Non-Analytic Solutions of Nonlinear Wave Models

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Abstract. This paper surveys recent work of the coauthors on nonanalytic solutions to nonlinear wave models. We demonstrate the connection between nonlinear dispersion and the existence of a remarkable variety of nonclassical solutions, including peakons, compactons, cuspons, and others. Nonanalyticity can only occur at points of genuine nonlinearity, where the symbol of the partial differential equation degenerates, and thereby provide singularities in the associated dynamical system for traveling waves. We propose the term "pseudo-classical" to characterize such solutions, and indicate how they are recovered as limits of classical analytic solutions.

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Historically, the mathematical study of physical phenomena has proceeded in three phases. The introductory phase relies on linear models, and is a necessary prerequisite to the deeper understanding of the actual nonlinear regime. Thus, for instance, the unidirectional linear wave equation $u_t + u_x = 0$ provides the prototype of linear hyperbolic waves. Any initial data propagates completely unchanged with speed $c = 1$. (For simplicity, we adopt units in which all physical parameters are 1 throughout this survey.) Restricting our attention to conservative systems, the next interesting case is the dispersive wave equation $u_t + u_x + u_{xxx} = 0$. Since different wave numbers propagate at different speeds, localized initial data will disperse in time, albeit with conservation of energy. In certain physical regimes, these simple linear models provide the satisfactory explanations of the physically observed wave propagation.

Inspired by the dramatic observations of J. Scott Russell, [26], Boussinesq initiated a systematic derivation of nonlinear wave models that could describe the weakly nonlinear regime, which is the second of the three historical phases. The most celebrated of Boussinesq's models is the equation

$$u_t + u_x + (u^2)_x + u_{xxx} = 0, \qquad (1.1)$$

which initially appears in both [4; eq. (30), p. 77] and [5; eqs. (283, 291)]. This well-studied equation, now known as Korteweg-de Vries (KdV) equation in honor of its rediscovery, [14], over two decades later, models the weakly nonlinear regime in which nonlinear effects are balanced by dispersion. As recognized by Boussinesq, its solitary wave solutions are described by analytic sech^2 functions — that have exponentially decaying tails. The discovery of the inverse scattering granted to the KdV a status of a paradigm, for it was the first equation shown to be completely integrable, and so a complete analysis can be effected, cf. [9]. Its solitary waves are nowadays referred to as "solitons" since they interact cleanly, except for a phase shift. It admits two distinct, compatible Hamiltonian structures, and hence an infinite hierarchy of local higher order symmetries and associated conservation laws, cf. [20]. Moreover, it is now understood that the KdV and its integrable cousins are not just esoteric equations describing particular phenomena, but ubiquitous descriptions of a weakly nonlinear regime wherein linear dispersion counterbalances the governing nonlinearity to form steadily propagating patterns.

Numerical studies indicate that much of the basic weakly nonlinear phenomenology associated with the KdV and other integrable models is shared with many nonintegrable models in the same physical regime. A good example is the BBM or regularized long wave model, [2],

$$u_t + u_x + uu_x - u_{xxt} = 0. \qquad (1.2)$$

Like the KdV, (1.2) also admits analytic solitary wave solutions, and localized initial data is observed to break up into a finite collection of solitary waves, [12]. Unlike the KdV, (1.2) has solitary waves of both positive and negative elevation; the positive waves interact almost cleanly — there is a very small but noticeable dispersive effect, [3, 19]. (On the other hand, collisions between positive and negative waves are quite dramatically inelastic, [25].) We are still a long way from being able to establish rigorously these numerical observations yet, one might venture that the basic phenomenology of wave mechanics in the weakly nonlinear regime is fairly well understood.

The final, and most difficult, phase in the modeling process is the transition to fully nonlinear models. Linear and weakly nonlinear models will only predict the physical behavior of weak disturbances — indeed, they are typically derived by rescaling the full physical system to include one or more small parameters, expanding in a power series, then truncating to eliminate some or all of the higher order terms. However, in a genuinely nonlinear regime, nonlinearity plays a dominant role rather than being a higher order correction, and so will introduce new complications, new phenomena, and require new analytical techniques. Weakly nonlinear models cannot describe the fully nonlinear, real world phenomena such as wave breaking, shocks, waves of maximal height, [**27**], large amplitude disturbances, and so on. The problem of how one derives realistic nonlinear models for such highly nonlinear effects is of fundamental importance but has not as yet received the attention it deserves.

One type of fully nonlinear phenomena — shock waves in gas dynamics and similar first order hyperbolic systems — already has a long history. The simplest case is the inviscid Burgers' — or dispersionless KdV — or Riemann equation $u_t + (u^2)_x = 0$, obtained by replacing the linear convective term in the unidirectional linear model by a fully nonlinear version. The hodograph transformation $(x, t, u) \mapsto (u, t, x)$ will linearize the equation, but the analytic solutions to the transformed equation revert to multiply-valued solutions of the Riemann equation. For the gas dynamics and continuum mechanics applications, a physical wave cannot be represented by a multiply-valued solution, and one must replace it with a discontinuous, nonanalytic shock wave, the placement of the shock governed by a suitable entropy condition, cf. [**29**]. On the other hand, if one uses this equation to model surface waves, then multiply-valued solutions are physically acceptable — the nonphysical solutions are when the curve crosses itself — and the weak characterization is no longer applicable. Therefore, *the analytical characterization of acceptable solutions depends on the physics underlying the model*.

