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Hamiltonian structure, symmetries and conservation laws for water waves

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An investigation on novel lines is made into the problem of water waves according to the perfect-fluid model, with reference to wave motions in both two and three space dimensions and with allowance for surface tension. Attention to the Hamiltonian structure of the complete nonlinear problem and the use of methods based on infinitesimal-transformation theory provide a systematic account of symmetries inherent to the problem and of corresponding conservation laws.

The introduction includes an outline of relevant elements from Hamiltonian theory (§1.1) and a brief discussion of implications that the present findings may carry for the approximate mathematical modelling of water waves (§1.2). Details of the hydrodynamic problem are recalled in §2. Then in §3 questions about the regularity of solutions are put in perspective, and a general interpretation is expounded regarding the phenomenon of wave-breaking as the termination of smooth Hamiltonian evolution. In §4 complete symmetry groups are given for several versions of the water-wave problem: easily understood forms of the main results are listed first in §4.1, and the systematic derivations of them are explained in §4.2. Conservation laws implied by the one-parameter subgroups of the full symmetry groups are worked out in §5, where a recent extension of Noether's theorem is applied relying on the Hamiltonian structure of the problem. The physical meanings of the conservation laws revealed in §5, to an extent abstractly there, are examined fully in §6 and various new insights into the water-wave problem are presented.

In Appendix 1 the parameterized version of the problem is considered, covering cases where the elevation of the free surface is not a single-valued function of horizontal position. In Appendix 2 a general method for finding the symmetry groups of free-boundary problems is explained, and the exposition includes the mathematical material underlying the particular applications in §§4 and 5.

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1. Introduction

The behaviour of Hamiltonian systems whose phase space has infinite dimensions is an old subject reinvigorated by much recent progress. Wide attention is paid to it both because various major problems of mathematical physics exemplify such systems and because certain model equations with interesting, comparatively accessible properties can also be so considered. The subject is moreover very appealing in mathematical respects, being particularly fertile in applications of variational methods and methods of infinite-dimensional differential geometry. An impressive, albeit still incomplete, chapter of the subject comprises applications to the abstract hydrodynamical problem for a perfect fluid, which can be considered as a Hamiltonian system whose configuration space is the group of volume-preserving diffeomorphisms of some (fluid-filled) manifold in $\mathbb{R}^2$ or $\mathbb{R}^3$ (e.g. see Arnold 1966; Ebin & Marsden 1970; Marsden 1974, ch. 4), and the present problem of water waves must be acknowledged to fall within the broad ambit of this previous work. The problem is quite special, however, being greatly complicated by the boundary conditions at the free surface, and the available general theory does not appear to be in any immediate way helpful. The need for a specialized theory is plain, of course, because the evolutionary nature of the water-wave problem can be appreciated to reside in the surface conditions rather than in the field equations for the fluid, which, under the hypothesis apposite to this problem, are reducible by Kelvin's theorem to a time-independent form (i.e. just Laplace's equation in fixed spatial coordinates, satisfied by the velocity potential).

The key to bringing notions of Hamiltonian mechanics usefully to bear on the water-wave problem is a remarkably concise interpretation of the free-surface conditions in terms of functional derivatives of the energy integral, an invariant of the problem given the role of Hamiltonian. In this interpretation two functions of time and horizontal position alone, namely the vertical elevation η of the free surface S and the evaluation Φ of the velocity potential at S, appear as canonical (Darboux) variables. In effect η is a generalized coordinate (of infinite dimension) and Φ a generalized momentum. This simplifying formalism appears to have been noticed first by Zakharov (1968). Representations of the nonlinear boundary conditions by functional derivatives of energy and momentum integrals were also considered by Benjamin (1974) in another, closely related light, and Miles (1977; see also Milder 1977) has made a deep study of this formalism, recognizing it to hold for wave motions of liquids contained in vessels of arbitrary shape. Further extensions of it applying to gravity-capillary waves with edge constraints, including waves of this type that are periodic in one horizontal coordinate, have been noted by Benjamin (1980).

In the present treatment of the water-wave problem, our principal aims are to examine its symmetry groups thoroughly and then to find the implications of these symmetries as regards conservation laws. Whereas the first aim is achievable without particular regard to the Hamiltonian structure of the problem, progress towards the second depends crucially on it, particularly on the variational characterizations provided. Many of the properties thus investigated are already well known (cf. Benjamin & Mahony 1971; Benjamin 1974; Longuet-Higgins 1974, 1975, 1980*a*). But to the best of our knowledge this is the first systematic study of the fundamental mathematical aspects that are covered.

The details of the hydrodynamic problem are set out in §2, which includes a brief rehearsal of the Hamiltonian formalism noted above. Surface tension is included in the account, and the only restrictive assumption about the wave motion is that η remains a single-valued function of horizontal position. In §3 some incidental commentary is presented concerning the regularity of solutions and the eventuality of wave-breaking. Folding of the free surface S, contrary to the aforementioned assumption, is considered as an essential precursor of breaking, and there is a review of the question how the evolutionary process may pass into this stage consistently with Hamiltonian principles. A reformulation of the hydrodynamic problem using parametric representations of S is given in Appendix 1, to which reference needs to be made in §3. A new general interpretation is proposed describing precisely the manner in which the applicability of perfect-fluid theory terminates.

The symmetry groups associated with several versions of the water-wave problem are presented in §4, being listed first in their simplest forms but then being derived systematically by identification of their infinitesimal generators. Explanations of some technical arguments needed are deferred to Appendix 2, where the account covers a generalized free-boundary problem including the water-wave problem as a particular case but also having many other prospective applications. All the symmetry groups established in §4, which total thirteen for the three-dimensional version of the problem in the absence of surface tension, are recognized to have immediate physical interpretations.

The conservation laws linked to these symmetries are derived in §5, where the treatment relies on a new generalization of Noether's theorem due to Olver (1980*a*). After an outline of the underlying theory, the pivotal result that is needed, establishing a one-to-one correspondence between the single-parameter symmetry groups of any Hamiltonian system and its conservative properties, is presented as Theorem 5.1. The inferences drawn from applications of this theorem are collected in Theorems 5.2 and 5.3, which, for the two-dimensional and three-dimensional versions of the problem, list respectively eight and twelve *conserved densities* on S (i.e. quantities that are not exact differentials but whose integrals over any horizontal domain depend only on boundary values – in a sense to be made clear). Certain of these results have straightforward physical interpretations, but others have delicate meanings which call for careful discussion. A finally comprehensive account of the conservation laws obeyed exactly by water waves is given in §6, the main findings being presented collectively as Theorems 6.1 (§6.1) and 6.2 (§6.4). The account includes full explanations of the conservation laws in physical terms (§6.2), and some original findings concerned with a quantity I^7, here termed virial, whose conservative properties are comparatively obscure (§6.3). The Lagrangian formulation of the water-wave problem and a certain connexion between I^7 and wave action are noted in §6.3. Finally, in §6.5, several delicate points of interpretation are clarified concerning waves on an ocean of infinite depth. In particular, a precise analysis on novel lines is made in order to elucidate the meaning of the momentum components in this case, which occur in certain kinematic identities but which appear at first sight to be indeterminate.

1.1. *Elements of Hamiltonian mechanics*

It will be helpful to recall a few of these elements in order to fix ideas. First, considering any Hamiltonian system with a *finite* number n degrees of freedom, take $x \in \mathbb{R}^n$ to denote the configuration variable and $y \in \mathbb{R}^n$ the (generalized) momentum variable. In terms of the Hamiltonian function $H(x, y) \in C^1(\mathbb{R}^{2n} \to \mathbb{R})$, the equations determining $(x, y)(t)$ from any initial value $(x, y)(t_0)$ are

$$\frac{dx}{dt} = H_y, \quad \frac{dy}{dt} = -H_x. \tag{1.1}$$

Here H_y is just the n-vector with components $\partial H/\partial y_i$ $(i = 1, \ldots, n)$, and H_x similarly. Regarding x and y as column vectors and writing

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{grad } H(u) = \begin{bmatrix} H_x(u) \\ H_y(u) \end{bmatrix},$$

we have that the $2n$ equations in (1.1) are equivalent to

$$u_t = J \text{ grad } H(u), \tag{1.2}$$

in which J is the skew-symmetric matrix expressible by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.3}$$

where I is the $n \times n$ identity matrix. The scalar H is commonly, but not necessarily, the total energy of the system, and (1.1) or (1.2) implies that $H(u) =$ const. along any solution curve $u = u(t)$ in $\mathbb{R}^{2n}$. Any system in the form (1.2) with J a nonsingular, skew-symmetric matrix of constants can be reduced locally, by a linear transformation of the dependent variable u, to a system in the canonical (Darboux) form with J given by (1.3).

Continuous systems having evolutionary equations in the form (1.2) also qualify as Hamiltonian. Then grad H is to be interpreted as a functional derivative in respect of some suitable inner product $(.,.)_P$ over the class P of functions (the infinite-dimensional phase space) to which the solution u can be attributed, and the skew-symmetric transformation J is correspondingly understood. Thus, formal definitions generalizing the preceding, finite-dimensional ones are

$$(\text{grad } H(u), v)_P = [(d/ds)\, H(u+sv)]_{s=0},$$

$$(Ju, v)_P = -(u, Jv)_P$$

$\forall\, u, v \in P$. By convention the label Hamiltonian is extended to any equation in the form (1.2) with J skew-symmetric, nonsingular and independent of u. As regards certain symmetry and conservative properties, however, the last two conditions on J are not absolutely essential (cf. Lax 1978; Gel'fand & Dorfman 1979). Accordingly, when either is not satisfied, the description *quasi-Hamiltonian* is justified. Equations in the form

$$Ku_t = \text{grad } H(u), \tag{1.4}$$

where K is a skew-symmetric operator that is not invertible, may also be so described. In Appendix 1, dealing with the parameterized version of the water-wave problem, we obtain an equation in this form. Plainly, it does not by itself fully describe an evolutionary process, for u_t is not determined uniquely by it. Rather, as the context makes clear, it is a sidelight, revealing incidental Hamiltonian structure, on a process that may be otherwise fully determined when arbitrary features of its representation are fixed. Note that either of the abstract equations (1.2) or (1.4) implies $H(u)$ = const.; for in the first case

$$\frac{dH(u)}{dt} = (\text{grad } H(u), u_t)_P = (\text{grad } H(u), J \text{ grad } H(u))_P = 0,$$

and in the other

$$\frac{dH(u)}{dt} = (\text{grad } H(u), u_t)_P = (Ku_t, u_t)_P = 0.$$

Many properties of Hamiltonian systems can be conveniently represented in terms of Poisson brackets. Respective to a particular J in (1.2), the Poisson bracket of any two sufficiently smooth functions $F: P \to \mathbb{R}$ and $G: P \to \mathbb{R}$ is defined as

$$\begin{aligned}[F, G] &= (\text{grad } F(u), J \text{ grad } G(u))_P \\ &= -[G, F].\end{aligned}$$

It follows immediately from this definition that $F(u)$ = const. along any solution curve of (1.2) in P if and only if $[F, H] = 0$. (We have already exemplified this proposition in noting that $H(u)$ = const. follows from the obvious identity $[H, H] = 0$.) Because $[F, H] = 0$ is a symmetrical relationship between F and H, this also implies that if $u_1(t)$ is any solution curve of the Hamiltonian system given by replacing H by F in (1.2), then $H(u_1)$ = const. Another, deeper implication of the condition $[F, H] = 0$ is that the *flows* of the respective Hamiltonian equations commute (i.e. $S_F \circ S_H = S_H \circ S_F$, where S_H and S_F are respectively the solution operators for (1.2) and its alternative; cf. Lax 1978; Olver 1980*a*). Ideas extending these basic ones will be developed in §5.

1.2. *Wider implications*

We have good reason to believe that the present investigation accounts for *all* conservation laws intrinsic to the complete water-wave problem. While proof that no other exists is deferred to a subsequent paper (Olver 1982), it is timely to appreciate the import of this proposition, particularly its bearing on the interpretation of approximate theories. The fundamental aspect now highlighted is comparable with that of the three-body problem clarified by the celebrated theorems of Bruns and of Poincaré (Whittaker 1937, Ch. 14), which establish the only invariant integrals of the dynamical equations to be those in respect of energy and the components of linear and angular momentum. In common with this classic example of a (finite-dimensional) Hamiltonian system, the limited number of conservation laws obeyed exactly by water waves contrasts with a greater, in fact unbounded number of them obeyed by some approximate models, which correspondingly have unnatural symmetries (cf. Ibragimov 1977) and need to be interpreted cautiously.

The point in question, that simplifications can introduce conservative properties not attributable to the complete version of the problem, is particularly obvious with regard to linearized theory. For instance, consider two-dimensional irrotational wave motions in an infinite layer of water free from surface tension lying on a horizontal plane. Let h denote the mean depth and $x \in \mathbb{R}$ the relevant horizontal coordinate.

