phi = 0$ is still a double zero of $F(\phi)$, the solution has exponential decay at $\pm\infty$.

(iii). When $\tilde{c} = 0$, (5.9) becomes

$$\phi^2\phi_x^2 = \phi^2(\phi - A_1)(\phi - A_2).$$

Using the fact that $0 < A_1 < A_2$, and $F(\phi) > 0$ when $\phi < A_1$ or $\phi > A_2$, one may show the singular point $\phi = 0$ is in fact degenerate, and thus there is no nontrivial solution satisfying the decay condition.

Case II. $\phi(x) < 0$ near $-\infty$. In this case, we are solving

$$\phi_x = -\sqrt{F(\phi)}, \qquad \phi(x_0) = -\epsilon$$

for some x_0 sufficiently negative and $\epsilon > 0$ sufficiently small. Then the same analysis as used in the proof of Case (1) leads to the conclusion that when $\tilde{c} > 0$, there is no solitary wave. When $\tilde{c} < 0$, $\phi = \tilde{c}$ is a pole of $F(\phi)$. Arguing as before, we obtain an anticusped solitary wave with $\min_{x\in\mathbb{R}} \phi = \tilde{c}$, which decays exponentially.

Finally, by the standard ODE theory and the fact that the equation (5.6) is invariant under the transformations $x \mapsto \pm x + d$ for any constant d, we conclude that the solitary waves obtained above are unique and even up to translations. Moreover, among all the cusped or anticusped solitary waves, those who arise from the subcase $0 < \tilde{c} = A_1$ or $\tilde{c} = A_2 < 0$ are caused by the simple pole at $\phi(\bar{x}) = \tilde{c}$. So, in contrast to the other cases, where the singular behavior follows (5.14) and (5.15), when $0 < \tilde{c} = A_1$ or $\tilde{c} = A_2 < 0$, the solution $\phi(x)$ will approach to $\tilde{c}$ in accordance with (5.12) and (5.13). □

In addition, we can derive an implicit formula for the cusped solitary waves. Let us consider only the case $A_1 > 0$. By Theorem 5.1, we know that the cusped solitary wave exists only when $2\sqrt{a} \le \beta < -c$. Choose $\bar{x}$ at the cusp, so that $\phi(\bar{x}) = \tilde{c}$. Since ϕ is positive, even with respect to $\bar{x}$, and decreasing on $[\,\bar{x}, +\infty)$, for $x > \bar{x}$, we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\phi\sqrt{(A_1-\phi)(A_2-\phi)}}{\phi - \tilde{c}},$$

which gives rise to

$$-(x-\bar{x}) = \int_{\tilde{c}}^{\phi} \frac{\tilde{c}-y}{y(A_1-y)}\sqrt{\frac{A_1-y}{A_2-y}}\,\mathrm{d}y.$$

Set

$$t = \sqrt{\frac{A_1-y}{A_2-y}}\,, \qquad t_0 = \sqrt{\frac{A_1-\tilde{c}}{A_2-\tilde{c}}}\,, \qquad \Delta = \sqrt{\frac{A_1-\phi}{A_2-\phi}}\,.$$

Then the above equation becomes

$$-(x-\bar{x}) = 2\int_{t_0}^{\Delta}\left(\frac{\tilde{c}}{A_2t^2-A_1} - \frac{1}{t^2-1}\right)\mathrm{d}t = \left(\frac{\tilde{c}}{\sqrt{A_1A_2}}\ln\left|\frac{\sqrt{A_2}t-\sqrt{A_1}}{\sqrt{A_2}t+\sqrt{A_1}}\right| - \ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_{t_0}^{\Delta},$$

which implies the following implicit formula for the cusped solitary wave:

$$-|x-\bar{x}| = \left(\frac{\tilde{c}}{\sqrt{A_1A_2}}\ln\left|\frac{\sqrt{A_2}t-\sqrt{A_1}}{\sqrt{A_2}t+\sqrt{A_1}}\right| - \ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_{t_0}^{\Delta}.$$

Theorem 5.2. *When $a = 0$, one has $A_1 = A_2 = -c$.*
Case 1. *$c < 0$, i.e., $A_1 = A_2 > 0$.*
1.1. *If $\beta < 0$, i.e., $\tilde{c} > A_1$, then there is a smooth traveling wave solution $\phi > 0$ such that $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = A_1$;*
1.2. *If $\beta = 0$, i.e., $\tilde{c} = A_1 = -c$, then the solitary wave $\phi > 0$ is peaked with $\max_{x\in\mathbb{R}}\phi(x) = \tilde{c}$;*
1.3. *If $0 < \beta < -c$, i.e., $0 < \tilde{c} < A_1$, then the solitary wave $\phi > 0$ is cusped with $\max_{x\in\mathbb{R}}\phi(x) = \tilde{c}$;*
1.4. *If $\beta > -c$, i.e., $\tilde{c} < 0$, then there is a smooth traveling wave solution $\phi > 0$ satisfying $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = A_1$, along with an anticusped wave $\phi < 0$ with $\min_{x\in\mathbb{R}}\phi(x) = \tilde{c}$.*
Case 2. *$c > 0$, i.e., $A_1 = A_2 < 0$.*
2.1. *If $\beta > 0$, i.e., $\tilde{c} < A_1$, then there is a smooth traveling wave solution $\phi < 0$ satisfying $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = A_1$;*
2.2. *If $\beta = 0$, i.e., $\tilde{c} = A_1$, then the solitary wave $\phi < 0$ is an antipeaked with $\min_{x\in\mathbb{R}}\phi(x) = \tilde{c}$;*
2.3. *If $-c < \beta < 0$, i.e., $A_1 < \tilde{c} < 0$, then the solitary wave $\phi < 0$ is anticusped with $\min_{x\in\mathbb{R}}\phi(x) = \tilde{c}$;*
2.4. *If $\beta < -c$, i.e., $\tilde{c} > 0$, then there is a smooth traveling wave solution $\phi < 0$ satisfying $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = A_1$ and a cusped solitary wave $\phi > 0$ with $\max_{x\in\mathbb{R}}\phi(x) = \tilde{c}$.*

Moreover, each kind of solitary waves in Cases 1.2, 1.3 and 2.2, 2.3 is unique and even up to translations. Cases 1.1, 1.4, 2.1 and 2.4 admit kink solutions which are unique up to *translations. All the solitary waves decay exponentially to zero at infinity, and all the cusped and anticusped solitary waves approach to* $\tilde{c}$ *according to* (5.14) *and* (5.15).