Just as linear dispersion dramatically alters the hyperbolic traveling wave, so one expects nonlinear dispersion to be capable of causing deep qualitative changes in the nature of dispersive phenomena. Mimicking the passage from the linear hyperbolic equation to the nonlinear Riemann equation, one is naturally led to introduce a simple nonlinearly dispersive model of the form

$$u_t \pm (u^2)_x + (u^2)_{xxx} = 0, \qquad (1.3\pm)$$

known as the $K(2, 2)$ equation. The $\pm$ sign plays an important role in the structure, and we shall refer to (1.3+) as the focusing model, while (1.3−) is the defocusing version. Unlike the KdV model, (1.1), the $K(2, 2)$ equation will model dispersive waves in the fully nonlinear regime. It is one of a large class of simple nonlinearly dispersive wave models, incorporating different powers of u in the nonlinear terms, first proposed in [**22**, **24**]. We shall use (1.3) as our initial paradigm for the structures that may appear in the fully nonlinear regime, and then indicate additional, stranger denizens, of the nonlinear *terra incognita*.

At this point, it is important to distinguish between a fully nonlinear model, in which the highest order terms in the partial differential equation are nonlinear (typically quasilinear), and the points in phase space where the nonlinearity is actually manifested. These

points will be determined by the complete degeneration of the symbol of the partial differential equation. For example, the fully nonlinear $K(2,2)$ equation has degenerate symbol when $u = 0$, and these will be the points of genuine nonlinearity in this case.

A crucial difference between the linear or weakly nonlinear models, versus the fully nonlinear models like (1.3) is that the latter admit *nonanalytic solutions*. However, unlike the abrupt discontinuities of shock waves in the first order hyperbolic regime, these discontinuities are more subtle, typically occurring at higher order derivatives; moreover, standard existence theorems, e.g., Cauchy–Kovalevskaya, indicate that one can only expect such singularities at the points of genuine nonlinearity in the phase space. This is in contrast to the nonanalytic shock solutions of nonlinear hyperbolic models, which can occur at any point in the phase space, but whose implementation requires additional physical assumptions, e.g., entropy conditions. The first observed manifestations of nonanalyticity were compactons — solitary wave solutions of compact support, [**24**], — and peakons — solitary wave solutions having peaks or corners, [**6**]. Both types of solutions appear in the basic $K(2,2)$ equation, as well as other nonlinear models of this type.

How does one characterize such nonclassical solutions? Motivated by the entropy characterization of shock waves, the first idea that might come to mind is to view them as weak solutions in the traditional sense, [**29**]. Although this is a valid formulation, and typically does serve to distinguish them, both compactons and peakons are solutions in a considerably stronger (although not quite classical) sense. In fact, they are piecewise analytic, satisfying the equation in a classical sense away from the points of genuine nonlinearity; moreover, each term (or, possibly, certain combinations of terms) in the equation is well-defined at the singularities. They remain solutions even when subjected to nonlinear changes of variables — a process that has disastrous consequences for the weak formulation. Consequently, one does not need to introduce an artificial (or physical) entropy condition to prescribe the type of singularity. In [**17**, **18**], we proposed the term *pseudo-classical solution* to distinguish such non-analytic solutions.

Yet another characterization is based on the observation that these solutions can be realized as the limits of classical, analytic solutions. Note that (1.3) is genuinely nonlinear when the undisturbed (asymptotic) height of the solution is $\lim_{|x| \to \infty} u = 0$, but becomes weakly nonlinear at nonzero amplitudes. Therefore, the solutions with $\lim_{|x| \to \infty} u = \alpha \neq 0$ are analytic, and, in favorable situations, will converge to a solution as $\alpha \to 0$. Alternatively, one can replace u by $u + \alpha$ in (1.3) — the parameter α then measures linear dispersion. The convergence of analytic solitary wave solutions under vanishing linear dispersion or asymptotic height distinguishes those non-analytic solitary wave solutions which are genuine pseudo-classical solutions.

Since the analysis of general solutions is much too difficult, we shall now concentrate on the traveling wave solutions to the equations under consideration. Substituting the basic ansatz $u = \phi(x - ct)$ for a traveling wave of speed c reduces the partial differential equation in x, t to an ordinary differential equation, which can then be analyzed by dynamical systems methods. The classical, analytic traveling wave solutions are given by classical analytic orbits to the dynamical system: solitary wave solutions correspond to homoclinic orbits at a single equilibrium point; periodic waves come from periodic orbits; while heteroclinic orbits connecting two equilibrium points yield analytic kink solutions.

Away from singular points of the dynamical system one can apply standard existence and analytic dependence results to prove that the corresponding solutions are analytic. Therefore, nonanalytic behavior will only arise at the singular points, which are precisely the points of genuine nonlinearity of the original partial differential equation. The admission of nonanalytic traveling wave solutions indicates that classical dynamical systems analysis must be significantly broadened to explain how a classical analytic orbit can give rise to a nonanalytic solution!