According to the linearized approximation which is very well known, the elevation $\eta = \eta(x, t)$ of the free surface above its undisturbed level satisfies an evolution equation in the form

$$\eta_{tt} = A\eta_{xx}, \tag{1.5}$$

where A is the symmetric linear operator, invariant with x and t, whose symbol is given by

$$\hat{A}(\xi) = (g/\xi)\tanh(\xi h) > 0$$

[i.e. if $\eta(x, t)$ is a smooth function vanishing together with all its derivatives as $x \to \pm\infty$ and its Fourier transform with respect to x is $\mathscr{F}\eta = \hat{\eta}(\xi, t)$, then $\mathscr{F}(A\eta) = \hat{A}\hat{\eta}$]. Treating η and $v = \eta_t$ as conjugate variables, we recognize (1.5) to exemplify a Hamiltonian system as represented by (1.2); the Hamiltonian function is

$$H = \int_{\mathbb{R}} (\tfrac{1}{2}\eta_x A\eta_x + \tfrac{1}{2}v^2)\, dx,$$

and $dH/dt = 0$. On the other hand, *any number* of such quadratic invariants are to be found trivially from (1.5). For, if $\mathbf{C}$ is any symmetric linear invariant operator whatever, for example $(-\partial_x^2)^n$ with $n = 1, 2, \ldots$, then (1.5) implies that

$$\int_{\mathbb{R}} (\mathbf{C}\eta_x A\eta_x + v\mathbf{C}v)\, dx = \text{const.}$$

In the approximation providing (1.5), the total energy of the wave motion is found to be

$$E = \int_{\mathbb{R}} \tfrac{1}{2}g\{\eta^2 + v(-A\partial_x^2)^{-1}v\}\, dx = \text{const.},$$

and its total momentum in the x-direction to be

$$-\int_{\mathbb{R}} g\eta_x(-A\partial_x^2)^{-1}v\, dx = \text{const.}$$

Having a clear physical meaning, unlike all the others, these two nonlinear invariants are in fact the only ones representative of the complete problem.

The infinite profusion of invariants for the linearized problem thus is a straightforward matter without particularly helpful implications. A comparable but much more remarkable property, however, is possessed by the Korteweg–de Vries (KdV) equation which serves as a rudimentary nonlinear model for unidirectional propagation of small-amplitude long waves in a uniform channel. This equation too is known to have an infinite number of conservation laws tied to it, accountable to infinitely many 'accidental' symmetries (e.g. see Olver 1977), and they have unquestionably interesting consequences. In particular, nonlinear interactions among solitary-wave solutions (solitons) are immaculate, allowing each component to emerge asymptotically in its exact original form. The Hamiltonian structure of the KdV equation has been discussed by Gardner (1971), Lax (1978) and others in the light of this special conservative behaviour. Roughly speaking, by analogy with Hamiltonian systems whose phase space is finite-dimensional, one can say that the KdV equation is completely integrable by virtue of its unbounded set of invariant integrals, and its amenability in this respect makes it particularly attractive as a model for physical processes. The special properties must nevertheless be judged as intrinsic to the model, rather than as accurately simulative of real phenomena. The complete water-wave problem for a uniform channel is known to have solitary-wave solutions (Friedrichs & Hyers 1954; Amick & Toland 1981), but perfect interactions akin to those of KdV solitons are unlikely since the problem appears to have only two nonlinear integral invariants. We may appropriately note that the alternative, regularized long-wave equation studied by Benjamin, Bona & Mahony (1972) has a formal status more or less equal to that of the KdV equation as an approximate model for water waves, but this equation was shown by Olver (1979*a*) to have only two such invariants. It too can be regarded as a Hamiltonian system (see Lax 1978), having comparatively few symmetry properties but, at least in respect of its amenability to numerical solution, being none the worse for this shortage.

2. The hydrodynamic problem

We first present the full problem with allowance for three-dimensional motion and for surface tension, leaving particular versions of it to be specified later. Let (x, y, z) be fixed cartesian axes with y vertical upwards. An incompressible inviscid liquid, having unit density, is considered to occupy a domain D_η whose upper boundary, the moving free surface denoted by S, is described by the equation

$$y = \eta(x, z, t). \tag{2.1}$$

Here η is assumed to be a single-valued function of x, z for all relevant t, but in §3 and Appendix 1 the contrary case where S is folded will be examined. The horizontal projection of S is the whole plane $\mathbb{R} \times \mathbb{R}$, to be denoted by S_0, and η is taken to be a smooth function vanishing together with all its derivatives as $(x^2+z^2)^{\frac{1}{2}} \to \infty$. Either D_η extends to infinite depth for all $(x, z) \in S_0$, or it is bounded by a fixed horizontal plane $y = -h$.

The motion of the liquid is supposed to have been generated from rest by conservative forces, being consequently irrotational according to Kelvin's theorem. The eulerian velocity field $\mathbf{u}: D_\eta \to \mathbb{R}^3$ is therefore given by $\mathbf{u} = \nabla\phi$; and since the incompressibility of the liquid requires that $\nabla . \mathbf{u} = 0$, the velocity potential $\phi = \phi(x, y, z, t)$ satisfies

$$\Delta\phi \, (= \phi_{xx} + \phi_{yy} + \phi_{zz}) = 0 \quad \text{in} \quad D_\eta. \tag{2.2}$$

The kinematic conditions of the problem include

$$|\nabla\phi| \to 0 \quad \text{as} \quad (x^2+y^2+z^2)^{\frac{1}{2}} \to \infty \tag{2.3}$$

and, in the case of finite depth, additionally

$$\phi_y = 0 \quad \text{on} \quad y = -h. \tag{2.3'}$$

To represent evaluations of ϕ and its derivatives at the free surface S, the following notations are used:

$$\Phi = \phi_S = \phi(x, \eta(x, z, t), z, t)$$

$$\Phi_{(x)} = (\phi_x)_S, \quad \Phi_{(y)} = (\phi_y)_S, \quad \Phi_{(z)} = (\phi_z)_S, \quad \Phi_{(t)} = (\phi_t)_S.$$

Note that

$$\Phi_t \, (= \partial_t \Phi) = \Phi_{(t)} + \Phi_{(y)} \eta_t,$$

and that corresponding expressions hold for Φ_x and Φ_z. We also write

$$\Phi_{(n)} = (\partial_n \phi)_S = R^{-1}(\Phi_{(y)} - \eta_x \Phi_{(x)} - \eta_z \Phi_{(z)}),$$

where $R = (1+\eta_x^2+\eta_z^2)^{\frac{1}{2}}$, and

$$q^2 = |\nabla\phi|_S^2 = \Phi_{(x)}^2 + \Phi_{(y)}^2 + \Phi_{(z)}^2.$$

In this notation, the kinematic condition applying at S is

$$\eta_t = R\Phi_{(n)} = \Phi_{(y)} - \eta_x \Phi_{(x)} - \eta_z \Phi_{(z)} \tag{2.4}$$

(cf. Lamb 1932, §9). The remaining, dynamical boundary condition expresses the fact that the pressure in the liquid at S is $-2\sigma H$, where σ is the coefficient of surface tension and

$$2H = (R^{-1}\eta_x)_x + (R^{-1}\eta_z)_z$$

is twice the mean curvature of S. Thus, using the Bernoulli integral of the dynamical equations to express the pressure, we have

$$\Phi_{(t)} + \tfrac{1}{2}q^2 + g\eta - 2\sigma H = 0 \tag{2.5}$$

(cf. Lamb, §20), which is the same as

$$\begin{aligned}\Phi_t &= -(\tfrac{1}{2}q^2 + g\eta - 2\sigma H) + \Phi_{(y)}\,\eta_t \\ &= -(\tfrac{1}{2}q^2 + g\eta - 2\sigma H) + \Phi_{(y)}\,R\Phi_{(n)}\end{aligned} \tag{2.5$'$}$$

by (2.4). In general a function of time alone is included in the Bernoulli integral, but here it is zero by virtue of the asymptotic conditions on S for $(x^2+z^2)^{\frac{1}{2}} \to \infty$.

The kinetic energy of the motion is given by

$$K = \int_{D_\eta} \tfrac{1}{2}|\nabla\phi|^2\,dx\,dy\,dz = \int_{S_0} \tfrac{1}{2}\Phi\Phi_{(n)}\,R\,dx\,dz, \tag{2.6}$$

and the associated potential energy by

$$V = \int_{S_0} \{\tfrac{1}{2}g\eta^2 + \sigma(R-1)\}\,dx\,dz. \tag{2.7}$$

The second integral expressing K follows from the first by Green's theorem combined with (2.2) and (2.3); and V is expressed relative to a state of rest with S everywhere horizontal. It may readily be confirmed that (2.2)–(2.5) imply the total energy $E = K + V$ to be conserved.

To obtain the Hamiltonian formulation of the problem, we first note that the motion is determined fully by the two functions η and Φ of x, z, t. That is, for each t, η determines the domain D_η and Φ determines the corresponding ϕ, which is the unique solution of the linear boundary-value problem comprised by (2.2), (2.3) and $(\phi)_S = \Phi$. We can accordingly consider $E = E(\eta, \Phi)$ and proceed to calculate the functional derivatives E_η and E_Φ. The calculation is done on the lines of the classical calculus of variations, using the formal definitions

$$(\dot\eta, E_\eta) = \left[\frac{d}{ds}E(\eta + s\dot\eta, \Phi)\right]_{s=0}, \quad (\dot\Phi, E_\Phi) = \left[\frac{d}{ds}E(\eta, \Phi + s\dot\Phi)\right]_{s=0},$$

where $(\,.\,,\,.\,)$ denotes the inner product for $L^2(S_0)$. The expression (2.6) for K gives

$$\begin{aligned}\dot K &= (\dot\eta, \tfrac{1}{2}q^2) + \int_{D_\eta} \nabla\phi\,.\,\nabla\dot\phi\,dx\,dy\,dz \\ &= (\dot\eta, \tfrac{1}{2}q^2) + ((\dot\phi)_S, \Phi_{(n)}\,R)\end{aligned}$$

by Green's theorem, and the second integral can be reduced to the required form by use of the obvious identity

$$\dot\Phi = (\dot\phi)_S + \Phi_{(y)}\,\dot\eta.$$

The expression for $V = V(\eta)$ gives at once

$$\dot V = (\dot\eta, g\eta) + \sigma\{(\dot\eta_x, R^{-1}\eta_x) + (\dot\eta_z, R^{-1}\eta_z)\},$$

and the terms proportional to σ are reducible through integrations by parts. Hence the final results are

$$E_\eta = \tfrac{1}{2}q^2 + g\eta - 2\sigma H - \Phi_{(y)} R\Phi_{(n)},$$

$$E_\Phi = R\Phi_{(n)}.$$

Comparison with (2.4) and (2.5′) now shows that the hydrodynamic problem is formally equivalent to the Hamiltonian system of equations

$$\eta_t = E_\Phi, \quad \Phi_t = -E_\eta, \tag{2.8}$$

with η and Φ as canonical variables (cf. Zakharov 1968; Miles 1977). It may be noted incidentally that the same formalism holds when D_η is bounded beneath by any smooth fixed surface Γ, not necessarily horizontal, but S_0 is still the whole horizontal plane. The only difference then is that the kinematical condition $\partial_n \phi = 0$ on Γ is included in the specifications of the linear boundary-value problem closing the system (2.8). In the case that D_η is bounded in all directions (i.e. the liquid is contained in a finite vessel), some modifications are needed. If surface tension is operative, a condition at the edge of S needs to be specified, which could reasonably be that the contact angle between S and the fixed part of ∂D_η takes a prescribed value, or that $\eta = 0$ at the edge (cf. Benjamin 1980). In this case, moreover, a non-zero constant, say λ, is generally needed on the right-hand side of the dynamical surface condition (2.5). Accordingly, the appropriate Hamiltonian function replacing E in (2.8) is $E + \lambda m$, where in keeping with (2.2) and (2.4)

$$m = \int_{S_0} \eta \, dx \, dz$$

is plainly a constant of the wave motion.

For easy reference later, let us note the two-dimensional version of the problem (2.2)–(2.5) in the case of infinite depth. Here $\eta = \eta(x, t)$, $\Phi = \Phi(x, t)$, etc., and the complete two-dimensional problem is

$$\left.\begin{aligned} \phi_{xx} + \phi_{yy} = 0 \quad \text{in} \quad D_\eta = \{(x, y) : x \in \mathbb{R},\, y \leqslant \eta(x, t)\}, \\ |\nabla\phi| \to 0 \quad \text{as} \quad (x^2 + y^2)^{\frac{1}{2}} \to \infty, \\ \eta_t = \Phi_{(y)} - \Phi_{(x)}\eta_x = R\Phi_{(n)}, \\ \Phi_{(t)} + \tfrac{1}{2}q^2 + g\eta - \sigma\{\eta_{xx}/(1+\eta_x^2)^{\frac{3}{2}}\} = 0. \end{aligned}\right\} \tag{2.9}$$

In this case, of course, we have $R = (1+\eta_x^2)^{\frac{1}{2}}$, and in the last of (2.9) the expression for the curvature of S is the same as $(\eta_x/R)_x$.