Proof. When $a = 0$, $A_1 = A_2 = -c \neq 0$, and equation (5.9) becomes

$$\phi_x^2 = \frac{\phi^2(\phi - A_1)^2}{(\phi - \tilde{c})^2} := G(\phi). \tag{5.17}$$

As before, we shall only consider the case $c < 0$, since the analysis used in this case can be directly applied to the case $c > 0$. It follows from (5.17) that ϕ cannot oscillate around zero near infinity. We thus consider the following two cases.

Case I. $\phi(x) > 0$ near $-\infty$. Because $\phi \to 0$ as $x \to -\infty$, there exists some x_0 sufficiently negative so that $\phi(x_0) = \epsilon > 0$, with ϵ sufficiently small and $\phi_x(x_0) > 0$.

(i). When $\tilde{c} > A_1 > 0$ or $A_1 > 0$, $\tilde{c} < 0$, $G(\phi)$ is locally Lipschitz. Hence, there is a local solution to

$$\phi_x = \sqrt{G(\phi)}, \qquad \phi(x_0) = \epsilon, \qquad \epsilon > 0,$$

on $[x_0 - \delta,\, x_0 + \delta]$ for some $\delta > 0$. Therefore, the smooth solution is increasing, which can be constructed until ϕ approaches A_1 as $x \to +\infty$. Furthermore, since both $\phi = 0$ and $\phi = A_1$ are double zeros, the smooth solution exhibits exponentially decay to zero as $x \to -\infty$ and to A_1 as $x \to +\infty$.

One can solve (5.17) to obtain the implicit formula for the solution:

$$\phi^{\tilde{c}/A_1}(A_1 - \phi)^{1-\tilde{c}/A_1} = \begin{cases} e^x, & \tilde{c} > A_1 > 0, \\ e^{-x}, & \tilde{c} < 0 < A_1. \end{cases}$$

(ii). When $0 < \tilde{c} < A_1$, similarly as above and using (5.17), one can derive the implicit formula

$$\phi^{\tilde{c}/A_1}(A_1 - \phi)^{1-\tilde{c}/A_1} = \tilde{c}^{\tilde{c}/A_1}(A_1 - \tilde{c})^{1-\tilde{c}/A_1} e^{-|x-\bar{x}|},$$

which implies that ϕ forms a cusped solitary wave caused by the singularity at $\tilde{c}$.

(iii). When $\tilde{c} = A_1 > 0$, then the smooth solution can be constructed until $\phi = \tilde{c} = A_1$. However, at $\phi = \tilde{c} = A_1$, it can make a sudden turn and thus give rise to a peak. The peaked solitary wave satisfies $\phi_x^2 = \phi^2$, which can be explicitly solved: $\phi(x) = \tilde{c}\exp(-|x - \bar{x}|)$.

(iv). When $0 = \tilde{c} < A_1$, it is easy to see that there is no solitary wave satisfying the decay condition.

Case II. $\phi(x) < 0$ near $-\infty$. In this case, we are solving

$$\phi_x = -\sqrt{G(\phi)}, \qquad \phi(x_0) = -\epsilon$$

for some x_0 sufficiently large negative and $\epsilon > 0$ sufficiently small.

When $\tilde{c} < 0$, then $\phi = \tilde{c}$ is a double pole of $G(\phi)$. Arguing as before, we obtain an anticusped solitary wave with $\min_{x\in\mathbb{R}} \phi = \tilde{c}$, which decays exponentially to zero and is implicitly defined by

$$\phi^{-\tilde{c}/A_1}(A_1 - \phi)^{\tilde{c}/A_1 - 1} = \tilde{c}^{-\tilde{c}/A_1}(A_1 - \tilde{c})^{\tilde{c}/A_1 - 1} e^{-|x-\bar{x}|}.$$

On the other hand, when $\tilde{c} \geq 0$, an argument similar to that was used in Theorem 5.1 shows that there is no solitary wave solution.

Finally, the conclusion that, up to translations, all solitary waves are unique and even and all kink solutions are unique, are direct consequences of the invariance of (5.17). □

We now shift our attention to the other component $\eta(x)$ of the traveling wave solution to (5.2), using the preceding classification for $\phi(x)$. The second equation in (5.4) implies

$$\eta_x = -\tfrac{1}{2}(\phi - \tilde{c})\phi_{xx} - \tfrac{1}{2}\phi_x^2 - \tfrac{1}{2}(2\phi + c)\phi_x,$$

hence, using (5.5),

$$\eta_x = -\frac{1}{2}\left((2\phi + c)\phi_x + \frac{2\phi^3 + 3c\phi^2 + (c^2 - 4a)\phi}{\phi - \tilde{c}}\right),$$

where, according to (5.9),

$$\phi_x = \pm\frac{\phi\sqrt{(\phi - A_1)(\phi - A_2)}}{\phi - \tilde{c}}. \tag{5.18}$$

Based on these identities, one may show that, corresponding to the "minus" and "plus" signs in (5.18),

$$\eta(x) = \eta^-(x) = -\frac{1}{2}\phi\left(\phi + c - \sqrt{(\phi - A_1)(\phi - A_2)}\right),$$
$$\eta_x(x) = \eta_x^-(x) = \frac{\phi}{2(\phi - \tilde{c})}(I(\phi) - J(\phi)),$$

while

$$\eta(x) = \eta^+(x) = -\frac{1}{2}\phi\left(\phi + c + \sqrt{(\phi - A_1)(\phi - A_2)}\right),$$
$$\eta_x(x) = \eta_x^+(x) = -\frac{\phi}{2(\phi - \tilde{c})}(I(\phi) + J(\phi)),$$

where

$$I(\phi) = (2\phi + c)\sqrt{(\phi - A_1)(\phi - A_2)}, \qquad J(\phi) = 2\phi^2 + 3c\phi + c^2 - 4a.$$