In the case of the $K(2,2)$ model (1.3), the traveling wave solutions satisfy a three-dimensional dynamical system

$$-c\phi' \pm (\phi^2)' + (\phi^2)''' = 0. \tag{1.4$\pm$}$$

Each point $(\alpha, 0, 0)$ lying on the ϕ-axis in the (ϕ, ϕ', ϕ'')–phase space provides a fixed point for the dynamical system (1.4) representing a constant (undisturbed) solution of height α. The equation (1.4) can be integrated twice by standard methods, leading to

$$\phi^2(\phi')^2 = \mp\tfrac{1}{4}\phi^4 + \tfrac{1}{3}c\phi^3 + a\phi^2 + b \qquad \text{or} \qquad (\phi')^2 = \frac{b}{\phi^2} + a + \frac{c}{3}\phi \mp \frac{1}{4}\phi^2. \tag{1.5}$$

where a, b are the integration constants. If $b \neq 0$ then the potential on the right hand side of (1.5) has a pole of order 2 at $\phi = 0$; on the other hand, if $b = 0$ then the pole disappears — nevertheless, $\phi = 0$ remains a singular point of the original form of the equation, and one still anticipates that nonanalytic behavior can occur when an orbit crosses the singular axis in the phase plane. Thus, even when the singular pole is no longer present owing to a cancellation of factors, its nonanalytic legacy will remain. Note particularly that the singular points of the dynamical system arise from the points where the symbol of the original partial differential equation (1.3) degenerates.

For simplicity, we only consider the case $c > 0$ of positive wave speed. In the focusing case (1.4+), for each $0 < \alpha < \frac{c}{2}$, there is a homoclinic orbit at the fixed point $(\alpha, 0, 0)$ that represents an analytic solitary wave solution with asymptotic height α. Since each homoclinic orbit can be obtained as the limit of periodic orbits, the solitary wave solution is the limit of periodic traveling wave solutions as the period becomes infinite. When $\alpha = 0$, the origin is a singular point for the dynamical system (1.4), which implies that the associated homoclinic orbit is traversed in finite time. Therefore, the corresponding traveling wave solution is no longer analytic, but has compact support. In the present case, the dynamical system can be integrated explicitly, leading to the *compacton*

$$\phi_c(x - ct) = \begin{cases} \dfrac{4c}{3}\cos^2\dfrac{x - ct}{4}, & |x - ct| \leq 2\pi, \\ 0, & |x - ct| > 2\pi, \end{cases} \tag{1.6}$$

which can be realized as the limit of solitary waves solutions as their asymptotic height $\alpha \to 0$. (See [**17**, **18**] for the basic method of proof of such results.) In the defocusing case (1.4−), the fixed point $(\alpha, 0, 0)$ has a homoclinic orbit and corresponding analytic solitary wave solution only when $\alpha > -\frac{2c}{3}$. At the limiting point $\alpha = -\frac{2c}{3}$ the homoclinic orbit passes through the singularity, and the result is a nonanalytic traveling wave that has a

corner at its crest $\phi = 0$, known as a *peakon*, which can also be realized as a limit of analytic solitary wave solutions. There is also a closed-form formula for the peakon:

$$\phi_p(x - ct) = \frac{2c}{3}\left(e^{-\frac{|x-ct|}{2}} - 1\right). \tag{1.7}$$

The peakon solution corresponds to the particular case of (1.5) where the integration constant $b = 0$, and so the pole is canceled out, but, as discussed above, the $\phi = 0$ singularity remains and permits the non-smooth corner.
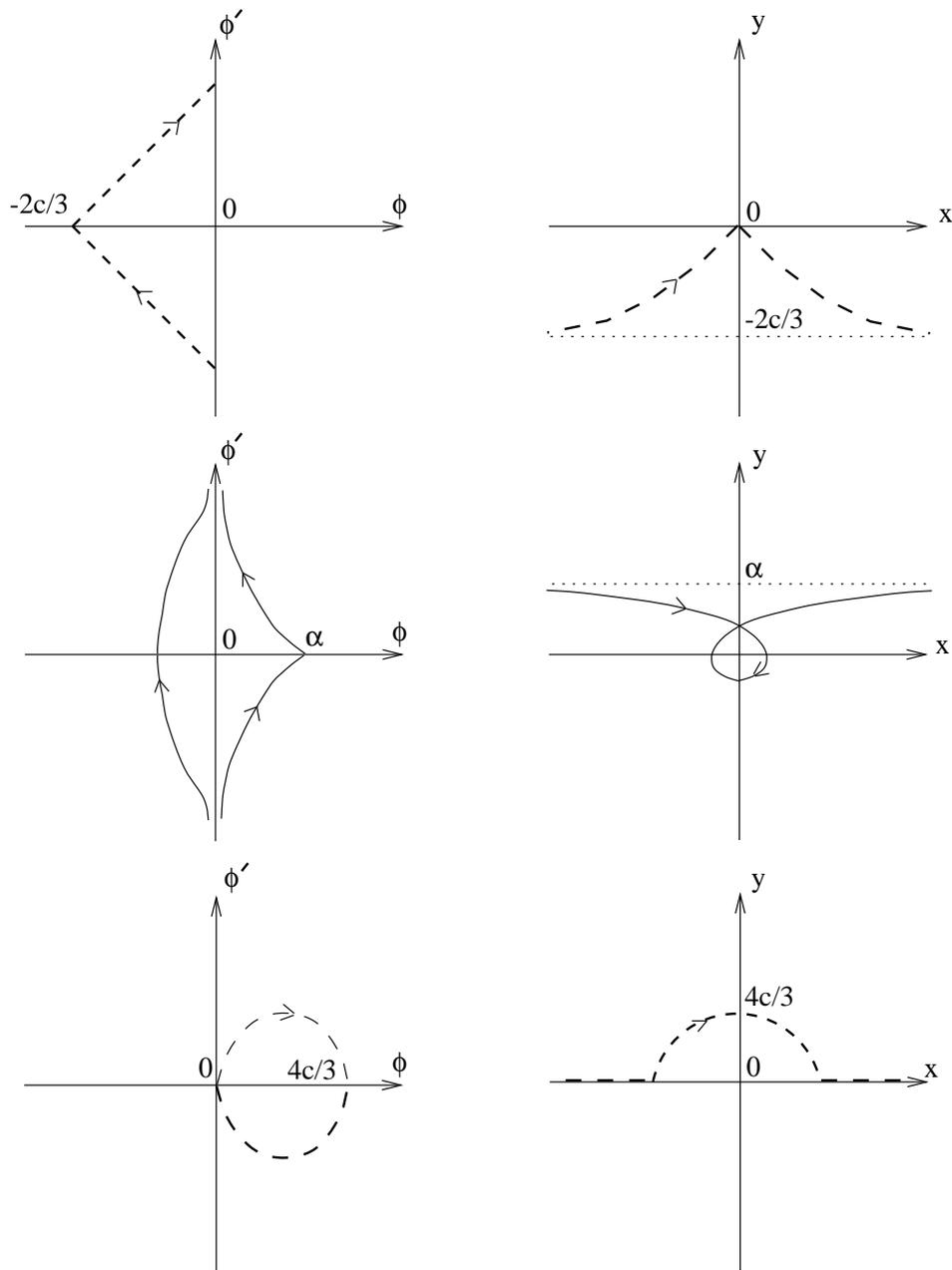