3. Regularity and wave-breaking

For continuous Hamiltonian systems it is usually difficult to verify that physically meaningful solutions exist over long intervals of time. In common with general three-dimensional vortex motions of a perfect fluid even when confined within finite fixed boundaries (cf. Ebin & Marsden 1970, §1), water waves are mathematically uncertain to the extent that no existence and regularity theory is yet established. Note that constancy of the positive Hamiltonian E, either as expressed in §2 or as more generally expressed in Appendix 1, is quite insufficient to guarantee acceptable regularity of the solution. Accordingly, like most others dealing exactly with unsteady water waves, our investigation proceeds on the unsupported presumption of solutions remaining smooth. While its tentativeness should be acknowledged, this line of approach is entirely reasonable for present purposes – and *faute de mieux* necessary.

On the other hand, a specially interesting aspect of the theoretical problem is indicated by the practical observation that water waves can break, producing vigorous turbulence and so passing beyond the reach of perfect-fluid theory. The evolutionary process with Hamiltonian structure is an abstraction, of course, but has been amply confirmed as an excellent approximation to nature before the onset of breaking. It is therefore evident that the process can eventually develop properties incompatible with nature, where the obviation of them depends essentially on physical factors not included in the ideal theory. Many different suggestions have been made about the mechanism of wave-breaking, but in our view the basic mathematical issue has not yet been adequately exposed.

One sort of explanation that has often been given refers to the possibility of a local 'instability', as the result of which small-scale features of the wave motion develop rapidly and real-fluid effects thereby become predominant somehow (for a recent appeal to this idea, see Stiassnie & Peregrine 1980, §5). With regard to a continuous evolutionary process the meaning of 'instability' can hardly be made precise, however, and the main theoretical issue is not illuminated by proposals on these lines. Rapidly developing motions such as occur near the crest of a wave prior to breaking may still be modelled accurately by the perfect-fluid equations, which can in principle be integrated forward in time until some ruinous singularity is developed. Until that event, the Hamiltonian structure and other conservative properties of the perfect-fluid model remain intact. In particular, the motion remains everywhere irrotational according to Kelvin's theorem and is unquestionably not turbulent.

Another explanation – or rather description – of the breaking mechanism has been suggested by proponents of catastrophe theory (Thom 1975, p. 94; Zeeman 1970). According to it, the crest of a wave evolves into a sharp-angled form, akin to the extreme form of *steady* waves in the absence of surface tension as discovered by Stokes (Lamb 1932, p. 418). The crest thereafter becomes cusped forwards, in the manner typified algebraically by the unfolding of the hyperbolic umbilic catastrophe. This characterization is not unlike what is sometimes seen, particularly in that a jet of water is thrown forward; but there is little evidence that this type of singularity is evolved precisely by the hydrodynamic system. More cogently, local perfect-fluid behaviour that simulates the formation of sharp corners in the free surface of a water wave and probably typifies the intermediate stages in the overturning of waves has been analysed by Longuet-Higgins (1980*b*, 1981). However, such behaviour if it did occur would presumably still be a stage of Hamiltonian evolution, so that it does not account for the final breakdown of the model and the appearance of turbulence.

The interpretation favoured by us is simple and definite, recognizing that the free-boundary problem can have smoothly evolving solutions that suddenly become incommensurate with reality. Consider first the stage of evolution illustrated in figure 1(*b*), which smoothly follows that in figure 1(*a*). The height η of the free surface S has ceased to be a single-valued function of position in the horizontal plane, so that the simplifying assumption introduced in §2 is violated. As will be shown in Appendix 1, the hydrodynamic problem can nevertheless easily be recast in a parametric form covering this situation; the equations can still be presumed to determine a smooth evolutionary process, and a quasi-Hamiltonian formalism is applicable. But, taken forward in time, the solution describing S will eventually

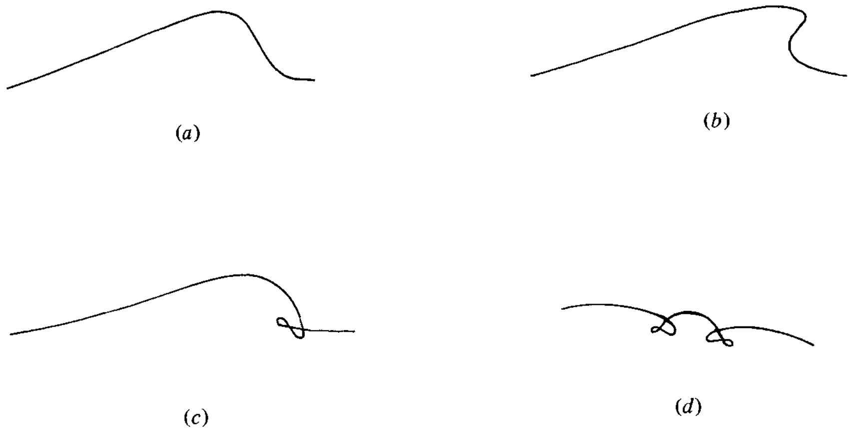

FIGURE 1. Eventualities in extreme stages of evolution.

present the situation in figure 1(*c*), where the same part of space is occupied twice by the fluid. Note that neither the mass-conservation condition $\operatorname{div}\mathbf{u} = 0$ nor any specification of the parameterized problem precludes such behaviour, although it is, of course, physically impossible.

Thus an extra, overriding constraint suddenly arrests the previously smooth evolutionary process. Unbounded decelerations of the liquid occur at the instant of impact between two different points of S, and it is highly plausible to explain the generation of turbulence, air-entrainment and other real-fluid effects as the sequel to the abrupt cancellation of the preceding hydrodynamic model. The situation is roughly comparable with the interruption of Hamiltonian structure when collisions occur in the N-body problem.

The suggested manner of breakdown is readily understood with regard to 'plunging breakers', which feature a definite, more or less two-dimensional jet of water being projected forward of the wave crest to fall on the part of S ahead. We emphasize, however, that in principle the explanation may relate equally well to three-dimensional as to two-dimensional situations, and to phenomena such as 'spilling breakers' where the impact of different parts of S is less plainly involved. Thus, according to our view, incipient breaking in any of its various manifestations (that is, whenever 'white water' first appears in practice) may be similarly accountable to the suppression of an impossible situation determined by Hamiltonian evolution. Such interruptions of a previously smooth, energy-conserving process may occur on a small, even microscopic scale, not merely in the prominent manner of a plunging breaker. Two localized impossibilities, such as may perhaps initially underlie the turbulent zone of a spilling breaker, are sketched in figure 1(*d*). Note that the reason for breakdown is not a singularity in the usual mathematical sense, nor in any precise sense an instability, but is rather a smooth departure of the Hamiltonian model from its approximate intersection with the physical world.

4. Symmetries of the water-wave problem

Our object here is to establish the complete symmetry group for the problem specified in §2, both its two-dimensional and three-dimensional versions and both with and without surface tension. The methods used to find the symmetries are developed from the theory of transformation groups and from prolongation theory, as will be summarized in Appendix 2. Accordingly, we postpone exposition of the underlying mathematics until the end of this section, first presenting the main findings in two theorems and proceeding to some discussion of their physical meaning, before taking up the technicalities of the proofs. In fact, once any particular group is given as in the theorems, it is fairly straightforward to check from the equations of the problem whether or not this is a symmetry group. The detailed and admittedly complex computations used to prove the theorems nevertheless have a twofold purpose. They provide a *systematic* means of finding symmetry groups, and they give assurance finally that all possible symmetries of the kind under consideration have been found.

For the sake of readers primarily interested in practical applications, we shall state the results first in the form of explicit group transformations, rather than their infinitesimal generators which are the prior outcome of our computations. The groups to be specified all depend on a finite number of continuous parameters. We thus ignore *discrete* symmetries, such as reflections, which are not connected continuously to the identity map (as recovered by the zero or unit value of parameters in the continuous case). This mild delimitation of our aims, being tied to our use of infinitesimal methods, is considered to miss nothing of importance.

The following theorems state that the full symmetry group respective to each specified version of the water-wave problem is 'generated' by a certain number of one-parameter subgroups. This means that for the respective problem any symmetry (of the kind with continuous connection to the identity) can be constructed by applying a finite number of the component symmetries in succession. [Recall, for example, that the group of rotations in $\mathbb{R}^3$ has a corresponding representation, being generated by the three one-parameter subgroups of rotations about three cartesian axes.]

The given subgroups may be considered to act geometrically on the four-dimensional space with coordinates (x, y, t, ϕ) for the two-dimensional physical problem, or the five-dimensional space with coordinates (x, y, z, t, ϕ) for the three-dimensional problem. Each group correspondingly induces a transformation in the space of solutions, in effect transforming the graphs of the free surface and of the velocity potential [cf. (A 9) in Appendix 2]. The key point is that transforming a given solution by any of the symmetries produces a continuous family of other solutions. The theorems first specify the geometrical transformations, listing expressions for the transformed independent and dependent variables. Then the corresponding new solutions found from the group transformations are expressed in terms of the original free-surface elevation and potential. In these expressions, $\epsilon \in \mathbb{R}$ denotes each *additive* group parameter (so that the composition of transformations determined respectively by ϵ and ϵ', say, is the same as the transformation determined by $\epsilon + \epsilon'$), and $\lambda > 0$ denotes each *multiplicative* parameter. Note that a multiplicative parameter λ can be considered as e^{ϵ}.

4.1. *The main results*

THEOREM 4.1. *The full symmetry group for the two-dimensional water-wave problem in the absence of surface tension (i.e. (2.9) with $\sigma = 0$) is generated by the following nine one-parameter subgroups:*

Horizontal translation

$$G_1: \ (x+\epsilon, y, t, \phi),$$

Time translation

$$G_2: \ (x, y, t+\epsilon, \phi),$$

Variation of base-level for potential

$$G_3: \ (x, y, t, \phi+\epsilon),$$

Vertical translation

$$G_4: \ (x, y+\epsilon, t, \phi-\epsilon g t),$$

Horizontal Galilean boost

$$G_5: \ (x+\epsilon t, y, t, \phi+\epsilon x+\tfrac{1}{2}\epsilon^2 t),$$

Vertical Galilean boost

$$G_6: \ (x, y+\epsilon t, t, \phi+\epsilon(y-\tfrac{1}{2}gt^2)+\tfrac{1}{2}\epsilon^2 t),$$

Vertical acceleration

$$G_7: \ (x, y+\tfrac{1}{2}gt^2(1-\lambda^{-2}), \lambda^{-1}t, \lambda\{\phi+gty(1-\lambda^{-2}) + \tfrac{1}{6}g^2t^3(1-3\lambda^{-2}+2\lambda^{-4})\}),$$

Gravity-compensated rotation

$$G_8: \ (x\cos\epsilon+(y+\tfrac{1}{2}gt^2)\sin\epsilon, \ -x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\cos\epsilon-\tfrac{1}{2}gt^2, \ t, \ \phi+gt\{x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\ (1-\cos\epsilon)\}),$$

Scaling

$$G_9: \ (\lambda x, \lambda y, \lambda^{\frac{1}{2}}t, \lambda^{\frac{3}{2}}\phi).$$

New solutions given by transforming the free-surface elevation $\eta(x, t)$ *and velocity potential* $\phi(x, y, t)$ *are as follows:*

$$G_1: \ \eta(x-\epsilon, t), \quad \phi(x-\epsilon, y, t),$$

$$G_2: \ \eta(x, t-\epsilon), \quad \phi(x, y, t-\epsilon),$$

$$G_3: \ \eta(x, t), \quad \phi(x, y, t)+\epsilon,$$

$$G_4: \ \eta(x, t)+\epsilon, \quad \phi(x, y-\epsilon, t)-\epsilon g t,$$

$$G_5: \ \eta(x-\epsilon t, t), \quad \phi(x-\epsilon t, y, t)+\epsilon x-\tfrac{1}{2}\epsilon^2 t,$$

$$G_6: \ \eta(x, t)+\epsilon t \quad \phi(x, y-\epsilon t, t)+\epsilon(y-\tfrac{1}{2}gt^2)-\tfrac{1}{2}\epsilon^2 t$$

$$G_7: \ \eta(x, \lambda t)-\tfrac{1}{2}gt^2(1-\lambda^2),$$

$$\lambda\phi(x, y+\tfrac{1}{2}gt^2(1-\lambda^2), \lambda t)-gty(1-\lambda^2)-\tfrac{1}{6}g^2t^3(1-3\lambda^2+2\lambda^4),$$

$$G_8: \ \eta(\hat{x}, t)\cos\epsilon-\hat{x}\sin\epsilon+\tfrac{1}{2}gt^2(\cos\epsilon-1),$$

$$\phi(x\cos\epsilon-(y+\tfrac{1}{2}gt^2)\sin\epsilon, \ x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\cos\epsilon-\tfrac{1}{2}gt^2, \ t) + gt\{x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\ (\cos\epsilon-1)\},$$

where $\hat{x}(x, t)$ *is determined implicitly by*

$$x = \hat{x}\cos\epsilon-\{\eta(\hat{x}, t)+\tfrac{1}{2}gt^2\}\sin\epsilon,$$

$$G_9: \ \lambda\eta(\lambda^{-1}x, \lambda^{-\frac{1}{2}}t), \quad \lambda^{\frac{3}{2}}\phi(\lambda^{-1}x, \lambda^{-1}y, \lambda^{-\frac{1}{2}}t).$$