Note that $|I(\phi)| = |J(\phi)|$ when $\phi = \left(4a - c^2\right)/(2c)$. In what follows, we let

$$B = \frac{4a - c^2}{2c}, \qquad C_1 = \frac{-3c - \sqrt{c^2 + 32a}}{4}, \qquad C_2 = \frac{-3c + \sqrt{c^2 + 32a}}{4}.$$

Then

$$J(\phi) = (\phi - C_1)(\phi - C_2).$$

One may show that if $0 < A_1 < A_2$, then $0 < C_1 < B < A_1 < C_2$, whereas, if $A_1 < A_2 < 0$, then $C_1 < A_2 < B < C_2 < 0$. Finally, we denote

$$x(A_1) = \min\{x\colon\ \phi(x) = A_1\}, \qquad x_1(B) = \min\{x\colon\ \phi(x) = B\}, \qquad x_2(B) = 2x(A_1) - x_1(B).$$

The symmetry of ϕ implies that $\phi(x_2(B)) = \phi(x_1(B)) = B$.

Theorem 5.3. *When $a > 0$, the solution $\eta(x)$ of (5.3) satisfies the asymptotic boundary condition*

$$\lim_{x\to\pm\infty}(\eta(x),\, \eta_x(x)) = (0,\, 0).$$

Case 1. $c < -2\sqrt{a}$, $(0 < A_1 < A_2)$.

1.1. *If $\beta < 2\sqrt{a}$, $(\tilde{c} > A_1)$, then there exists a smooth solution $\eta(x) > 0$, which is increasing on $(-\infty, x_1(B)]$ and decreasing on $[x_1(B), +\infty)$, attaining its maximum $\max_{x\in\mathbb{R}}\eta(x) = \eta_1(B) = \left(c^2 - 4a\right)/4$ at $x_1(B)$.*

1.2. *If $2\sqrt{a} < \beta < -c$, $(0 < \tilde{c} < A_1)$, then there is a solution $\eta(x) > 0$, which is smooth except at the jump discontinuity $\bar{x}$. Moreover,*

(a) *If $-(c^2 + 4a)/(2c) < \beta < -c$, i.e., $0 < \tilde{c} < B < A_1$, then $\eta(x)$ is increasing on $(-\infty, \bar{x}]$ and decreasing on $[\bar{x}, +\infty)$, with*

$$\lim_{x\uparrow\bar{x}}(\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, +\infty\right), \quad \lim_{x\downarrow\bar{x}}(\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, -\infty\right). \tag{5.19}$$

(b) *If $\beta = -(c^2 + 4a)/(2c)$, i.e., $0 < \tilde{c} = B < A_1$, then $\eta(x)$ is increasing on $(-\infty, \bar{x}]$ and decreasing on $[\bar{x}, +\infty)$, satisfying*

$$\lim_{x\uparrow\bar{x}}(\eta(x),\, \eta_x(x)) = \left(\eta_1(B),\, \tfrac{1}{2}c^2\right), \qquad \lim_{x\downarrow\bar{x}}(\eta(x),\, \eta_x(x)) = (\eta_2(B),\, -\infty), \tag{5.20}$$

where $\eta_2(B) = a - 4a^2/c^2$.

(c) *If* $2\sqrt{a} < \beta < -(c^2+4a)/(2c)$, *i.e.,* $0 < B < \tilde{c} < A_1$, *then* $\eta(x)$ *is increasing on* $(-\infty,\, x_1(B)\,]$ *and decreasing on* $[\,x_1(B),\, +\infty\,)$, *with*

$$\begin{aligned} &\lim_{x\to x_1(B)} (\eta(x),\, \eta_x(x)) = (\eta_1(B),\, 0) \quad \text{and} \\ &\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, -\infty\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, -\infty\right). \end{aligned} \tag{5.21}$$

1.3. *If* $\beta = 2\sqrt{a}$, $(0 < \tilde{c} = A_1)$, *then* $\eta(x) > 0$ *is smooth for all* $x \in \mathbb{R}$ *except at the point* $\bar{x}$, *increasing on* $(-\infty,\, x_1(B)\,]$ *and decreasing on* $[\,x_1(B),\, +\infty\,)$, *and satisfying*

$$\lim_{x\to x_1(B)} (\eta(x),\, \eta_x(x)) = (\eta_1(B),\, 0)\,, \qquad \lim_{x\to \bar{x}} (\eta(x),\, \eta_x(x)) = (\eta(A_1),\, -\infty)\,,$$

where $\eta(A_1) = \sqrt{a}\, A_1 = -(c+2\sqrt{a})\sqrt{a}$.

1.4. *If* $\beta > -c$, $(\tilde{c} < 0)$, *then, corresponding to the smooth solitary wave* $\phi(x)$ *in this subcase, there exists a smooth solution* $\eta(x) > 0$, *which is increasing on* $(-\infty,\, x_2(B)\,]$ *and decreasing on* $[\,x_2(B),\, +\infty\,)$, *attaining its maximum* $\max_{x\in\mathbb{R}} \eta(x) = \eta_1(B)$ *at* $x_2(B)$. *On the other hand, corresponding to the anticusped solitary wave, there is a solution* $\eta(x) < 0$ *which has a jump discontinuity at* $\bar{x}$*; more precisely, it is decreasing on* $(-\infty,\, \bar{x}\,]$ *and increasing on* $[\,\bar{x},\, +\infty\,)$, *with*

$$\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, -\infty\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, +\infty\right).$$

Case 2. $c = -2\sqrt{a}$ *and* $\beta > 2\sqrt{a}$, $(A_1 = 0 < A_2$ *and* $\tilde{c} < 0)$.

There exists a solution $\eta(x) < 0$, *which has a jump discontinuity at* $\bar{x}$. *Moreover,* $\eta(x)$ *is decreasing on* $(-\infty,\, \bar{x}\,]$, *increasing on* $[\,\bar{x},\, +\infty\,)$, *and with*

$$\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, -\infty\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, +\infty\right).$$

Case 3. $c = 2\sqrt{a}$ *and* $\beta < -2\sqrt{a}$, $(A_1 < A_2 = 0$ *and* $\tilde{c} > 0)$.

There exists a solution $\eta(x) < 0$, *which takes the same profile as the solution* $\eta(x)$ *in Case 2 and satisfies*

$$\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, -\infty\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, +\infty\right).$$

Case 4. $c > 2\sqrt{a}$, $(A_1 < A_2 < 0)$.