Fig. 1. *Trajectories and graphs of the peakon, loopon and compacton*

solutions of the $K(2,2)$ equation.

When $\alpha > 0$, the solutions that pass through the singular axis $\phi = 0$ take the form of multiply-valued looped solitary waves — *loopons*. Loopons first appeared in an integrable model of [**28**], which, in contrast to the $K(2,2)$ equation, required a changeable sign in the equation. One can justify such solutions either by skewing the coordinate axes, replacing x by $\tilde{x} = x + k\phi$, or by parametrizing the graph by arc length instead of the horizontal coordinate x. As with the Riemann equation, their admissibility as physical solutions depends on the physical system being modeled; unlike breaking surface waves, it seems unlikely that these could emerge from standard initial data. We shall defer the detailed analysis of these loopons until a subsequent publication.

In Figure 1, we display the phase-plane portrait and the corresponding solution for the peakon, compacton and loopon solutions. Note that the phase plane axes are labeled by ϕ, ϕ' and the curves represent the integral curves of the dynamical system (1.5) with appropriate integration constants a, b. The reader should compare these with the more standard phase portrait for the KdV equation — in that case, changing the integration constants merely translates the picture; in contrast, here the change of integration constants fundamentally alters the potential, allowing new families of solutions to emerge.

The $K(2,2)$ example already indicates the possibility of significantly more interesting phenomena in the fully nonlinear regime. An even richer example is provided by the $K(3,3)$ equation

$$u_t \pm (u^3)_x + (u^3)_{xxx} = 0. \tag{1.8$\pm$}$$

The resulting dynamical system for its traveling wave solutions $u = \phi(x - ct)$ is twice integrated, with constants a, b, yielding

$$(\phi')^2 = \frac{b}{\phi^4} + \frac{a}{\phi} + \frac{c}{6} \mp \frac{1}{9}\phi^2. \tag{1.9}$$

The occurrence of two different types of poles at the singularity $\phi = 0$ already demonstrates that the $K(3,3)$ equation has, potentially, a much richer structure. Let us indicate the types of solutions associated with different fixed points $(\phi, \phi') = (\alpha, 0)$ in the phase space.

Consider first the focusing case (1.8+). For positive wave speeds $c > 0$, if $0 < \alpha < \sqrt{\frac{c}{3}}$, there is an analytic solitary wave of elevation having asymptotic height α. As $\alpha \to 0$ these solitary waves converge to a compacton

$$\phi_c(x - ct) = \begin{cases} \sqrt{\dfrac{3c}{2}} \cos \dfrac{x - ct}{3}, & |x - ct| \leq \dfrac{3\pi}{2}, \\ 0, & |x - ct| > \dfrac{3\pi}{2} \end{cases}. \tag{1.10}$$

Even though (1.10) has discontinuous first derivatives, its cube is C^2, and so ϕ_c is a pseudo-classical (and hence weak) solution to (1.8+). In addition, when $0 < \alpha < \sqrt{\frac{c}{3}}$, the homoclinic "tails" have the singular ϕ' axis as an asymptote, and hence there is a corresponding solution which attains an infinite slope in a finite time. But, as indicated in the phase portrait in figure 2, the two points of infinite slope can be connected through an analytic solution. The resulting nonanalytic solution forms a symmetric wave of depression that is analytic except for two points of infinite slope, called a *tipon*, [**23**]. In Figure 2,

the solid curve in the phase portrait is the homoclinic orbit that gives rise to the analytic solitary wave, while the dashed curve indicates the homoclinic tails and connecting orbit that provide the nonanalytic tipon.
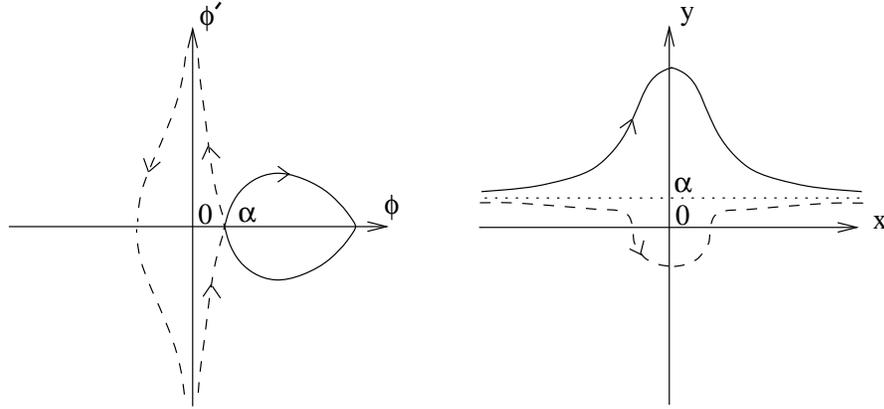

Fig. 2. *Trajectories and graphs of the analytic and tipon solitary wave solutions of the $K(3,3)$ equation.*

Further analysis leads to the following sequence of solutions as we move through the fixed points $(\alpha, 0, 0)$:

(a) $\alpha = 0$ — a compacton.