When surface tension is operative ($\sigma > 0$), *all the above groups except* G_7 *and* G_9 *remain symmetries. In this case, moreover, the following 'combination' of* G_7 *and* G_9 *(in effect* $G_9 \circ G_7$ *with* $\lambda_7 = \lambda^{-1}$, $\lambda_9 = \lambda$) *remains a symmetry:*

Scaled acceleration

$$\hat{G}_7: \ (\lambda x, \lambda\{y+\tfrac{1}{2}gt^2(1-\lambda^2)\}, \quad \lambda^{\frac{3}{2}}t, \lambda^{\frac{1}{2}}\{\phi+gty(1-\lambda^2)+\tfrac{1}{6}g^2t^3(1-3\lambda^2+2\lambda^4)\}),$$

whose action on solutions is

$$\lambda\eta(\lambda^{-1}x,\ \lambda^{-\frac{3}{2}}t)-\tfrac{1}{2}gt^2(1-\lambda^{-2}),$$
$$\lambda^{\frac{1}{2}}\phi(\lambda^{-1}x,\ \lambda^{-1}\{y+\tfrac{1}{2}gt^2(1-\lambda^{-2})\},\ \lambda^{-\frac{3}{2}}t)-gty(1-\lambda^{-2})-\tfrac{1}{6}g^2t^3(1-3\lambda^{-2}+2\lambda^{-4}).$$

The physical interpretations attaching to the various symmetry groups noted in this theorem are reasonably clear. The first four may seem trivial, being recognizable with little thought to be symmetries; however, it is crucial to list them since they will be shown in §5 to underline significant conservation laws. The Galilean boosts G_5 and G_6 represent the effects observed from frames of reference moving uniformly in the horizontal and vertical directions respectively. The group G_7 represents the corresponding effects for a frame that is accelerating uniformly in the vertical direction, so modifying the effective gravity constant. To understand the rotation group G_8, it is helpful to consider the special case where $g = 0$. Plainly, since there is then no preferred direction, any solution will remain a solution after being rotated about an arbitrary point in the (x, y)-plane. It can hence be appreciated that when $g \neq 0$ the group G_8 transforms solutions by first accelerating vertically to cancel the effect of gravity and then rotating. Finally, the scaling group G_9 is easily understood, being evident from dimensional considerations. For example, when $\sigma = 0$ and so g is the only physical parameter including time in its dimensions, periodic waves of permanent form on water of unbounded depth evidently have a speed proportional to $g^{\frac{1}{2}}$, and hence to $\lambda^{\frac{1}{2}}$ in the case that both their amplitude and wavelength are varied in proportion to λ.

Several remarks are pertinent concerning the influence of further boundary conditions, both conditions at the bottom if the depth of water is finite and asymptotic conditions as $|x| \to \infty$. First note that the imposition of such additional constraints on solutions can only decrease the number of symmetries. For example, under the restriction that $|\eta| \to 0$ as $|x| \to \infty$, only G_1, G_2, G_3, G_5 and G_9 are symmetry groups, preserving this property of solutions upon transformation. For waves on infinitely deep water that are periodic in x (not necessarily in t), all the groups except G_8 are relevant, although the mean height of the free surface changes according to G_4, G_6 and G_7. For finite geometries the reduction in symmetries is even more severe. In the case of a horizontal rigid bottom at finite depth, only G_1, G_2, G_3 and G_5 are symmetry groups; and for waves on water contained in a closed basin of any shape, only G_2 and G_3 remain. The non-physical character of some of the transformations should not, however, be judged to disqualify them from serious consideration. The relationship of all the present symmetry groups with conservation laws will justify our attention to them.

Corresponding results for the three-dimensional problem are given next:

THEOREM 4.2. *The full symmetry group for the three-dimensional water-wave problem in the absence of surface tension is generated by the following thirteen subgroups*:

Horizontal translations

$$G_1:\ (x+\epsilon,\ y,\ z,\ t,\ \phi),\quad G_2:\ (x,\ y,\ z+\epsilon,\ t,\ \phi),$$

Time translation

$$G_3:\ (x,\ y,\ z,\ t+\epsilon,\ \phi),$$

Variation of base-level for potential

$$G_4:\ (x,\ y,\ z,\ t,\ \phi+\epsilon),$$

Vertical translation

$$G_5:\ (x,\ y+\epsilon,\ z,\ t,\ \phi-\epsilon gt),$$

Horizontal rotation

$$G_6: \ (x\cos\epsilon - z\sin\epsilon,\ y,\ x\sin\epsilon + z\cos\epsilon,\ t,\ \phi),$$

Horizontal Galilean boosts

$$G_7: \ (x+\epsilon t,\ y,\ z,\ t,\ \phi+\epsilon x+\tfrac{1}{2}\epsilon^2 t),$$

$$G_8: \ (x,\ y,\ z+\epsilon t,\ t,\ \phi+\epsilon z+\tfrac{1}{2}\epsilon^2 t),$$

Vertical Galilean boost

$$G_9: \ (x,\ y+\epsilon t,\ z,\ t,\ \phi+\epsilon(y-\tfrac{1}{2}gt^2)+\tfrac{1}{2}\epsilon^2 t),$$

Vertical acceleration

$$G_{10}: \ (x,\ y+\tfrac{1}{2}gt(1-\lambda^{-2}),\ z,\ \lambda^{-1}t,\ \lambda\{\phi+gty(1-\lambda^{-2})+\tfrac{1}{6}g^2t^3(1-3\lambda^{-2}+2\lambda^{-4})\}),$$

Gravity-compensated rotations

$$G_{11}: \ (x\cos\epsilon+(y+\tfrac{1}{2}gt^2)\sin\epsilon,\ -x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\cos\epsilon-\tfrac{1}{2}gt^2,$$
$$z,\ t,\ \phi+gt\{x\sin\epsilon+(y+\tfrac{1}{2}gt^2)\ (1-\cos\epsilon)\}),$$

$$G_{12}: \ (x,\ -z\sin\epsilon+(y+\tfrac{1}{2}gt^2)\cos\epsilon-\tfrac{1}{2}gt^2,\ z\cos\epsilon+(y+\tfrac{1}{2}gt^2)\sin\epsilon,$$
$$t,\ \phi+gt\{z\sin\epsilon+(y+\tfrac{1}{2}gt^2)\ (1-\cos\epsilon)\}),$$

Scaling

$$G_{13}: \ (\lambda x,\ \lambda y,\ \lambda z,\ \lambda^{\frac{1}{2}}t,\ \lambda^{\frac{3}{2}}\phi).$$

When surface tension is operative, all the above groups except G_{10} and G_{13} remain symmetries, and the following 'combination' of G_{10} and G_{13} remains a symmetry:

Scaled acceleration

$$\hat{G}_{10}: \ (\lambda x,\ \lambda\{y+\tfrac{1}{2}gt^2(1-\lambda^2)\},\ \lambda z,\ \lambda^{\frac{3}{2}}t,\ \lambda^{\frac{1}{2}}\{\phi+gty(1-\lambda^2)+\tfrac{1}{6}g^2t^3(1-3\lambda^2+2\lambda^4)\}).$$

Note that, except for the horizontal-rotation group G_6, all the symmetry groups noted in this theorem are analogues of those for the two-dimensional problem. With the one exception, expressions for transformed solutions exactly correspond to those given in the second part of Theorem 4.1, and so can be omitted for brevity. The new solutions in the remaining case are

$$G_6: \ \eta(x\cos\epsilon+z\sin\epsilon,\ -x\sin\epsilon+z\cos\epsilon,\ t),$$
$$\phi(x\cos\epsilon+z\sin\epsilon,\ y,\ -x\sin\epsilon+z\cos\epsilon,\ t).$$

4.2. *Derivations*

We now take up the proofs of the above theorems, applying the infinitesimal-transformation methods that are explained generally in Appendix 2. Since here and in §5 the mathematical arguments are necessarily quite technical, it is advised that the reader mainly interested in the physical bearing of the demonstrated symmetries and conservation laws can skip ahead to §6. For simplicity of illustration we concentrate on the two-dimensional problem without surface tension, but derivations of the other results proceed in a precisely similar way.

Consider the free-boundary problem (2.9) with $\sigma = 0$. Referring to the context of (A 10) in Appendix 2 for explanations of terms, take the vector field

$$\mathbf{v} = \alpha\partial_x+\beta\partial_y+\tau\partial_t+\gamma\partial_\phi,$$

where α, β, τ, γ are functions of (x, y, t, ϕ), to be the infinitesimal generator of a one-parameter group of symmetries for this problem. According to (A 21), the required invariance of Laplace's equation in the problem implies that the prolongation

$$\mathrm{pr}\ \mathbf{v}(\Delta\phi) = \delta\phi_{xx}+\delta\phi_{yy} = 0 \quad \text{whenever} \quad \Delta\phi = 0. \tag{4.1}$$

The functions $\delta\phi_{xx}$ and $\delta\phi_{yy}$ are computable from (A 13); but (4.1) simply means, of course, that **v** is the generator of a group of conformal transformations in (x, y)-space. It can be shown (cf. Ovsiannikov 1982) that consequently **v** is a *projectable* vector field, which means that α, β, τ are independent of ϕ. Since this fact greatly simplifies the remaining calculations, we shall at once resort to it. Although with a little extra trouble it can be confirmed directly in the present example, we choose rather to rely on Ovsjannikov's general result that symmetries tied to Laplace's equation are projectable. Moreover, since the conformal group in two dimensions is much larger than the group of symmetries expected to be delimited mainly by the free-boundary conditions of the problem (2.9), we can concentrate instead on the symmetry criteria that apply at the free surface S.

The prolongation of **v** to S is

$$\mathrm{pr}\,\mathbf{v}_S = \alpha_S\,\partial_x + \beta_S\,\partial_y + \tau_S\,\partial_t + \gamma_S\,\partial\phi + (\delta\phi_x)_S\left(\frac{\partial}{\partial\Phi_{(x)}}\right)$$
$$+ (\delta\phi_y)_S\left(\frac{\partial}{\partial\Phi_{(y)}}\right) + (\delta\phi_t)_S\left(\frac{\partial}{\partial\Phi_{(t)}}\right) + \delta\eta_x\left(\frac{\partial}{\partial\eta_x}\right) + \delta\eta_t\left(\frac{\partial}{\partial\eta_t}\right),$$

where the subscripts S and (x), etc., have the meanings introduced in §2 and

$$\begin{aligned}
\delta\phi_x &= D_x\gamma - \phi_x\alpha_x - \phi_y\beta_x - \phi_t\tau_x,\\
\delta\phi_y &= D_y\gamma - \phi_x\alpha_y - \phi_y\beta_y - \phi_t\tau_y,\\
\delta\phi_t &= D_t\gamma - \phi_x\alpha_t - \phi_y\beta_t - \phi_t\tau_t,\\
\delta\eta_x &= D_x(\beta_S) - \eta_x D_x(\alpha_S) - \eta_t D_x(\tau_S),\\
\delta\eta_t &= D_t(\beta_S) - \eta_x D_t(\alpha_S) - \eta_t D_t(\tau_S).
\end{aligned}$$

In these expressions D denotes total derivatives: for instance, $D_x\gamma = \gamma_x + \gamma_\phi\phi_x$. Note that

$$D_x(\beta_S) = (D_x\beta)_S + (\beta_y)_S\,\eta_x = (\beta_x)_S + (\beta_y)_S\,\eta_x,$$

and similarly for $D_t(\beta_S)$. Note also that total derivatives of γ only are needed in the first three expressions, since α, β, τ are known to be independent of ϕ.

Now, for the two nonlinear equations in (2.9) applying at the free surface (with $\sigma = 0$), the symmetry conditions represented in general by (A 22) become

$$\delta\eta_t - \delta\phi_y + \eta_x\,\delta\phi_x + \phi_x\,\delta\eta_x = 0, \tag{4.2}$$

$$\delta\phi_t + \phi_x\,\delta\phi_x + \phi_y\,\delta\phi_y + g\beta = 0, \tag{4.3}$$

evaluated at S. We proceed to analyse (4.2) and (4.3) by use of the above prolongation formula, substituting for η_t and $\Phi_{(t)}$ from (2.9) wherever they occur. Since these conditions are required to hold over the whole class of solutions to (2.9), the coefficients of each mononomial term in powers of η, η_x, $\Phi_{(x)}$ and $\Phi_{(y)}$ must vanish. The calculations on this basis are quite straightforward, and their outcome is as follows.

In (4.2) the highest-order terms, in $\eta_x^2\Phi_{(x)}^2$, cancel independently of α, β, τ, γ and so reveal nothing, as also do terms in $\eta_x^2\Phi_{(x)}$. For the only other cubic mononomials, the coefficients of $\eta_x\Phi_{(x)}^2$ and $\eta_x\Phi_{(y)}^2$ are non-zero multiples of τ_x, while that of $\eta_x\Phi_{(x)}\Phi_{(y)}$ is a non-zero multiple of τ_y. Therefore τ can depend only on t, which fact is also implied because τ_x occurs as a coefficient of $\Phi_{(x)}\Phi_{(y)}$ and $g\eta\eta_x$, while τ_y occurs as a coefficient of $\Phi_{(x)}^2$, $\Phi_{(y)}^2$ and $g\eta$.