4.1. *If* $\beta > -2\sqrt{a}$, $(\tilde{c} < A_2)$, *then there exists a smooth solution* $\eta(x) > 0$, *which is increasing on* $(-\infty,\, x_1(B)\,]$ *and decreasing on* $[\,x_1(B),\, +\infty\,)$, *attaining its maximum* $\max_{x\in\mathbb{R}} \eta(x) = \eta_1(B)$ *at* $x_1(B)$.

4.2. *If* $-c < \beta < -2\sqrt{a}$, $(A_2 < \tilde{c} < 0)$, *then there is a solution* $\eta(x) > 0$, *which is smooth except at the jump discontinuity* $\bar{x}$. *Moreover, we have the following:*

(a) *If* $-c < \beta < -(c^2+4a)/(2c)$, *i.e.,* $A_2 < B < \tilde{c} < 0$, *then* $\eta(x)$ *is increasing on* $(-\infty,\, \bar{x}\,]$ *and decreasing on* $[\,\bar{x},\, +\infty\,)$, *with*

$$\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^+(\tilde{c}),\, +\infty\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta^-(\tilde{c}),\, -\infty\right).$$

(b) *If* $\beta = -(c^2+4a)/(2c)$, *i.e.,* $A_2 < B = \tilde{c} < 0$, *then* $\eta(x)$ *is increasing on* $(-\infty,\, \bar{x}\,]$ *and decreasing on* $[\,\bar{x},\, +\infty\,)$, *with*

$$\lim_{x\uparrow \bar{x}} (\eta(x),\, \eta_x(x)) = \left(\eta_1(B),\, \tfrac{1}{2}c^2\right), \qquad \lim_{x\downarrow \bar{x}} (\eta(x),\, \eta_x(x)) = (\eta_2(B),\, -\infty)\,.$$

(c) *If* $-(c^2+4a)/(2c) < \beta < -2\sqrt{a}$, *i.e.,* $A_2 < \tilde{c} < B < 0$, *then* $\eta(x)$ *is increasing on* $(-\infty,\, x_1(B)\,]$ *and decreasing on* $[\,x_1(B),\, +\infty\,)$, *satisfying*

$$\lim_{x\to x_1(B)} (\eta(x),\, \eta_x(x)) = (\eta_1(B),\, 0)$$

and

$$\lim_{x\uparrow\bar{x}} (\eta(x),\, \eta_x(x)) = \big(\eta^+(\tilde{c}),\, -\infty\big)\,, \qquad \lim_{x\downarrow\bar{x}} (\eta(x),\, \eta_x(x)) = \big(\eta^-(\tilde{c}),\, -\infty\big)\,.$$

4.3. *If $\beta = -2\sqrt{a}$, $(A_2 = \tilde{c} < 0)$, then $\eta(x)$ is smooth for all $x \in \mathbb{R}$ with $x \neq x(A_2)$. Moreover, $\eta(x)$ is increasing on $(-\infty,\, x_1(B)\,]$ and decreasing on $[\,x_1(B),\, +\infty\,)$, satisfying*

$$\lim_{x\to x_1(B)} (\eta(x),\, \eta_x(x)) = (\eta_1(B),\, 0)\,, \qquad \lim_{x\to x(A_2)} (\eta(x),\, \eta_x(x)) = (\eta(A_2),\, -\infty)\,,$$

where $\eta(A_2) = -\sqrt{a}A_2 = (c - 2\sqrt{a})\sqrt{a}$.

4.4. *If $\beta < -c$, $(\tilde{c} > 0)$, then, firstly, there exists a smooth solution $\eta(x) > 0$ corresponding to the smooth solitary wave $\phi(x)$, and it is increasing on $(-\infty,\, x_2(B)\,]$ and decreasing on $[\,x_2(B),\, +\infty\,)$, attaining its maximum $\max_{x\in\mathbb{R}} \eta(x) = \eta_1(B)$ at $x_2(B)$; Secondly, corresponding to the cusped solitary wave, there is a solution $\eta(x) < 0$, which has a jump discontinuity at $\bar{x}$, more precisely, it is decreasing on $(-\infty,\, \bar{x}\,]$ and increasing on $[\,\bar{x},\, +\infty\,)$, with*

$$\lim_{x\uparrow\bar{x}} (\eta(x),\, \eta_x(x)) = \big(\eta^-(\tilde{c}),\, -\infty\big)\,, \qquad \lim_{x\downarrow\bar{x}} (\eta(x),\, \eta_x(x)) = \big(\eta^+(\tilde{c}),\, +\infty\big)\,.$$

Proof. **Case I.** $0 < A_1 < A_2$.

When $0 < A_1 < \tilde{c}$, according to the monotonicity of ϕ and using the fact that in the present case $0 < \phi \le A_1 < \tilde{c}$, we conclude that

$$(\,\eta(x),\ \eta_x(x)\,) = \begin{cases} \big(\,\eta^-(x),\ \eta_x^-(x)\,\big)\,, & -\infty < x \le x(A_1) \\ \big(\,\eta^+(x),\ \eta_x^+(x)\,\big)\,, & x(A_1) < x < +\infty, \end{cases}$$

which immediately implies that both $\eta(x)$ and $\eta_x(x)$ decay to zero at infinity. Furthermore, note that $I(B) = J(B) < 0$ in this situation, which, when combined with the relation $0 < B < A_1$, implies that

$$\lim_{x\to x_1(B)} \eta_x(x) = \eta_x^-(B) = 0, \qquad \lim_{x\to x_1(B)} \eta(x) = \eta^-(B) = \frac{c^2 - 4a}{4}.$$

As $-\infty < x \le x_1(B)$, $0 < \phi \le B$, we have $|I(\phi)| > |J(\phi)|$. Furthermore, using the inequality

$$I(\phi) < \frac{4a}{c}\sqrt{(\phi - A_1)(\phi - A_2)} < 0,$$

one may show that, no matter whether $J(\phi) < 0$ or $J(\phi) > 0$, the function $\eta_x^-(x)$ will be positive everywhere, and so $\eta(x)$ is strictly increasing on this interval. As $x_1(B) < x < x(A_1)$, $0 < C_1 < B < \phi < A_1$, we thus conclude that $|I(\phi)| < |J(\phi)|$ and $J(\phi) < 0$. Therefore, $\eta_x^-(x) < 0$, meaning that $\eta(x)$ is decreasing on $[\,x_1(B),\, x(A_1)\,]$. Next, by a similar analysis as above, one may show that $\eta_x^+(x) < 0$ for $x \in (A_1,\, +\infty)$, and then $\eta(x)$ will continue to decrease on $[\,x(A_1),\, +\infty\,)$. Finally, we deduce that both $\eta^-(x)$ and $\eta^+(x)$ are positive in their respective interval, resulting from the fact that $|\phi + c| > \sqrt{(\phi - A_1)(\phi - A_2)}$ and $\phi + c \le c + A_1 = -2\sqrt{a} < 0$.