(b) $0 < \alpha < \sqrt{\frac{c}{3}}$ — analytic solitary waves of elevation and solitary tipons of depression. As $\alpha \to 0$ the solitary waves converge to the compacton, while the amplitudes of the solitary tipons decrease to zero. On the other hand, when $\alpha \to \sqrt{\frac{c}{3}}$, the analytic solitary waves shrink to zero while solitary tipons produce a solitary tipon.

(c) $\alpha = \sqrt{\frac{c}{3}}$ — there is no analytic solitary wave, but a solitary tipon of depression remains.

(d) $\sqrt{\frac{c}{3}} < \alpha < \sqrt{c}$ — a family of analytic periodic waves and a second family of *periodic tipons*, which are even, periodic pseudo-classical solutions having isolated points of infinite slope. As the period of the waves increases to ∞, they converge, respectively, to the solitary wave and the solitary tipon at the "complementary" fixed point $\widetilde{\alpha} = \frac{1}{2}(-\alpha + \sqrt{4c - \alpha^2})$. As $\alpha \to \sqrt{\frac{c}{3}}$ from the right, the analytic periodic waves converge to zero while the periodic tipons converge to the solitary tipon. As $\alpha \to \sqrt{c}$, the analytic periodic waves converge to the compacton at $\alpha = 0$, whereas the periodic tipons shrink to zero.

(e) $\alpha = \sqrt{c}$ — analytic periodic waves, which converge to the compacton at $\alpha = 0$ as the period becomes infinite.

(f) $\alpha > \sqrt{c}$ — a family of periodic waves that begin as analytic, then, at a critical period converge to a *periodic cuspon* (also called a billiard solution because the cusps are all upside down touching the real axis) and then become periodic tipons.

The defocusing case $(1.8-)$ is also interesting. For $c > 0$ there are no periodic or solitary traveling waves at all. For $c < 0$, the following sequence of solutions successively

emerges as we move through the fixed points $(\alpha, 0, 0)$:

(a) $0 < \alpha < \sqrt{-\frac{c}{3}}$ — analytic periodic waves.

(b) $\alpha = \sqrt{-\frac{c}{3}}$ — no solitary or periodic waves.

(c) $\sqrt{-\frac{c}{3}} < \alpha < \sqrt{-\frac{c}{2}}$ — a unique analytic solitary wave of depression for each α. As $\alpha \to \sqrt{-\frac{c}{2}}$, the solitary waves converge to a *cuspon*, which is a wave of depression having an infinite slope at its crest.

(d) $\alpha = \sqrt{-\frac{c}{2}}$ — the homoclinic orbit reaches the singularity $\phi = 0$ and becomes a cuspon solution.

(e) $\sqrt{-\frac{c}{2}} < \alpha < \sqrt{-c}$ — the cuspon mutates into a tipon (of depression) whose width becomes larger and larger. (In the $K(3,3)$ equation, we obtain tipons instead of the $K(2,2)$ loops because the associated potential has a pole of order 4 here.)

(f) $\alpha = \sqrt{-c}$ — two *tipon kinks*, each analytic except for a single point of infinite slope.

(g) $\alpha > \sqrt{-c}$ — no solitary or periodic waves.

Unlike the KdV equation, the general $K(m, n)$ equations are not integrable. For example, when $m = n$ they admit only four local conservation laws[†]. Moreover, a zero-mass ripple is left after a collision of, say, two compactons, although their shape remains the same after they collide, [**24**]. This robustness cannot be explained in terms known to us from the conventional soliton theory. There are, however, a wide variety of integrable nonlinearly dispersive systems, and admit analytic multi-soliton solutions. Historically, the first work in this direction was by Wadati *et. al.*, [**28**], on cusped and looped solitons. Presently, the most well-studied is the Camassa-Holm equation

$$u_t + u_x - \nu u_{xxt} + 3(u^2)_x - \nu(u^2)_{xxx} + \nu(u_x^2)_x = 0, \qquad (1.11)$$

which was derived as a model for ocean dynamics, [**6, 7**], and shown to be integrable by inverse scattering; see also [**1, 8**]. The biHamiltonian structure of (1.11) had, in fact, been first written down (albeit with an error in one of the coefficients) by Fuchssteiner, [**11**; (5.3)]. The equation for traveling waves $u = \phi(x - ct)$ becomes, upon two integrations,

$$\nu \left(\frac{c}{2} - \phi \right) (\phi')^2 = \left(\frac{c}{2} - \phi \right) \left[\left(\phi + \tfrac{1}{4} \right)^2 + a \right] + b, \qquad (1.12)$$

where a, b are the integration constants. In contrast to the dynamical system (1.5), the location of the singularity is not universal, but depends on the wave speed; this is reflected in the fact that the characteristic equation $\xi^2 \tau + 2u\xi^3 = 0$ for the original partial differential equation (1.11) has solution-dependent degeneracies, and so the points of genuine nonlinearity are solution-dependent. It is worth re-emphasizing that both cases are fundamentally different from what occurs in the formation of shock waves or more general weak solutions, whose discontinuities are not at all associated with points of genuine nonlinearity. In the focusing case $\nu = +1$, the solitons are *peakons*,

$$\phi_p(x - ct) = \left(\frac{c}{2} + \frac{1}{4} \right) e^{-|x - ct|} - \frac{1}{4}, \qquad (1.13)$$

[†] This has not, in fact, been rigorously established, but seems almost certain.