It next follows from terms in $\eta_x\Phi_{(x)}$ that

$$\tau_t + \gamma_\phi = 2\alpha_x, \tag{4.4}$$

and from terms in either $\eta_x \Phi_{(y)}$ or $\Phi_{(x)}$ that

$$\alpha_y + \beta_x = 0. \tag{4.5}$$

Finally from (4.2), terms in $\Phi_{(y)}$, in η_x and then the remaining terms which do not depend on any derivatives of η or ϕ lead to

$$\tau_t + \gamma_\phi = 2\beta_y, \tag{4.6}$$

$$\alpha_t = \gamma_x, \tag{4.7}$$

$$\beta_t = \gamma_y. \tag{4.8}$$

A similar analysis of (4.3) is found to give equations that, with a single exception, duplicate ones already established. The new equation, obtained from the terms not involving derivatives of η or ϕ, is

$$\gamma_t + g\beta = gy(\gamma_\phi - \tau_t). \tag{4.9}$$

Proceeding to solve these equations, we first recognize (4.4) to imply that

$$\gamma = c_0(x, y, t)\,\phi + \chi(x, y, t),$$

where $c_0 = 2\alpha_x - \tau_t$. Then, since α, β and τ are independent of ϕ, equations (4.7)–(4.9) show that c_0 is a constant, and that

$$\chi_t = g\{(c_0 - \tau_t)\,y - \beta\}. \tag{4.10}$$

Hence (4.4)–(4.6) lead at once to

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2}(\tau_t + c_0)\,x + c_8(t)\,y + \mu(t), \\ \beta &= \tfrac{1}{2}(\tau_t + c_0)\,y - c_8(t)\,x + \nu(t), \end{aligned}\right\} \tag{4.11}$$

in which c_8, μ and ν depend on t alone. But (4.7) and (4.8) imply that $\alpha_{yt} = \beta_{xt}$, showing that c_8 is in fact a constant. [The choice of numeration for constants c_i will become clear presently.] Hence (4.7) and (4.8) show further that

$$\chi = \tfrac{1}{4}(x^2 + y^2)\,\tau_{tt} + x\mu_t + y\nu_t + \xi(t), \tag{4.12}$$

where ξ is another function of t alone.

By substituting (4.12) and the second of (4.11) into (4.10), we obtain the equations

$$\tau_{ttt} = 0, \qquad \mu_{tt} = gc_8,$$
$$\nu_{tt} = g(\tfrac{1}{2}c_0 - \tfrac{3}{2}\tau_t), \quad \xi_t = -g\nu,$$

whose general solution is

$$\begin{aligned} \tau &= c_{10}\,t^2 + (-c_7 + \tfrac{1}{2}c_9)\,t + c_2, \\ \mu &= \tfrac{1}{2}gc_8\,t^2 + c_5\,t + c_1, \\ \nu &= -\tfrac{1}{2}gc_{10}\,t^3 + gc_7\,t^2 + c_6\,t + c_4, \\ \xi &= \tfrac{1}{8}g^2c_{10}\,t^4 - \tfrac{1}{3}g^2c_7\,t^3 - \tfrac{1}{2}gc_6\,t^2 - gc_4\,t + c_3, \end{aligned}$$

where $c_1, \ldots, c_{10}$ are constants with $c_7 + \frac{3}{2}c_9 = c_0$. The use of these results to evaluate (4.11) and (4.12) for α, β and χ gives the most general infinitesimal symmetry of the free-boundary conditions; but we must verify finally that these symmetries generate conformal transformations. When the general expression obtained for χ is substituted into (4.1), we thus expose the requirement that $c_{10} = 0$.

A basis for the Lie algebra of infinitesimal symmetries is given by setting each of

$c_1, \ldots, c_9$ in turn equal to 1, with the rest equal to 0. We thus establish the precursory version of Theorem 4.1:

THEOREM 4.3. *The Lie algebra of infinitesimal symmetries for the two-dimensional water-wave problem in the absence of surface tension is spanned by the following nine vector fields:*

$$\begin{aligned}
&\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = \partial_\phi, \\
&\mathbf{v}_4 = \partial_y - gt\partial_\phi, \quad \mathbf{v}_5 = t\partial_x + x\partial_\phi, \\
&\mathbf{v}_6 = t\partial_y + (y - \tfrac{1}{2}gt^2)\,\partial_\phi, \\
&\mathbf{v}_7 = gt^2\partial_y - t\partial_t + (\phi + 2gty - \tfrac{1}{3}g^2t^3)\,\partial_\phi, \\
&\mathbf{v}_8 = (y + \tfrac{1}{2}gt^2)\,\partial_x - x\,\partial_y + gtx\,\partial_\phi, \\
&\mathbf{v}_9 = x\partial_x + y\partial_y + \tfrac{1}{2}t\partial_t + \tfrac{3}{2}\phi\partial_\phi.
\end{aligned}$$

Exponentiation of each vector field $\mathbf{v}_j$ specified in this theorem, which operation amounts to solving the corresponding system of ordinary differential equations (A 11), gives the one-parameter symmetry groups G_j of Theorem 4.1. The most general one-parameter symmetry group for the two-dimensional problem can be found by exponentiating an arbitrary linear combination of these nine vector fields; however, no further enlightenment is to be gained by spelling out this formula.

To obtain the corresponding result allowing for surface tension, the simplest way is to test which of the given symmetries or linear combinations of them are unaffected by the term $\sigma\eta_{xx}/(1+\eta_x^2)^{\frac{3}{2}}$ added to the dynamical free-boundary condition. While the sufficiency of such an approach is in general not automatically ensured, it is straightforward to confirm in the present case that no extra symmetry is induced by the additional term. The calculations covering the three-dimensional problem are very similar to those establishing Theorem 4.3, and they are marginally simplified by the fact that the conformal group in three dimensions admits only ten parameters.

5. Derivation of conservation laws

Noether's famous theorem states that every one-parameter group of symmetries for a variational problem determines a conservation law satisfied by solutions of the corresponding Euler–Lagrange equations. The general principle pinpointed by the theorem will now be applied to find conservation laws for the water-wave problem, but there are two basic drawbacks to a direct use of Noether's theorem for this purpose. First, the equations of the problem are in Hamiltonian form, and it is inexpedient to recast them as Euler–Lagrange equations. Second, not every symmetry tied to Euler–Lagrange equations (particularly any scaling symmetry) is necessarily a symmetry of the respective variational problem (for a full discussion of this point, see Olver 1980*b*, §4.1). These disadvantages have been obviated in a new, comprehensive theory that has been developed by Olver (1980*a*) concerning the relationship between symmetries and conservation laws for Hamiltonian systems. We present as follows an outline of this theory, covering needs for the present application.

5.1. *Needed generalities*

Consider an evolution equation in the abstract Hamiltonian form

$$\boldsymbol{\phi}_t(\mathbf{x}, t) = J \operatorname{grad} H(\boldsymbol{\phi}), \tag{5.1}$$

where $\boldsymbol{\phi} = (\phi^1, \ldots, \phi^q)$ is the set of dependent variables, $\mathbf{x} = (x^1, \ldots, x^p)$ the independent space variables, and $J = (J_{ij})$ is a $q \times q$ skew-symmetric matrix of constant-coefficient differential operators. [The conclusions to be summarized can be extended to certain cases of J depending on $\boldsymbol{\phi}$ and its spatial derivatives. An additional assumption, ensuring a closure condition on an associated symplectic form, is then needed to make (5.1) Hamiltonian, but we can pass over this generalization here.] A scalar C^∞ function T of $\mathbf{x}$, t, $\boldsymbol{\phi}$ and spatial derivatives of $\boldsymbol{\phi}$ is called a *conserved density* in a *conservation law* for (5.1) if, for any region $\Omega \subset \mathbb{R}^p$ with piecewise smooth boundary $\partial\Omega$ and any interval $t_0 \leqslant t \leqslant t_1$, all solutions $\boldsymbol{\phi}$ of (5.1) satisfy

$$\int_\Omega T(\mathbf{x}, t, \boldsymbol{\phi})\, dx \bigg|_{t=t_0}^{t=t_1} = \int_{t_0}^{t_1} \left\{ \int_{\partial\Omega} \mathbf{A}.d\mathbf{S} \right\} dt, \tag{5.2}$$

for some p-tuple of functions $\mathbf{A}$. In other words, the difference in a spatial integral of T at two different instants depends only on the behaviour of $\boldsymbol{\phi}$ and its derivatives at the boundary of the region in question. In particular, if $\boldsymbol{\phi}$ and its derivatives vanish on $\partial\Omega$, then $\int_\Omega T\,dx$ will be independent of time t. Note that according to (5.2), if T is a conserved density, so also is $T + \mathrm{Div}\,\mathbf{P}$ for any smooth p-tuple $\mathbf{P}$. Conserved densities thus differing by a divergence will be called *equivalent*.

We also need to consider conservation properties associated with one-forms. A one-form is a finite sum having the representation

$$\omega = \sum_{j,k} P_k^j\, d\phi_k^j,$$

where a standard notation explained below (A 14) in Appendix 2 is used (ϕ_k^j with $j = 1, \ldots, q$ and the multi-indices k is a representation of all the kth and lower derivatives of ϕ^j) and where the P_k^j are functions of $\mathbf{x}$, t, $\boldsymbol{\phi}$ and its derivatives as above. The one-form ω is said to be conserved if, for any region Ω and any $t_0 < t_1$ as before, an equality

$$\int_\Omega \int_0^1 P_k^j \frac{\partial \phi_k^j}{\partial \lambda}\, d\lambda\, dx \bigg|_{t=t_0}^{t=t_1} = \int_{t_0}^{t_1} \left\{ \int_{\partial\Omega} \mathbf{A}.d\mathbf{S} \right\} dt \tag{5.3}$$

holds for every one-parameter family of solutions of (5.1), written $\boldsymbol{\phi}(\mathbf{x}, t, \lambda)$ with $0 \leqslant \lambda \leqslant 1$. Again as before, if ω is conserved, so also is $\omega' = \omega + \mathrm{Div}\,\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is an arbitrary p-tuple of one-forms, and ω' is said to be equivalent to ω. [Note that the p terms in the divergence here can be worked out according to $D_i(P^j d\phi^j) = (D_i P^j)\, d\phi^j + P^j d\phi_i^j$, etc.] Hence it is easily seen from integrations by parts that every one-form is equivalent to another with the special, generally simpler representation

$$\omega = \sum_j P^j\, d\phi^j.$$

Corresponding to a given conserved density T, there is a conserved one-form defined by

$$\omega = dT = \Sigma \left(\frac{\partial T}{\partial \phi_k^j} \right) d\phi_k^j, \tag{5.4}$$

which is called the *exterior derivative* of T. Conversely, if ω is a conserved one-form equivalent to dT for some T, then this T is a conserved density for solutions of (5.1). The present method of deriving conservation laws for (5.1) accordingly consists in associating a conserved one-form with every symmetry group of (5.1), and then seeing which of these one-forms are equivalent to the exterior derivative of some density. It may be noted that these conserved one-forms are generalizations of wave action as defined by Hayes (1970) in that they provide conservation properties of one-parameter families rather than single solutions.

The needed theorem, proved in Olver (1980*a*), can be stated as follows:

THEOREM 5.1. *Let*

$$\mathbf{v} = \sum_i \alpha^i \frac{\partial}{\partial x^i} + \tau \frac{\partial}{\partial t} + \sum_j \gamma^j \frac{\partial}{\partial \phi^j}$$

be the infinitesimal generator of a one-parameter symmetry group of (5.1) [*see context of* (A 10) *in Appendix* 2]. *Then, if*

$$\sum_j J_{ij} P^j = \gamma^i - \sum_j \alpha^j \phi^i_j - \tau \phi^i_t, \tag{5.5}$$

the one-form $\omega = \sum P^i d\phi^i$ *is conserved.* [*In* (5.5), ϕ^i_t *can be evaluated from* (5.1).] *Moreover, if a T exists such that* ω *is equivalent to dT, or correspondingly*

$$P^i = \delta T / \delta \phi^i$$

[*here* $\delta/\delta\phi^i$ *denotes the variational derivative with respect to* ϕ^i, *i.e. the gradient, or functional derivative, of* $\iint T\, d\mathbf{x}\, dt$ *in the sense used in* §2], *then T is a conserved density for* (5.1).

Note that in the general case where J is a matrix of differential operators, $\mathbf{P}$ is not always well defined by (5.5). In the present application, however, J is an invertible matrix of constants, and so this difficulty does not arise.