When $0 < \tilde{c} < A_1$, as we have seen in the above discussion,

$$(\,\eta(x),\ \eta_x(x)\,) = \begin{cases} \big(\,\eta^-(x),\ \eta_x^-(x)\,\big)\,, & -\infty < x \le \bar{x}, \\ \big(\,\eta^+(x),\ \eta_x^+(x)\,\big)\,, & \bar{x} < x < +\infty. \end{cases} \tag{5.22}$$

Therefore, both $\eta(x)$ and $\eta_x(x)$ decay at infinity. To analyze its monotonicity, we need to distinguish three subcases:

(1). If $0 < \tilde{c} < B < A_1$, one may use an argument similar to that was used in the case when $0 < A_1 < \tilde{c}$ to verify that $\eta(x) > 0$ for all $x \in \mathbb{R}$ and $\eta_x^-(x) > 0$ on $(\infty,\, \bar{x}\,]$, $\eta_x^+(x) < 0$ on $[\,\bar{x},\, +\infty\,)$, and then (5.19) follows. Furthermore, it is easy to see that $\eta^-(\tilde{c}) > \eta^+(\tilde{c})$, meaning that the singular point $\tilde{c}$ will give rise to the jump discontinuity $\bar{x}$ for $\eta(x)$.

(2). If $0 < \tilde{c} = B < A_1$, then $\eta(x) > 0$ for all $x \in \mathbb{R}$ and has the same monotonicity as in the previous subcase. The limiting behavior at the point $\bar{x}$ in formulae (5.20) follows directly from (5.22).

(3). If $0 < B < \tilde{c} < A_1$, note that $I(B) = J(B)$ and using the fact that $|I(\phi)| > |J(\phi)|$when $0 < \phi < B$, and $|I(\phi)| < |J(\phi)|$ when $\phi > B$, one can directly show that

$$\eta_x(x) = \begin{cases} \eta_x^-(x) > 0, & -\infty < x < x_1(B), \\ \eta_x^-(B) = 0, & x = x_1(B), \\ \eta_x^-(x) < 0, & x_1(B) < x < \bar{x}, \\ \eta_x^+(x) < 0, & \bar{x} < x < +\infty, \end{cases} \tag{5.23}$$

and then the asympotic behavior given in (5.21) follows.

When $0 < B < \tilde{c} = A_1$, one may show that (5.23) is still valid, meaning that $\eta(x)$ will have the same monotonicity as in Case 1.2 (c). However, in contrast to the previous subcase, although the first-order derivative η_x goes to negative infinity as x approaches to $\bar{x}$ from both left- and right-hand sides, since $\lim_{x\uparrow\bar{x}} \eta^-(x) = \lim_{x\downarrow\bar{x}} \eta^+(x) = \sqrt{a}A_1$, $\eta(x)$ is continuous at $\bar{x}$.

When $\tilde{c} < 0$, there are two kinds of solitary waves. For the smooth solitary wave ϕ, which lies in $(0, A_1]$, utilizing its monotonicity and the fact that it decays exponentially to zero at infinity again, we can conclude that the corresponding solution $\eta(x)$ and its first-order derivative $\eta_x(x)$ is smooth for all $x \in \mathbb{R}$ and satisfies

$$\eta(x) = \begin{cases} \eta^+(x) > 0, & -\infty < x \le x(A_1), \\ \eta^-(x) > 0, & x(A_1) < x < +\infty, \end{cases} \qquad \eta_x(x) = \begin{cases} \eta_x^+(x) > 0, & -\infty < x \le x(A_1), \\ \eta_x^-(x) > 0, & x(A_1) < x < x_2(B), \\ \eta_x^-(B) = 0, & x = x_2(B), \\ \eta_x^-(x) < 0, & x_2(B) < x < +\infty. \end{cases}$$

On the other hand, for the anticusped solitary wave $\phi(x)$ which lies in $(\tilde{c}, 0)$, one may show that $\eta(x)$ and $\eta_x(x)$ are determined by

$$\eta(x) = \begin{cases} \eta^+(x) < 0, & -\infty < x \le \bar{x}, \\ \eta^-(x) < 0, & \bar{x} < x < +\infty, \end{cases} \qquad \eta_x(x) = \begin{cases} \eta_x^+(x) < 0, & -\infty < x \le \bar{x}, \\ \eta_x^-(x) > 0, & \bar{x} < x < +\infty. \end{cases}$$

Applying the same analysis as used in Case 1.1 and Case 1.2 (b) to the present two subcases respectively, yields all the corresponding conclusions.

Case II. When $A_1 = 0 < A_2$, as we have pointed out in Theorem 4.1, solitary wave solution $\phi(x)$ exists if and only if $\tilde{c} < 0$ and $\phi(x)$ turns out to be an anticusped with the singular point $\tilde{c}$. Hence, we may safely draw the conclusion that $\eta(x)$ in this case shall present the similar configuration as the anticusped solitary wave in Case 1.4. And then all the conclusions obtained from Case 1.4 remain valid in the present case.