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which can be realized as limits of analytic solitary wave solutions associated with fixed points $(\alpha, 0)$ in the phase plane as $\alpha \to -\frac{1}{4}$ from the right. When $\alpha < -\frac{1}{4}$ or $\alpha > \frac{1}{2}c$, the solutions are cuspons. As $\alpha \to -\frac{1}{4}$ from the left, the cusps suddenly open up into a peak in the limit, so the solitary waves, peakon and cuspons form a single family of solutions. In the defocusing case $\nu = -1$, the singular point $\alpha = \frac{c}{2}$ gives rise to a compacton

$$
\phi_{\boldsymbol{c}}(x - ct) = \begin{cases} \frac{1}{2}c - \left(c + \frac{1}{2}\right)\cos^2 \dfrac{x - ct}{2}, & |x - ct| \leq \pi, \\ 0, & |x - ct| \geq \pi, \end{cases} \tag{1.14}
$$

which is a limit of analytic solitary wave solutions as the asymptotic height goes to zero. However, in this case, (1.11) is not well-posed. See [17, 18] for details.

The Camassa-Holm equation (1.11) can be associated to the Korteweg-deVries equation (1.1) by a process called "triHamiltonian duality", which was applied in [10, 21] to systematically additional nonlinearly dispersive, integrable systems associated with many other classical weakly nonlinear soliton equations. A particularly interesting example is the generalized Boussinesq system

$$
\begin{aligned}
v_t + 2w_t - 2v_{xt} &= w_x - v_{xx} + \left(\tfrac{1}{2}v^2 + vw\right)_x - \left(\tfrac{1}{2}v^2\right)_{xx}, \\
w_t + 2w_{xt} &= w_{xx} + \left(vw + \tfrac{1}{2}w^2 + vw_x\right)_x,
\end{aligned} \tag{1.15}
$$

that forms the integrable dual to a Boussinesq system arising in shallow water theory that was studied by Whitham, [29], and solved by inverse scattering methods in [13, 15]. Let us summarize the types of solutions found in [16] for the case $c < -1$, focusing on novel features not seen in the preceding examples.

The dynamical system for traveling waves $v = \phi(x - ct)$, $w = \psi(x - ct)$ is

$$
\begin{aligned}
-c\phi' - 2c\psi' + 2c\phi'' &= \psi' - \phi'' + \left(\tfrac{1}{2}\phi^2 + \phi\psi - \phi\phi'\right)', \\
-c\psi' - 2c\psi'' &= \psi'' + \left(\phi\psi + \tfrac{1}{2}\psi^2 + \phi\psi'\right)'.
\end{aligned} \tag{1.16}
$$

The fixed points of (1.16) all lie in the (ϕ, ψ)–plane; moreover, the line $\phi = -2c - 1$ consists of singular points. Let Γ denote the triangle formed by the three lines $\phi = -2c - 1$, $\psi = 0$ and $\psi = 1 - \phi$. Let Ω be the inscribed ellipse defined by the equation

$$
\phi^2 + \phi\psi + \psi^2 + 2c\phi - \psi + c^2 = 0.
$$

The curves Γ and Ω divide the (ϕ, ψ)–plane into regions containing fixed points with different properties:

(a) inside the ellipse Ω — periodic traveling wave solutions,

(b) between Ω and Γ — solitary wave solutions,

(c) on the line segments $\psi = 0$ or $\psi = 1 - \phi$ of Γ — two kink solutions.
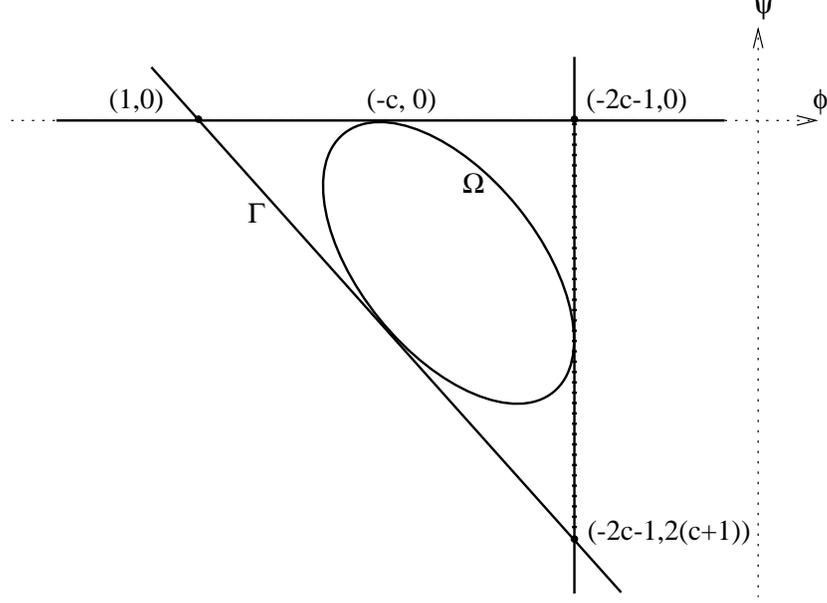
Fig. 3. *The phase plane for the Boussinesq system.*