5.2. *Present application*

Returning now to the water-wave problem, we recollect that the dependent variables are η and $\Phi = \phi_S$, which is the restriction, to the free surface S, of the underlying solution of $\Delta\phi = 0$ in D_η. Also, according to (2.8), the operation J is just the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applied to this free-boundary problem, as explained generally in Appendix 2, Theorem 5.1 tells us that if the vector field

$$\mathbf{v} = \alpha\partial_x + \beta\partial_y + \tau\partial_t + \gamma\partial_\phi$$

is the infinitesimal generator of a symmetry, then the one-form

$$\omega = -(\gamma - \alpha\Phi_x - \tau\Phi_t)\, d\eta + (\beta - \alpha\eta_x - \tau\eta_t)\, d\Phi$$

is conserved. [Note that here $\Phi_x = \Phi_{(x)} + \Phi_{(y)}\eta_x$, and similarly for the t-derivative.] Using this criterion, we proceed to test the various infinitesimal symmetries listed in Theorem 4.3.

To recover $\mathbf{v}_1$, we take $\alpha = 1$, $\beta = \tau = \gamma = 0$, and hence from (5.5) find the respective conserved one-form to be $\omega_1 = \Phi_x\, d\eta - \eta_x\, d\Phi$. As plainly appears from an integration by parts, this one-form is equivalent to

$$\hat{\omega}_1 = -\Phi d\eta_x - \eta_x\, d\Phi = -d(\eta_x \Phi),$$

showing $T_1 = -\eta_x\Phi$ to be a conserved density.

Similarly, the conserved one-form associated with $\mathbf{v}_2$ is seen to be

$$\omega_2 = \Phi_t\, d\eta - \eta_t\, d\Phi = -E_\eta\, d\eta - E_\Phi\, d\Phi$$

in the notation of (2.8). By a simple argument essentially retracing the steps that led to (2.8), it hence appears that ω_2 is equivalent to $-d\mathscr{H}$, where $\mathscr{H} = \frac{1}{2}\Phi\Phi_{(n)} R + \frac{1}{2}g\eta^2$ is the Hamiltonian density function whose integral over $\mathbb{R}$ equals E. This conclusion reproduces the fundamental fact that for any Hamiltonian system (5.1), the Hamiltonian density is conserved, which fact is of course attributable to the time-invariance (autonomy) of such a system.

The one-form corresponding to $\mathbf{v}_3$ is evidently $\omega_3 = d\eta$, which confirms $T_3 = \eta$ to be a conserved density; and that corresponding to $\mathbf{v}_4$ is $\omega_4 = gt\,d\eta + d\Phi$, which gives $T_4 = gt\eta + \Phi$ as the conserved density. In the same way, $\mathbf{v}_5$ leads to $T_5 = x\eta + t\eta_x\Phi$, $\mathbf{v}_6$ to $T_6 = \frac{1}{2}\eta^2 - t\Phi - \frac{1}{2}gt^2\eta$, and $\mathbf{v}_8$ to $T_8 = x\Phi + \eta\eta_x\Phi + gtx\eta + \frac{1}{2}gt^2\eta_x\Phi$.

On the other hand, with $\sim$ denoting equivalence, the one-forms generated by $\mathbf{v}_7$ and $\mathbf{v}_9$ are found to be

$$\begin{aligned}\omega_7 &= -(\Phi + 2gt\eta - \tfrac{1}{3}g^2t^3 + t\Phi_t)\,d\eta + (gt^2 + t\eta_t)\,d\Phi \\ &\sim d(t\mathscr{H} - gt\eta^2 + gt^2\Phi + \tfrac{1}{3}g^2t^3\eta) - \Phi d\eta\end{aligned}$$

and

$$\begin{aligned}\omega_9 &= -(\tfrac{3}{2}\Phi - x\Phi_x - \tfrac{1}{2}t\Phi_t)\,d\eta + (\eta - x\eta_x - \tfrac{1}{2}t\eta_t)\,d\Phi \\ &\sim d(-\tfrac{1}{2}t\mathscr{H} - x\eta_x\Phi + \eta\Phi) - \tfrac{7}{2}\Phi\,d\eta,\end{aligned}$$

neither of which is the exact derivative of a density. Plainly, however, the linear combination $\omega_9 - (7/2)\,\omega_7$ has the required property, and thus it transpires that

$$\begin{aligned}T_7 &= (\eta - x\eta_x)\Phi - 4t\mathscr{H} + \tfrac{7}{2}gt\eta^2 - \tfrac{7}{2}gt^2\Phi - \tfrac{7}{6}g^2t^3\eta \\ &= (\eta - x\eta_x)\Phi - t(4\mathscr{H} - 7gT_6) + \tfrac{7}{2}gt^2T_4 - \tfrac{7}{6}g^2t^3T_3\end{aligned}$$

is a conserved density.

Collecting the results and anticipating physical connotations of the first three conserved densities, we have:

THEOREM 5.2. *The two-dimensional water-wave problem* (2.9) *in the absence of surface tension has the following eight conserved densities:*

$$\begin{aligned}
T_1 &= -\eta_x\Phi \quad \text{(horizontal impulse)},\\
T_2 &= \mathscr{H} = \tfrac{1}{2}\Phi(\Phi_{(y)} - \eta_x\Phi_{(x)}) + \tfrac{1}{2}g\eta^2 \quad \text{(energy)},\\
T_3 &= \eta \quad \text{(mass)},\\
T_4 &= \Phi + gtT_3,\\
T_5 &= x\eta - tT_1,\\
T_6 &= \tfrac{1}{2}\eta^2 - tT_4 + \tfrac{1}{2}gt^2T_3,\\
T_7 &= (\eta - x\eta_x)\Phi - t(4\mathscr{H} - 7gT_6) + \tfrac{7}{2}gt^2T_4 - \tfrac{7}{6}g^2t^3T_3\\
T_8 &= (x + \eta\eta_x)\Phi + gtT_5 + \tfrac{1}{2}gt^2T_1.
\end{aligned}$$

When surface tension is operative, $\mathbf{v}_9$ *is no longer a symmetry, and consequently the density* T_7 *is no longer conserved. Also, the appropriate term* $\sigma\{(1+\eta_x^2)^{\frac{1}{2}} - 1\}$ *is added to the Hamiltonian density* $\mathscr{H}$. *Apart from these two changes, the results remain as above.*

The eight functions $T_j = T_j(x, t)$ specified in this theorem are conserved densities in the general sense indicated by (5.2), which can now be appropriately particularized as follows. Take any time-dependent domain in $\mathbb{R}^2$ whose boundary is $\mathscr{S} \cup \Gamma$, comprising an interval of the free surface, $\mathscr{S} = \{(x, y) : y = \eta(x, t), a(t) \leqslant x \leqslant b(t)\}$ with end points $\partial\mathscr{S} = \{(a, \eta(a, t)), (b, \eta(b, t))\}$, and the remainder Γ underwater. Then the theorem can be read to mean that

$$\int_{a(t)}^{b(t)} T_j\,dx\,\Big|_{t=t_0}^{t=t_1} = \int_{t_0}^{t_1}\left\{A_j|_{\partial\mathscr{S}} + \int_\Gamma (B_j\,dx + C_j\,dy)\right\}dt \tag{5.6}$$

for certain A_j, B_j and C_j. It should *not* be supposed, however, that in the case of a localized wave motion at the surface of an infinitely deep ocean, both the velocity potential ϕ and surface elevation η necessarily decay fast enough with distance for all the boundary terms on the right-hand sides of the identities (5.6) to make no contribution when Γ is removed to infinity, so that for each T_j

$$\int_{-\infty}^{\infty} T_j \, dx$$

is independent of time. Such a supposition would lead to some absurd conclusions, for example concerning the conserved density T_6. The explicit interpretations of (5.6) will be examined carefully in the next section.

It should be noted that for a uniform ocean T_1 is in effect the density of horizontal *impulse*, in the sense named after Kelvin (cf. Lamb 1932, §119). For the case that the water lies on a rigid bottom $y = -h$ and is at rest ($|\nabla\phi| = 0$) in the limits $x \to \pm\infty$, integration of the obvious identity

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} \phi \, dy = \int_{-h}^{\eta} \phi_x \, dy + \Phi\eta_x = \int_{-h}^{\eta} u \, dy - T_1$$

shows that the total horizontal momentum of the wave motion is given by

$$M = \int_{-\infty}^{\infty}\int_{-h}^{\eta} u \, dy \, dx = (\phi_\infty - \phi_{-\infty})\, h + \int_{-\infty}^{\infty} T_1 \, dx. \tag{5.7}$$

For free wave motions (i.e. subsequent to the removal of external forces that generated them from rest), the difference between the asymptotic (constant) values ϕ_∞ and $\phi_{-\infty}$ of ϕ is independent of time, and is generally non-zero but finite. Thus M may be well defined in the case of finite depth, and the Kelvin impulse given by the integral of T_1 is then merely the balance between a determinate total momentum and the opposing 'reaction' of the fluid at infinity. In the limit $h \to \infty$, however, both these quantities become indeterminate, as is well known to be the outcome of an attempt to calculate directly the momentum of an unbounded expanse of incompressible fluid; but the difference between them, defining the Kelvin impulse, remains determinate. The significance of T_2 as an energy density has already been explained, and T_3 is evidently a density of mass relative to the motionless state of the fluid. Further physical connotations will be noted in §6.

Let us now state the result obtained in a precisely similar way for the three-dimensional problem:

THEOREM 5.3. *When not affected by surface tension, the three-dimensional water-wave problem* (2.2, 4, 5) *has the following twelve conserved densities:*

$$\begin{aligned}
T_1 &= -\Phi\eta_x, \quad T_2 = -\Phi\eta_z \quad \text{(horizontal impulses)},\\
T_3 &= \mathscr{H} = \tfrac{1}{2}\Phi\Phi_{(n)}R + \tfrac{1}{2}g\eta^2 \quad \text{(energy)},\\
T_4 &= \eta \quad \text{(mass)},\\
T_5 &= \Phi + gtT_4,\\
T_6 &= x\eta - tT_1, \quad T_7 = z\eta - tT_2,\\
T_8 &= \tfrac{1}{2}\eta^2 - tT_5 + \tfrac{1}{2}gt^2T_4,\\
T_9 &= (x\eta_z - z\eta_x)\,\Phi,\\
T_{10} &= (x + \eta\eta_x)\,\Phi + gtT_6 + \tfrac{1}{2}gt^2T_1,\\
T_{11} &= (z + \eta\eta_z)\,\Phi + gtT_7 + \tfrac{1}{2}gt^2T_2,\\
T_{12} &= (\eta - x\eta_x - z\eta_z)\,\Phi + t(9gT_8 - 5\mathscr{H}) + \tfrac{9}{2}gt^2T_5 - \tfrac{3}{2}g^2t^3T_4.
\end{aligned}$$

When surface tension is included, so that a term $\sigma(R-1)$ is added to T_3, all except T_{12} remain conserved densities.

The functions $T_j = T_j(x, z, t)$ listed in this theorem are conserved densities in the sense that twelve identities akin to (5.6) hold. The integrals on the left-hand sides are replaced by integrals over a two-dimensional horizontal domain, namely the projection of a patch $\mathscr{S}$ of the free surface; the terms evaluated $\partial\mathscr{S}$ in the integrands on the right-hand sides become line integrals; and the integrals over Γ become surface integrals.

It should be acknowledged finally that once the forms of the T_j are established through the consideration of symmetry groups, direct confirmation that they are conserved densities is a fairly straightforward matter. This aspect is to be exemplified below. We re-emphasize that present means of derivation has the prime advantage of being systematic, so presumably identifying all the relevant conservation laws; and we can assert the absence of unlimited hidden (artificial) symmetries such as might perhaps have been expected by analogy with, for example, the KdV equation.

6. Further investigation of conservation laws

The foregoing analysis revealed sets of conservation laws that hold in several versions of the water-wave problem, but it was not determined specifically how the functions shown to be conserved densities depend on the behaviour of solutions at the boundaries of arbitrary 'control volumes'. The matter will now be treated, including the delicate aspect pointed out after Theorem 5.2. Needless to say, we shall in part merely recover representations of mass, energy and momentum conservation on already well-known lines. However, special care is needed in dealing with the less familiar conserved densities later in the lists, and our approach extends and is guided by the systematic development of the preceding results. For the sake of brevity, only the case of two-dimensional wave motions uninfluenced by surface tension will be studied, but it will be clear how analogous findings can be obtained for the other cases that have so far been included.

In the first place, deferring ticklish questions about the asymptotic behaviour of solutions at large distances, we refer the conservation laws to an arbitrary bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D as indicated in figure 2. One possibility admitted is that D is the space within a rigid container partially filled by the fluid, but the more apposite possibility is that D includes only part of the total domain occupied by the fluid and part of the free surface. In general, fluid occupies a time-dependent subdomain $\mathscr{D}(t) \subset D$ whose boundary $\partial\mathscr{D}(t)$ consists of (i) that part of the free surface lying inside D, to be denoted by $\mathscr{S}$, and (ii) the intersection of $\partial\mathscr{D}$ with ∂D, to be denoted by Γ. It is assumed that $\mathscr{D}(t)$ is simply connected for all t, so that the velocity potential for the motion in $\mathscr{D}$ is single-valued. The intersection of the free surface with ∂D is supposed to comprise a finite even number of (moving) points p_n $(n = 1,2,\ldots)$, denoted collectively by $\partial\mathscr{S}$, and it is assumed that in a neighbourhood of each p_n respectively ∂D is described by $(x, \beta_n(x))$ with $\beta_n'(x) \neq 0$. (The simple case where Γ is vertical at any of the points $\partial\mathscr{S}$ will be treated separately.) The total number of points p_n may be greater than two, as exemplified in the figure. For example, we allow that fixed rigid obstacles may dip into the fluid.