Finally, the remaining two subcases may be analyzed in a similar way as has been done in Cases 2 and 1, respectively. In this manner, we have completed the proof of the theorem. □

Remark 5.1. *When $0 < \tilde{c} < A_1$, there exists a jump discontinuity occurring at the singular point $\bar{x}$, such that*

$$\lim_{x\downarrow\bar{x}} \eta(x) = \lim_{x\downarrow\bar{x}} \eta^+(x) = \eta^+(\tilde{c}) = -\frac{1}{2}(c+\beta)\left(\beta - \sqrt{\beta^2 - 4a}\right),$$
$$\lim_{x\uparrow\bar{x}} \eta(x) = \lim_{x\uparrow\bar{x}} \eta^-(x) = \eta^-(\tilde{c}) = -\frac{1}{2}(c+\beta)\left(\beta + \sqrt{\beta^2 - 4a}\right).$$

It is easy to show that $\eta^+(\tilde{c})$ is increasing with respect to $\tilde{c}$ as $0 < \tilde{c} \le A_1$, while $\eta^-(\tilde{c})$ is increasing with respect to $\tilde{c}$ as $0 < \tilde{c} < B$, and decreasing as $B < \tilde{c} < A_1$; its maximum is $\eta_1(B)$, attained when $\tilde{c} = B$.

Another fact worth mentioning is that $\eta^-(\tilde c) \geq \eta^+(\tilde c)$, and the difference $\Gamma(\tilde c) = \eta^-(\tilde c) - \eta^+(\tilde c)$ increases as $\tilde c$ goes from 0 to C_1, and decreases as $\tilde c$ goes from C_1 to A_1, until $\Gamma(\tilde c)$ vanishes and $\eta^-(\tilde c) = \eta^+(\tilde c) = \eta(A_1)$ occurs when $\tilde c = A_1$. In this situation, as we have mentioned in Theorem 5.3, $\eta(x)$ is smooth for all $x \in \mathbb{R}$.

Figure 1 illustrates the different types of discontinuous non-analytic solitary wave solutions $\eta(\cdot)$ constructed in the Case 1.2 of Theorem 5.3 and a special cusped solitary wave solution described in Case 1.3.

In what follows we investigate the properties of $\eta(x)$ when $a = 0$, so $A_1 = A_2 = -c,\ B = -c/2$. Formula (5.18) becomes

$$\phi_x = \pm\frac{\phi(\phi - A_1)}{\phi - \tilde c}. \tag{5.24}$$

Consequently, $\eta(x)$ and $\eta_x(x)$ can be written in terms of ϕ by

$$\eta(x) = -\tfrac{1}{2}\phi(\phi - A_1)(1 \pm 1), \qquad \eta_x(x) = -\frac{\phi(2\phi + c)(\phi - A_1)(1 \pm 1)}{2(\phi - \tilde c)},$$

respectively. Obviously, $\eta(x)$ and $\eta_x(x)$ will vanish when ϕ satisfies (5.24) in the "minus" sign case. The corresponding results are given as follows.

Theorem 5.4. *When $a = 0$, we have*
Case 1. $0 < A_1 = A_2$.

1.1. *If $\tilde c > A_1$, then there is a smooth solution $\eta(x) > 0$, which is increasing on $(-\infty,\, x_1(B)\,]$ and decreasing on $[\,x_1(B),\, +\infty\,)$. It attains its maximum* $\max_{x\in\mathbb{R}} \eta(x) = c^2/4$ *at $x_1(B)$ and satisfies*

$$\lim_{x\to\pm\infty} (\eta(x),\, \eta_x(x)) = (0,\, 0)\,.$$

1.2. *If $\tilde c = A_1$, then there is a continous solution*

$$\eta(x) = \begin{cases} \phi(A_1 - \phi), & -\infty < x \leq \bar x \\ 0, & \bar x < x < +\infty. \end{cases}$$

$\eta(x)$ is increasing on $(-\infty,\, x_1(B)\,]$ and decreasing on $[\,x_1(B),\, \bar x\,]$, attaining its maximum $\max_{x\in\mathbb{R}} \eta(x) = c^2/4$ *at $x_1(B)$. Moreover,*

$$\lim_{x\to-\infty} (\eta(x),\, \eta_x(x)) = (0,\, 0)\,, \qquad \lim_{x\uparrow\bar x} (\eta(x),\, \eta_x(x)) = \left(0,\, -c^2\right).$$

1.3. *If $0 < \tilde c < A_1$, then there is a solution*

$$\eta(x) = \begin{cases} \phi(A_1 - \phi), & -\infty < x < \bar x \\ 0, & \bar x < x < +\infty, \end{cases}$$

which is increasing on $(-\infty,\, \bar x\,]$, and satisfies

$$\lim_{x\uparrow\bar x} (\eta(x),\, \eta_x(x)) = (-\beta(c + \beta),\, +\infty)\,, \qquad \lim_{x\to-\infty} (\eta(x),\, \eta_x(x)) = (0,\, 0)\,.$$

1.4. *If $\tilde c < 0$, then there is a trivial solution $\eta(x) = 0$ corresponding to the kink solution $\phi(x)$ in the present subcase, while, corresponding to the anticusped solitary wave $\phi(x)$, the solution $\eta(x)$ is governed by*