The most interesting point is the vertex $V_* = (1,0)$ of Γ, which corresponds to an infinite family of different nonanalytic solutions! Each of these solutions can be realized as a limit of analytic solitary wave solutions associated with points on the interior of Γ, but the limiting solution depends on how the interior points converge to V_*. More specifically, let $\mathcal{P}$ be a curve inside Γ (but outside Ω) which terminates at V_*. The solitary wave solutions (ϕ_a, ψ_a) corresponding to $a \in \mathcal{P}$ will converge to a nonanalytic traveling wave (ϕ_*, ψ_*) as $a \to V_*$, but this solution *depends on the asymptotics of the curve $\mathcal{P}$ at V_**. For example, if $\mathcal{P}$ is not tangent to Γ as it approaches V_*, then the solitary wave solutions converge to a peakon of the form

$$\begin{aligned}
\phi_p(\xi) &= 1 - 2(c+1)e^{-|\xi|/2}, \\
\psi_p(\xi) &= (c+1)e^{-|\xi|/2}(\text{sign}(\xi)+1),
\end{aligned} \qquad \text{where} \qquad \xi = x - ct. \qquad (1.17)$$

On the other hand, for the path defined by the equation $\psi(\psi + \phi - 1) + e^{-2\lambda/(\phi-1)} = 0$, which tangent to the sides of Γ, the solitary wave solutions converge to a solution that we call a *mesaon* owing to its flat top; the width $2\mu = -2\lambda/(c+1)$ of the top is governed by the parameter λ; in the zero-width limit $\mu \to 0$, the mesaon becomes a peakon. There are two different versions, depending on the branch of the path chosen by the quadratic equation (1.17); one of these is

$$(\phi_m, \psi_m)(\xi) = \begin{cases} \left(1 - 2(c+1)e^{(\mu+\xi)/2}, 0\right), & \xi < -\mu, \\ (-2c-1, 0) & |\xi| < \mu, \\ \left(1 - 2(c+1)e^{(\mu-\xi)/2}, 2(c+1)e^{(\mu-\xi)/2}\right), & \xi > \mu. \end{cases} \qquad (1.18)$$

Note that the mesaon (1.18) is constant at the fixed point $(-2c-1, 0)$ of the dynamical system, which *lies on the line of singularities*. (The other mesaon is constant at the second

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fixed point $(-2c-1, 2c+2)$ on the singular line.) The occurrence of fixed points which are also singularities is the key property that allows the peakon to open up into a flat topped mesaon of arbitrary width. Indeed, in contrast, the line of singularities of the Camassa-Holm dynamical system (1.12) contains no embedded fixed points, and this prevents the Camassa-Holm peakons from opening up.

If our path $\mathcal{P}$ coincides with a side of the triangle, the associated analytic kink solutions converge to yet another nonanalytic solution at V_*, called a *rampon* or *semi-compact kink*, of which an example is

$$(\phi_r, \psi_r)(\xi) = \begin{cases} (-2c-1, 0), & \xi < -2\log(-2(c+1)), \\ \left(e^{-\xi/2}+1, -e^{-\xi/2}\right), & \xi > -2\log(-2(c+1)). \end{cases} \quad (1.19)$$

Finally, points on the singular line $\phi = -2c-1$ admit compactons which are found as limits of solitary waves; an example is

$$\phi_c(\xi) = \begin{cases} 2\sqrt{-b(b-2c-2)}\cosh\frac{\xi}{2}+1, & |\xi| < \mu, \\ -2c-1, & |\xi| > \mu, \end{cases} \quad (1.20)$$

where $\mu = 2\cosh^{-1}\frac{-c-1}{\sqrt{-b(b-2c-2)}}$ determines the support of the compacton.
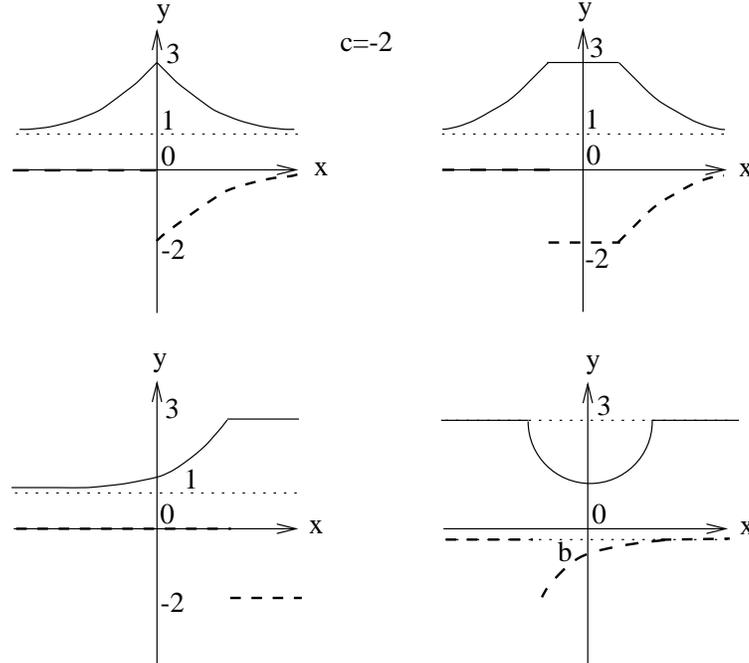

Fig. 4. *Graphs of the peakon, mesaon, rampon and compacton solutions of the dual Boussinesq system.*

Figure 4 shows the different types of nonanalytic solitary wave solutions admitted by (1.15). From left to right, they show a peakon, a mesaon, a rampon and a compacton. The solid line is $v = \phi$, while the dashed line shows $w = \psi$. Note particularly that w is not

continuous — it jumps back to zero at a certain point in the first three cases. Nevertheless, the solution is still quasi-classical and does not require any sort of entropy condition to fix the position of the discontinuity. It is clearly beyond the capacity of the classical dynamical systems theory to explain this phenomenon, and so passage to the fully nonlinear regime introduces new and unexpected phenomena even in such well plowed fields as ordinary dynamical systems!