Line integrals over the bounding curves Γ and $\mathscr{S}$ will be taken in the sense indicated by the arrows in figure 2, such that the interior of $\mathscr{D}$ lies on the right as the curves are traversed. This reversal of the customary orientation has the advantage that the

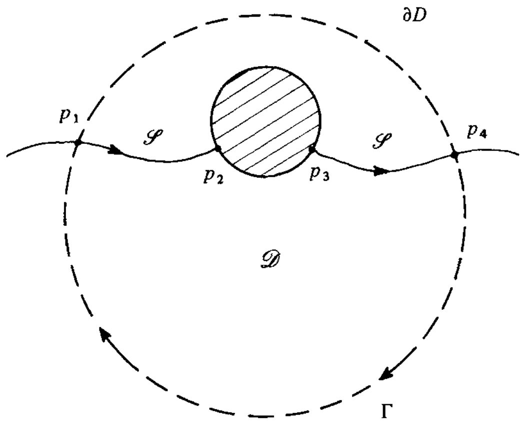

FIGURE 2. Illustration of arbitrary domain $D \subset \mathbb{R}^2$, whose time-dependent part $\mathscr{D}(t)$ is occupied by fluid.

free surface $\mathscr{S}$ is traversed in the direction of increasing x. We shall need to consider various line integrals in the form

$$G = G_\Gamma + G_{\mathscr{S}} = \oint_{\partial\mathscr{D}} (P\,dx + Q\,dy), \tag{6.1}$$

which vanish since $P_y = Q_x$ identically in $\mathscr{D}$. These are

$$\begin{aligned}
G^1 &= \oint\{-uv\,dx + \tfrac{1}{2}(u^2-v^2)\,dy\},\\
G^3 &= \oint(-v\,dx + u\,dy) = -\int(\partial_n\,\phi)\,ds,\\
G^4 &= \oint\{\tfrac{1}{2}(u^2-v^2)\,dx + uv\,dy\},\\
G^5 &= \oint\{-xv\,dx + (xu-\phi)\,dy\},\\
G^6 &= \oint\{(\phi - yv)\,dx + yu\,dy\},\\
G^7 &= \oint\{\tfrac{1}{2}(u^2-v^2)\,(y\,dx + x\,dy) - uv(x\,dx - y\,dy)\},\\
G^8 &= \oint\{\tfrac{1}{2}(u^2-v^2)\,(x\,dx - y\,dy) + uv(y\,dx + x\,dy)\},
\end{aligned}$$

being numbered in a way whose appropriateness will appear presently. The nullity of these integrals over $\partial\mathscr{D} = \Gamma \cup \mathscr{S}$, implying that $G^j_{\mathscr{S}} = -G^j_\Gamma$, is evident since the velocity field (u, v) is both solenoidal and irrotational (i.e. $u_x = -v_y$, $u_y = v_x$).

We shall also consider various non-vanishing integrals in the form (6.1). To find their t-derivatives, care is needed when dealing with them directly in this form; and it turns out that contributions arising from the variable end-points p_n of $\mathscr{S}$ and Γ are cancelled upon integrations by parts. The following derivation is perhaps simpler:

$$\begin{aligned}
\frac{d}{dt}\oint_{\partial\mathscr{D}} (P\,dx + Q\,dy) &= \frac{d}{dt}\iint_{\mathscr{D}} (P_y - Q_x)\,dx\,dy\\
&= \iint_{\mathscr{D}} (P_{yt} - Q_{xt})\,dx\,dy + \int_{\mathscr{S}} (P_y - Q_x)\,\eta_t\,dx\\
&= \int_\Gamma (P_t\,dx + Q_t\,dy) + \int_{\mathscr{S}} \{P_t + Q_t\,\eta_x + (P_y - Q_x)\,\eta_t\}\,dx. \qquad (6.2)
\end{aligned}$$

It is here assumed for simplicity that η is a single-valued, continuously differentiable function of x (although this assumption is in fact unessential to the main results that follow). Accordingly, dy in integrals over $\mathscr{S}$ is replaceable by $\eta_x\,dx$.

6.1. *Interpretation of conserved densities*

All except one of the following line integrals have straightforward physical meanings, made plain by the respective equivalent integrals over $\mathscr{D}$:

$$I^1 = -\oint_{\partial\mathscr{D}} \phi\, dy \quad \text{(horizontal momentum)},$$

$$I^2 = \oint_{\partial\mathscr{D}} \{\tfrac{1}{2}\phi(\partial_n \phi)\, ds + \tfrac{1}{2}gy^2\, dx\} \quad \text{(energy)},$$

$$I^3 = \oint_{\partial\mathscr{D}} y\, dx \quad \text{(mass)},$$

$$I^4 = \oint_{\partial\mathscr{D}} \phi\, dx \quad \text{(vertical momentum)},$$

$$I^5 = \oint_{\partial\mathscr{D}} xy\, dx \quad \text{(horizontal coordinate of mass centroid times } I^3\text{)},$$

$$I^6 = \oint_{\partial\mathscr{D}} \tfrac{1}{2}y^2\, dx \quad \text{(height of mass centroid times } I^3\text{)},$$

$$I^7 = \oint_{\partial\mathscr{D}} \phi(y\, dx - x\, dy),$$

$$I^8 = \oint_{\partial\mathscr{D}} \phi(x\, dx + y\, dy) \quad \text{(angular momentum)}.$$

Each integral has two components, thus

$$I^j = I^j_{\mathscr{S}} + I^j_{\Gamma}, \quad j = 1, \ldots, 8;$$

and when expressed as an integral with respect to x, each $I^j_{\mathscr{S}}$ is seen to be just the integral of the part of the respective conserved density T_j, as given in Theorem 5.2, that is not explicitly dependent on t. Therefore, according to Theorem 5.2, temporal changes of each $I^j_{\mathscr{S}}$, and hence of each I^j, depend only on (i) the behaviour of ϕ at the fixed part Γ of the boundary $\partial\mathscr{D}$ and (ii) for $j > 3$ the values of other conserved densities lower in the hierarchy.

To represent the boundary effects explicitly, we define

$$B^1 = G^1_{\Gamma} - \int_{\Gamma} (\phi_t + gy)\, dy = \int_{\Gamma p} \{p\,dy - u(v\,dx - u\,dy)\},$$

$$B^2 = \int_{\Gamma} \phi_t(\partial_n \phi)\, ds = \int_{\Gamma} \phi_t(v\, dx - u\, dy)$$

$$= -\int_{\Gamma} \{p + gy + \tfrac{1}{2}(u^2 + v^2)\}\,(v\, dy - u\, dx),$$

$$B^3 = G^3_{\Gamma},$$

$$B^4 = G^4_{\Gamma} + \int_{\Gamma} (\phi_t + gy)\, dx = -\int_{\Gamma} \{p\, dx + v(v\, dx - u\, dy)\},$$

$$B^5 = G^5_{\Gamma} + \int_{\Gamma} \phi\, dy = -\int_{\Gamma} x(v\, dx - u\, dy),$$

$$B^6 = G^6_\Gamma - \int_\Gamma \phi\, dx = -\int_\Gamma y(v\, dx - u\, dy),$$

$$B^7 = G^7_\Gamma + \int_\Gamma \{(\phi_t + gy)(y\, dx - x\, dy) - 2\phi(\partial_n \phi)\, ds\}$$

$$= -\int_\Gamma \{(xu + yv + 2\phi)(v\, dx - u\, dy) + p(y\, dx - x\, dy)\},$$

$$B^8 = G^8_\Gamma + \int_\Gamma (\phi_t + gy)(x\, dx + y\, dy)\}$$

$$= -\int_\Gamma \{(xv - yu)(v\, dx - u\, dy) - p(x\, dx + y\, dy)\}.$$

Here we use the identity $(\partial_n \phi)\, ds = (v\, dx - u\, dy)$ on Γ, and introduce the pressure p through the Bernoulli integral of the dynamical equations. The results in question can now be stated:

THEOREM 6.1. *Let η, ϕ be a solution of the two-dimensional water-wave problem in the absence of surface tension. Let the chosen domain $\mathscr{D}(t)$ be simply connected (cf. figure 2). Then the eight integrals I^j satisfy*

$$dI^1/dt = B^1, \quad dI^2/dt = B^2,$$
$$dI^3/dt = B^3, \quad dI^4/dt = -gI^3 + B^4,$$
$$dI^5/dt = I^1 + B^5, \quad dI^6/dt = I^4 + B^6,$$
$$dI^7/dt = 4I^2 - 7gI^6 + B^7 = 4K - 3V + B^7,$$
$$dI^8/dt = -gI^5 + B^8.$$

Proof. Each of these eight equations is demonstrable by use of the formula (6.2), the boundary conditions in (2.9) and Green's theorem as exemplified by the identities $G^j_{\mathscr{S}} = -G^j_\Gamma$. First, taking $P = 0$ and $Q = -\phi$ in (6.2) we obtain

$$\frac{dI^1}{dt} = -\int_\Gamma \phi_t\, dy - \int_{\mathscr{S}} (\phi_t \eta_x - \eta_t u)\, dx$$

$$= -\int_\Gamma \phi_t\, dy - \int_{\mathscr{S}} \{\tfrac{1}{2}(u^2 - v^2)\, dy - (uv + g\eta\eta_x)\, dx\}$$

$$= -\int_\Gamma \phi_t\, dy - G^1_{\mathscr{S}} + \int_{\mathscr{S}} gy\, dy = B^1. \tag{6.3}$$

Second, putting $P = \frac{1}{2}\phi\phi_y + \frac{1}{2}gy^2$, $Q = -\frac{1}{2}\phi\phi_x$ in (6.2) and noting that $P_y - Q_x = \frac{1}{2}(u^2 + v^2) + g\eta = -\phi_t$ on $\mathscr{S}$, we obtain

$$\frac{dI^2}{dt} = \int_\Gamma \tfrac{1}{2}\{\phi_t(\partial_n \phi) + \phi(\partial_n \phi_t)\}\, ds + \int_{\mathscr{S}} \tfrac{1}{2}\{(\phi\phi_y)_t - (\phi\phi_x)_t \eta_x - \eta_t \phi_t\}\, dx$$

$$= \int_\Gamma \tfrac{1}{2}\{\phi_t(\partial_n \phi) + \phi(\partial_n \phi_t)\}\, ds + \int_{\mathscr{S}} \tfrac{1}{2}\{\phi(\partial_n \phi_t) - \phi(\partial_n \phi)\}\, ds$$

$$= \int_\Gamma \phi_t(\partial_n \phi)\, dx = B^2, \tag{6.4}$$

where again the last step follows by Green's theorem. In a similar way it is found that

$$\frac{dI^3}{dt} = \int_{\mathscr{S}} (v - u\eta_x)\, dx = -G^3_{\mathscr{S}} = B^3. \tag{6.5}$$

For the next case the use again of (6.2) gives

$$\frac{dI^4}{dt}+gI^3 = \int_\Gamma (\phi_t+gy)\,dx+\int_{\mathscr{S}}(\phi_t+v\eta_t+g\eta)\,dx$$

$$= \int_\Gamma (\phi_t+gy)\,dx-G^4_{\mathscr{S}} = B^4. \tag{6.6}$$

Similarly,

$$\frac{dI^5}{dt}-I^1 = \int_{\mathscr{S}}(x\eta_t\,dx+\phi\,dy)+\int_\Gamma \phi\,dy$$

$$= -G^5_{\mathscr{S}}+\int_\Gamma \phi\,dy = B^5, \tag{6.7}$$

and

$$\frac{dI^6}{dt}-I^4 = \int_{\mathscr{S}}(\eta\eta_t-\phi)\,dx-\int_\Gamma \phi\,dx$$

$$= -G^6_{\mathscr{S}}-\int_\Gamma \phi\,dx = B^6. \tag{6.8}$$

By some cancellation of terms in the integral over $\mathscr{S}$, it is next found that

$$\frac{dI^7}{dt}-4K+3V = \int_{\mathscr{S}}\{\phi_t(\eta-x\eta_x)+\eta_t(xu+yv)+\tfrac{3}{2}g\eta^2\}\,dx$$

$$+\int_\Gamma\{\phi_t(y\,dx-x\,dy)-2\phi(\partial_n\phi)\,ds+\tfrac{3}{2}gy^2\,dx\}$$

$$= -G^7_{\mathscr{S}}+\int_\Gamma\{(\phi_t+gy)\,(y\,dx-x\,dy)-2\phi(\partial_n\phi)\,ds\} = B^7, \tag{6.9}$$

where integrations by parts have reduced the terms proportional to g. Finally we obtain

$$\frac{dI^8}{dt}+gI^5 = \int_{\mathscr{S}}\{\phi_t(x+\eta\eta_x)+\eta_t(xv-yu)+gx\eta\}\,dx+\int_\Gamma\{\phi_t(x\,dx+y\,dy)+gxy\,dy\}$$

$$= -G^8_{\mathscr{S}}+\int_\Gamma(\phi_t+gy)\,(x\,dx+y\,dy) = B^8, \tag{6.10}$$

where again two integrations have reduced the terms in g. Thus the proof of the theorem is complete.