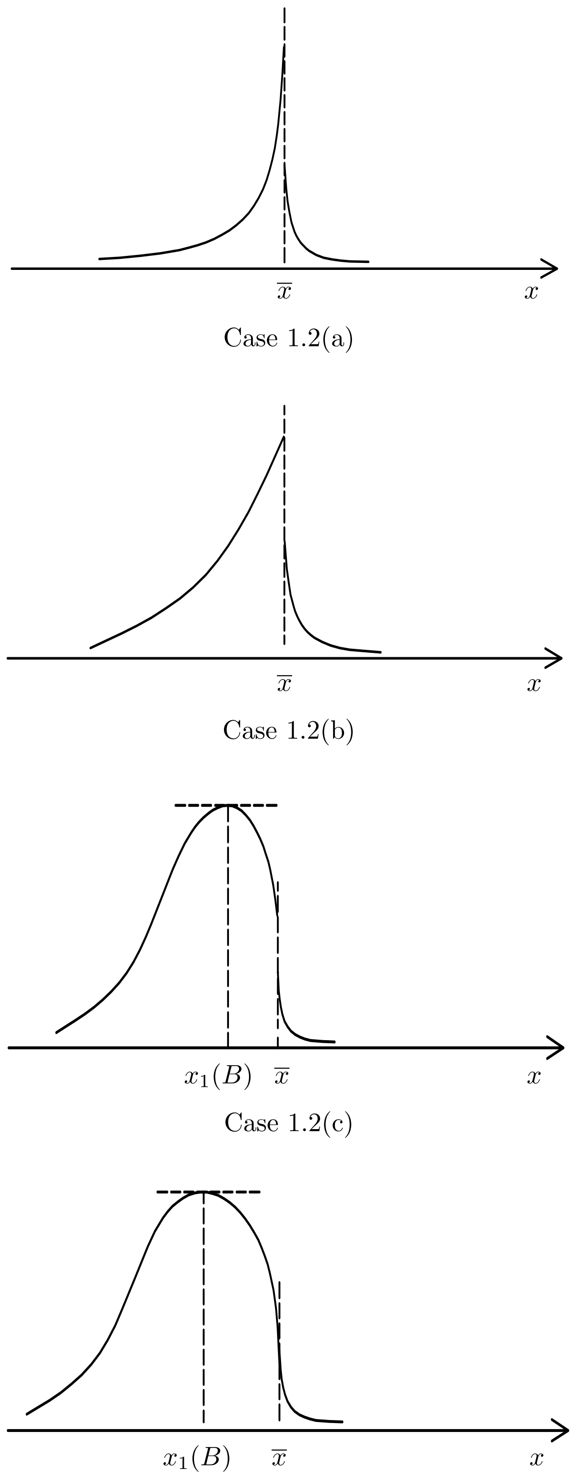
$$\eta(x) = \begin{cases} 0, & -\infty < x \leq \bar x \\ -\phi(\phi - A_1), & \bar x < x < +\infty, \end{cases}$$

and satisfies

$$\lim_{x\downarrow\bar x} (\eta(x),\, \eta_x(x)) = (-\beta(c + \beta),\, +\infty)\,, \qquad \lim_{x\to+\infty} (\eta(x),\, \eta_x(x)) = (0,\, 0)\,.$$

Case 2. $A_1 = A_2 < 0$.

For $\tilde c < A_1$, $\tilde c = A_1$, $A_1 < \tilde c < 0$ and $\tilde c > 0$, their corresponding solutions $\eta(x)$ have the same properties as the solutions in Cases 1.1, 1.2, 1.3 and 1.4, respectively.



Case 1.3

FIGURE 1. Discontinuous solitary waves and a special "cusped" solitary wave

Finally, to understand the effect of the nonlinear dispersive terms, it is worth comparing the properties of the traveling wave solutions of two systems (3.9) and (3.12). As far as system (3.9) is concerned, we consider $\alpha = 0$ and $\beta = 1$. We assume that system (3.9) has the solitary wave solution

$$(q(t,x),\, r(t,x)) = (\psi(x-c\,t),\, \phi(x-c\,t))\,, \quad c \in \mathbb{R},$$

such that $\phi \to 0$, $\psi \to a\,(a \geq 0)$ as $|x| \to \infty$. Replacing ψ by $\psi + a$, we arrive at the following system of ordinary differential equations

$$\begin{cases} c\psi' + 2a\phi' + 2(\phi\psi)' - \psi'' = 0, \\ c\phi' + 2\psi' + 2\phi\phi' + \phi'' = 0. \end{cases}$$

We thus deduce that $\phi(x)$ is governed by the equation

$$\phi_x^2 = \phi^2(\phi - A_1)(\phi - A_2),$$

with $A_1 = -c - 2\sqrt{a},\ A_2 = -c + 2\sqrt{a}$, from which one may classify the traveling waves as follows. If $a > 0$, then there exists a smooth solitary wave solution

$$\phi = \begin{cases} \dfrac{A_1A_2}{A_2\cosh^2\left(-\frac{1}{2}\sqrt{A_1A_2}|\bar{x}-x|\right) - A_1\sinh^2\left(-\frac{1}{2}\sqrt{A_1A_2}|\bar{x}-x|\right)}, & 0 < A_1 < A_2, \\ \dfrac{-A_1A_2}{A_2\sinh^2\left(-\frac{1}{2}\sqrt{A_1A_2}|\bar{x}-x|\right) - A_1\cosh^2\left(-\frac{1}{2}\sqrt{A_1A_2}|\bar{x}-x|\right)}, & A_1 < A_2 < 0. \end{cases}$$

If $a = 0$, then $A_1 = A_2 = -c$, and system (3.9) has a kink solution

$$\phi = \frac{-ce^{|c|x}}{1+e^{|c|x}}.$$

Acknowledgements. The work of J. Kang is supported by NSF-China grant-11631007 and grant-11471260. The work of X.C. Liu is supported by NSF-China grant-11631007, grant-11401471 and Ph.D. Programs Foundation of Ministry of Education of China grant-20136101120017. The work of P.J. Olver is partially supported by NSF grant DMS-1108894. The work of C.Z. Qu is supported by the NSF-China grant-11631007 and grant-11471174.

References

[1] M. Antonowicz and A.P. Fordy, Coupled KdV equations with multi-Hamiltonian structures, *Physica D*, **28** (1987), 345-357.
[2] M. Antonowicz and A.P. Fordy, Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems, *Commun. Math. Phys.*, **124** (1989), 465-486.
[3] R. Beals, D.H. Sattinger, and J. Szmigielski, Multipeakons and the classical moment problem, *Adv. Math.*, **154** (2000), 229-257.
[4] L.J.F. Broer, Approximate equations for long water waves, *Appl. Sci. Res.*, **31** (1975), 377-395.
[5] R. Camassa and D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661-1664.
[6] R. Camassa, D.D. Holm, and J. Hyman, An new integrable shallow water equation, *Adv. Appl. Mech.*, **31** (1994), 1-33.
[7] M. Chen, S.Q. Liu, and Y.J. Zhang, A two-component generalization of the Camassa-Holm equation and its solutions, *Lett. Math. Phys.*, **75** (2006), 1-15.