These examples provide concrete evidence that both the integrable and nonintegrable, nonlinearly dispersive examples treated to date are merely the tip of the proverbial iceberg. Even though, the nature of the nonanalyticity changes in each case, these differences are technical, showing merely the different facets of nonlinearity. All in all, we have not even begun to understand the nature of the nonlinear interactions that govern these equations.

References

[1] Beals, R., Sattinger, D.H., and Szmigielski, J., Acoustic scattering and the extended Korteweg–deVries hierarchy, preprint, University of Minnesota, 1998.

[2] Benjamin, T.B., Bona, J.L., and Mahoney, J.J., Model equations for long waves in nonlinear dispersive systems, *Phil. Trans. Roy. Soc. London A* **272** (1972), 47–78.

[3] Bona, J.L., Pritchard, W.G., and Scott, L.R., Solitary–wave interaction, *Phys. Fluids* **23** (1980), 438–441.

[4] Boussinesq, J., Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, *J. Math. Pures Appl.* **17** (2) (1872), 55–108.

[5] Boussinesq, J., Essai sur la théorie des eaux courants, *Mém. Acad. Sci. Inst. Nat. France* **23** (1) (1877), 1–680.

[6] Camassa, R., and Holm, D.D., An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[7] Camassa, R., Holm, D.D., and Hyman, J.M., A new integrable shallow water equation, *Adv. Appl. Mech.* **31** (1994), 1–33.

[8] Constantin, A., and McKean, H., The integrability of a shallow water equation with periodic boundary conditions, preprint, Courant Institute, 1997.

[9] Drazin, P.G., and Johnson, R.S., *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.

[10] Fokas, A.S., Olver, P.J., and Rosenau, P., A plethora of integrable bi-Hamiltonian equations, in: *Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman*, A.S. Fokas and I.M. Gel'fand, eds., Progress in Nonlinear Differential Equations, vol. 26, Birkhäuser, Boston, 1996, pp. 93–101.

[**11**] Fuchssteiner, B., The Lie algebra structure of nonlinear evolution equations admitting infinite dimensional abelian symmetry groups, *Prog. Theor. Phys.* **65** (1981), 861–876.

[**12**] Gardner, L.R.T., and Gardner, G.A., Solitary waves of the regularised long wave equation, *J. Comp. Phys.* **91** (1990), 441–459.

[**13**] Kaup, D.J., A higher-order water-wave equation and the method for solving it, *Prog. Theor. Physics* **54** (1975), 396–408.

[**14**] Korteweg, D.J., and de Vries, G., On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves, *Phil. Mag.* (5) **39** (1895), 422–443.

[**15**] Kupershmidt, B.A., Mathematics of dispersive waves, *Comm. Math. Phys.* **99** (1985), 51–73 .

[**16**] Li, Y.A., Weak solutions of a generalized Boussinesq system, *J. Dyn. Diff. Eq.*, to appear.

[**17**] Li, Y.A., and Olver, P.J., Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system. I. Compactons and peakons, *Discrete Cont. Dyn. Syst.* **3** (1997), 419–432.

[**18**] Li, Y.A., and Olver, P.J., Convergence of solitary-wave solutions in a perturbed bi-Hamiltonian dynamical system. II. Complex analytic behavior and convergence to non-analytic solutions, *Discrete Cont. Dyn. Syst.* **4** (1998), 159.

[**19**] Makhankov, V.G., Dynamics of classical solitons (in non-integrable systems), *Phys. Rep.* **35** (1978), 1–128.

[**20**] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, Springer–Verlag, New York, 1993.

[**21**] Olver, P.J., and Rosenau, P., Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E* **53** (1996), 1900–1906.

[**22**] Rosenau, P., Nonlinear dispersion and compact structures, *Phys. Rev. Lett.* **73** (1994), 1737–1740.

[**23**] Rosenau, P., On nonanalytic waves formed by a nonlinear dispersion, *Phys. Lett. A* **230** (1997), 305–318.

[**24**] Rosenau, P., and Hyman, J.M., Compactons: solitons with finite wavelength, *Phys. Rev. Lett.* **70** (1993), 564–567.

[**25**] Santarelli, A.R., Numerical analysis of the regularized long-wave equation: anelastic collision of solitary waves, *Il Nuovo Cim..* **46B** (1978), 179–188.

[**26**] Scott Russell, J., *Report on Water Waves*, British Assoc. Report, 1844.

[**27**] Toland, J.F., On the existence of a wave of greatest height and Stokes' conjecture, *Proc. Roy. Soc. London A* **363** (1978), 469–485.

[**28**] Wadati, M., Ichikawa, Y.H., and Shimizu, T., Cusp soliton of a new integrable nonlinear evolution equation, *Prog. Theor. Phys.* **64** (1980), 1959–1967.

[**29**] Whitham, G.B., *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.