[8] Robin M. Chen, Y. Liu, and Q. Qiao, Stability of solitary waves and global existence of a generalized two-componen Camassa-Holm system, *Commun. Partial Differential Equations*, **36** (2011), 2162-2188.
[9] K.S. Chou and C.Z. Qu, Integrable equations arising from motions of plane curves, *Physica D*, **162** (2002), 9-33.
[10] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321-362.
[11] A. Constantin, V. Gerdjikov, and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse Problems*, **22** (2006), 2197-2207.
[12] A. Constantin and R. Ivanov, On an integrable two-component Camassa-Holm shallow water system, *Phys. Lett. A*, **372** (2008), 7129-7132.
[13] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rational Mech. Anal.*, **192** (2009), 165–186.
[14] J. Escher, M. Kohlmann, and J. Lenells, The geometry of the two-component Camassa-Holm and Degasperis-Procesi equations, *J. Geometry and Phys.*, **61** (2011), 436-452.
[15] A.S. Fokas, The Korteweg-de Vries equation and beyond, *Acta Appl. Math.*, **39** (1995), 295-305.
[16] A.S. Fokas, On a class of physically important integrable equations, *Physica D*, **87** (1995), 145-150.
[17] A.S. Fokas and B. Fuchssteiner, Bäcklund transformations for hereditary symmetries, *Nonlinear Anal.*, **5** (1981), 423-432.
[18] A.S. Fokas, P.J. Olver, and P. Rosenau, A plethora of integrable bi-Hamiltonian equations, in: *Algebraic aspects of integrable systems*, Progr. Nonlinear Differential Equations Appl., vol. 26, Birkhäuser Boston, Boston, MA, 1997, pp. 93-101.
[19] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D*, **95** (1996), 229-243.
[20] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4** (1981/1982), 47-66.
[21] G.L. Gui and Y. Liu, On the Cauchy problem for the two-component Camassa-Holm system, *Math. Z.*, **268** (2010), 45-66.
[22] G.L. Gui, Y. Liu, P.J. Olver, and C.Z. Qu, Wave-breaking and peakons for a modified Camassa-Holm equation, *Commun. Math. Phys.*, **319** (2013), 731-759.
[23] P. Guha and P.J. Olver, Geodesic flow and two (super) component analog of the Camassa-Holm equation, *Symmetry, Integrability and Geometry: Methods and Appl.*, **2** (2006), 054, 9 pages.
[24] Y.W. Han, F. Guo, and H.J. Gao, On solitary waves and wave-breaking phenomena for a generalized two-component integrable Dullin-Gottwald-Holm system, *J. Nonlinear Sci.*, **23** (2013), 617-656.
[25] D.D. Holm and R. Ivanov, Two-component CH system: inverse scattering, peakons and geometry, *Inverse Problems*, **27** (2011), 045013, 19pp.
[26] M. Ito, Symmetries and conservation laws of a coupled nonlinear wave equation, *Phys. Lett. A*, **91** (1982), 335-338.
[27] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, **455** (2002), 63-82.
[28] J. Kang, X.C. Liu, P.J. Olver, and C.Z. Qu, Liouville correspondence between the modified KdV hiearchy and its dual integrable hierarchy, *J. Nonlinear Sci.*, **26** (2016), 141-170.
[29] S. Kichenassamy and P.J. Olver, Existence and non–existence of solitary wave solutions to higher order model evolution equations, *SIAM J. Math. Anal.*, **23** (1992) 1141-1166.
[30] S. Kouranbaeva, The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.*, **40** (1999), 857-868.
[31] B. Kupershmidt, Mathematics of dispersive water waves, *Commun. Math. Phys.*, **99** (1985), 51-73.

[32] J. Lenells, Traveling wave solutions of the Camassa-Holm equation, *J. Differential Equations*, **217** (2005), 393-430.
[33] Y.A. Li and P.J. Olver, Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system, Part I. Compactons and peakons, *Discrete Cont. Dyn. Syst.*, **3** (1997), 419-432.
[34] Y.A. Li and P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Differential Equations*, **162** (2000), 27-63.
[35] Y.A. Li, P.J. Olver, and P. Rosenau, Non-analytic solutions of nonlinear wave models, in: *Nonlinear Theory of Generalized Functions*, M. Grosser, G. Höormann, M. Kunzinger, and M. Oberguggenberger, eds., Research Notes in Mathematics, vol. 401, Chapman and Hall/CRC, New York, 1999, pp. 129-145.
[36] Y. Liu, P.J. Olver, C.Z. Qu, and S.H. Zhang, On the blow-up of solutions to the integrable modified Camassa-Holm equation, *Anal. Appl.* (Singap.), **12** (2014), 355-368.
[37] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.*, **19** (1978), 1156-1162.
[38] P.J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.*, **18** (1977), 1212-1215.
[39] P.J. Olver, On the Hamiltonian structure of evolution equations, *Math. Proc. Cambridge Philos. Soc.*, **88** (1980), 71-88.
[40] P.J. Olver, Hamiltonian perturbation theory and water waves, *Fluids and plasmas: geometry and dynamics (Boulder, Colo., 1983)*, 231-249, Contemp. Math., **28**, Amer. Math. Soc., Providence, RI, 1984.
[41] P.J. Olver, Applications of Lie Groups to Differential Equations, Second edition, *Graduate Texts in Mathematics*, **107**, Springer-Verlag, New York, 1993.
[42] P.J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E*, **53** (1996), 1900-1906.
[43] C.Z. Qu, J.F. Song, and R.X. Yao, Multi-component integrable systems and invariant curve flows in certain geometries, *Symmetry, Integrability and Geometry: Methods and Appl.*, **9** (2013), 001, 19 pages.
[44] C. Rogers and W.F. Shadwick, Bäcklund Transformations and Their Applications, Academic Press, New York, 1982.
[45] J. Schiff, Zero curvature formulations of dual hierarchies, *J. Math. Phys.*, **37** (1996), 1928-1938.

Jing Kang
Center for Nonlinear Studies and School of Mathematics, Northwest University, Xi'an 710069, P. R. China
E-mail address: `jingkang@nwu.edu.cn`

Xiaochuan Liu
Center for Nonlinear Studies and School of Mathematics, Northwest University, Xi'an 710069, P. R. China
E-mail address: `liuxc@nwu.edu.cn`

Peter J. Olver
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: `olver@umn.edu`

Changzheng Qu
Center for Nonlinear Studies, Ningbo University, Ningbo 315211, P. R. China
E-mail address: `quchangzheng@nbu.edu.cn`