It is evident from this proof that a corresponding set of identities holds for the integrals over the free surface, denoted by $I^j_{\mathscr{S}}$, which form part of the complete contour integrals I^j. That is, there are corresponding boundary functions $B^j_{\mathscr{S}}(t)$ such that the equations for $I^j(t)$ given in Theorem 6.1 are duplicated for $I^j_{\mathscr{S}}(t)$. Plainly, the new boundary functions are just

$$B^j_{\mathscr{S}} = B^j-\frac{d}{dt}\int_\Gamma (P_j\,dx+Q_j\,dy)-\int_\Gamma (p_j\,dx+q_j\,dy),$$

where the $P_j\,dx+Q_j\,dy$ are the integrands in the definitions of the I^j and the $p_j\,dx+q_j\,dy$ ($j = 4,\ldots,8$) are the integrands of those integrals appearing as coefficients of t in the respective expressions for conserved densities (e.g. $p_3 = gy$, $q_4 = 0$). Unlike the B^j, the $B^j_{\mathscr{S}}$ include some terms that cannot be expressed as integrals over Γ, being due to the moving end-points p_n of $\mathscr{S}$. To express these terms concisely, we use the notation

$$\sum_{\partial\mathscr{S}} f = \sum_n (-1)^n f(p_n),$$

thus attaching signs in the sum (over the elements p_n of $\partial\mathscr{S}$) so that

$$\sum_{\partial\mathscr{S}} f = \int_{\mathscr{S}} df = -\int_{\Gamma} df$$

when such a representation is meaningful. We also write $\xi = \xi(t)$ for the x-coordinate of a point $p_n \in \partial\mathscr{S}$, so that its y-coordinate is represented by $\beta(\xi)$; and since therefore $\eta(\xi(t), t) = \beta(\xi(t))$, it follows that $\eta_t = (\beta_x - \eta_x)\xi_t$. For the moment, each β_x is taken to be bounded (i.e. Γ is not vertical at any of the points $\partial\mathscr{S}$ of intersection with $\mathscr{S}$); but a simple modification covering the contrary case will be noted presently.

Hence the boundary functions in the new statement of Theorem 6.1 are found to be

$$B^1_{\mathscr{S}} = G^1_{\Gamma} - \sum_{\partial\mathscr{S}} (\phi\beta_x\xi_t - \tfrac{1}{2}g\eta^2),$$

$$B^2_{\mathscr{S}} = \int_{\Gamma} \tfrac{1}{2}\{\phi_t(\partial_n\phi) - \phi(\partial_n\phi_t)\}\,dx + \sum_{\partial\mathscr{S}} \tfrac{1}{2}\{\phi(v - u\beta_x) + \tfrac{1}{2}g\eta^2\}\,\xi_t,$$

$$B^3_{\mathscr{S}} = G^3_{\Gamma} + \sum_{\partial\mathscr{S}} \eta\xi_t, \qquad B^4_{\mathscr{S}} = G^4_{\Gamma} + \sum_{\partial\mathscr{S}} \phi\xi_t,$$

$$B^5_{\mathscr{S}} = G^5_{\Gamma} + \sum_{\partial\mathscr{S}} \eta\xi\xi_t, \qquad B^6_{\mathscr{S}} = G^6_{\Gamma} + \sum_{\partial\mathscr{S}} \tfrac{1}{2}\eta^2\xi_t,$$

$$B^7_{\mathscr{S}} = G^7_{\Gamma} + \sum_{\partial\mathscr{S}} \{\phi(\eta - \xi\beta_x)\,\xi_t + \tfrac{1}{2}g\xi\eta^2\},$$

$$B^8_{\mathscr{S}} = G^8_{\Gamma} + \sum_{\partial\mathscr{S}} \{\phi(\xi + \eta\beta_x)\,\xi_t - \tfrac{1}{3}g\eta^3\}.$$

In the case that Γ is vertical at any of the points $\partial\mathscr{S}$, these results are to be modified as follows: put $\xi_t = 0$ except where ξ_t is multiplied by β_x, and there put $\beta_x\xi_t = \eta_t$.

The original and alternative forms of Theorem 6.1 embrace the same information, of course, and they fall in line with our general definition of conservation laws which was given in §5.1 and equation (5.6). The method of proof has been chosen to conform with our systematic derivation of the conserved densities, and the method will serve further in answering certain delicate mathematical questions to be treated in §6.5. It should be noted, however, that the following is a marginally more direct method of verifying equations (6.3)–(6.10), although not their alternative forms in terms of $I^j_{\mathscr{S}}$ and $B^j_{\mathscr{S}}$. Identifying each contour integral I^j with an integral over $\mathscr{D}$, say of H^j, one uses the well-known general formula

$$\frac{d}{dt}\iint_{\mathscr{D}} H\,dx\,dy = \iint_{\mathscr{D}} H_t\,dx\,dy + \int_{\mathscr{S}} \eta_t H\,dx$$
$$= \iint_{\mathscr{D}} \frac{DH}{Dt}\,dx\,dy - \int_{\Gamma} (\mathbf{u}.\mathbf{n})\,H\,ds,$$

where $DH/Dt = H_t + \mathbf{u}.\nabla H$ and the second identity follows from the kinematical boundary condition on $\mathscr{S}$ and the fact that $\operatorname{div}\mathbf{u} = \Delta\phi = 0$. Equations (6.3)–(6.10) are thus obtainable by use of $D(x, y)/Dt = (u, v)$ and the dynamical equation $D(u, v)/Dt = -\nabla p - (0, 1)g$.

6.2. *Physical interpretations*

The eight identities in Theorem 6.1 can be appreciated to conform also with physical principles whose applications to water waves are for the most part well known. These interpretations are spotlighted by the second, equivalent forms for the boundary terms listed before Theorem 6.1.

Thus, according to (6.3), the rate of change of horizontal momentum in the domain $\mathscr{D}(t)$ is equated as expected to the sum of two terms comprising B^1, first the net pressure force in the x-direction acting across the submerged boundary Γ, and second the rate at which horizontal momentum is convected inwards across Γ (recall the adopted, clockwise sense of line integrals over Γ). Similarly, (6.4) shows the rate of change of energy in $\mathscr{D}$ to equal the rate of working by the pressure plus the rate of energy convection across Γ. Equation (6.5) is a plain expression of mass conservation; and (6.6) shows as expected that the rate of change of vertical momentum in $\mathscr{D}$ equals the rate of convection of vertical momentum across Γ plus the difference between the upward pressure force and the weight of the fluid in $\mathscr{D}$.

The remaining results have less obvious physical meanings. Noting the listed form of B^5, we recognize (6.7) to show that the x-moment of the mass in $\mathscr{D}$ changes at a rate equal to I^1 plus the x-moment of the mass flux across Γ. Correspondingly, from the form of B^6, (6.8) is seen to show that the y-moment changes at a rate equal to I^4 plus the y-moment of the mass flux across Γ. These two relationships are kinematic, being demonstrable independently of the dynamical boundary condition at the free surface; and in fact they remain true whatever dynamical equations apply inside the (incompressible) fluid or at the surface (cf. Benjamin & Mahony 1971). The coincidental roles of I^1 and I^4 as components of the total momentum in $\mathscr{D}$ nevertheless provide an interesting link with the dynamics of the problem, as also does the fact that $gI^6 = V$, the potential energy.

The last two results in Theorem 6.1 are recognizable as constituents of a virial theorem for the motion in $\mathscr{D}$ (cf. Truesdell & Toupin 1960, §§216, 219). This interpretation applies even more plainly to the corresponding set of four results for the three-dimensional problem (i.e. those relating to the conserved densities T^9 to T^{12} of Theorem 5.3, the last of which is the counterpart of the present result for I^7). Contraction of the virial tensor recovers the identity (6.9) for dI^7/dt, and the off-diagonal components give, as is usual, an identity expressing conservation of angular momentum. Among our eight results the seventh is the least amenable to simple physical explanation, apparently in keeping with its origin in a curious combination of the vertical-acceleration and scaling symmetries, and some further comment on it will be made in §6.3 below. Writing out this result as

$$\frac{dI^7}{dt} = -\int_\Gamma (xu+yv+2\phi)(v\,dx-u\,dy) - \int_\Gamma p(y\,dx-x\,dy)+4K-3V, \qquad (6.9')$$

we note that the first of the line integrals represents the rate of convection, inwards across Γ, of the quantity whose integral over $\mathscr{D}$ is equal to I^7. The combination of the remaining terms on the right-hand side, which generally does not vanish when Γ is impermeable, has no immediate interpretation. In contrast, the eighth result which can be rewritten

$$\frac{dI^8}{dt} = -\int_\Gamma (xv-yu)(v\,dx-u\,dy) - \int_\Gamma p(x\,dx+y\,dy)-gI^5 \qquad (6.10')$$

is seen to have a simple meaning. As expected it equates the rate of increase of angular momentum in $\mathscr{D}$, about the origin of (x, y), to the rate of convection of angular momentum inwards across Γ plus the difference between the moment of the pressure force on Γ and the moment of the weight of the fluid in $\mathscr{D}$.

It should be noted from these results that when Γ is a finite impermeable (solid) boundary, I^2 and I^3 are independent of t. Thus, as obviously expected, total energy and mass are completely conserved by the wave motion. But no other of the quantities I^j is so conserved. However, I^1 is also conserved in the case that the fluid is bounded below by a solid plane $y = -h$ and extends to the limits $x \to \pm\infty$ where it is at rest. In this case, I^1 is evidently equal to M given by (5.7).

6.3. *Discussion of* I^7

In general the main application of identities derived from the trace of a virial tensor are to periodic motions or to other motions presumed to remain regular for indefinitely long times. Relations between the mean kinetic and mean potential energies may thus be established. Equation (6.9) provides such a relation; however, since it includes a term involving the values of pressure on Γ, it seems unlikely to give much useful information except in special cases. Supposing Γ to be a solid boundary and writing $\langle . \rangle$ for averages over one period of a periodic motion, or otherwise for asymptotic, long-time averages, we may conclude from (6.9) that

$$\left\langle \int_\Gamma p(y\,dx - x\,dy) \right\rangle = 4\langle K \rangle - 3\langle V + V_0 \rangle, \tag{6.11}$$

where K is the kinetic energy and $V_0 + V = gI^6$ the potential energy of the fluid in $\mathscr{D}$. Note that this result is compatible with a state of rest ($K = 0$), for then $p = -gy$ and an integration by parts shows the left-hand side to equal $-3V_0$.

For example, in respect of *infinitesimal* standing waves in a tank of whatever shape, it is well known that $\langle K \rangle = \langle V \rangle$. Hence the averaged integral on the left-hand side of (6.11) must equal $\langle K \rangle - 3V_0$. In the case of a rectangular tank, this conclusion can be confirmed directly by a simple calculation using Bernoulli's theorem, but for tanks of other shapes a direct verification is hardly straightforward. The formula (6.11) is thus an incidental, albeit possibly interesting expression for the pressure integral, rather than a means of establishing facts about the more significant physical quantities $\langle K \rangle$ and $\langle V \rangle$. For waves of any amplitude in a finite tank, our other results show that $I^2 = K + V + V_0 = \text{const.}$ and $I^3 = \text{const.}$, also that

$$\left\langle \int_\Gamma p\,dy \right\rangle = 0, \quad \left\langle -\int_\Gamma p\,dx \right\rangle = gI^3,$$

$$\langle I^1 \rangle = 0, \quad \langle I^4 \rangle = 0,$$

and finally that

$$\left\langle -\int_\Gamma p(x\,dx + y\,dy) \right\rangle = g\langle I^5 \rangle.$$

But these equalities generally will not help to evaluate the left-hand side of (6.11).

The following special use of the virial equation (6.9), serving as a check on its validity, may nevertheless be of interest. Anticipating some of the results to be gathered below in Theorem 6.2, for an infinite ocean lying on a solid plane $y = -h$, let us consider the case of a solitary wave which travels at velocity c in the x-direction without change of form. Both η and $|\nabla\phi|$ decay exponentially with large $|x|$, but $C = \phi_\infty - \phi_{-\infty} > 0$ (cf. (5.7)). For the infinite contour $S \cup \Gamma$ comprising the entire free surface, the bottom and the two ends at infinity, I_4 is written for the respective version of I^4, to which no contribution is made from the parts of the contour taken to the limits $x \to \pm\infty$. On the other hand, to have a convergent integral for the counterpart of (6.9), we take $I_7 = I^7_S$. The quantities $I_3 = I^3_S$ and $I_6 = I^6_S = V/g$ are also finite for a solitary wave. Equation (6.9) now reduces to

$$\frac{dI_7}{dt} = 4K - 3V + h\int_{-\infty}^{\infty} \tfrac{1}{2}u_B^2\, dx,$$

where u_B denotes the values of u on the bottom (along which the integral is taken here in the forward direction). Similarly, (6.6) reduces to

$$\frac{dI_4}{dt} = -\int_{-\infty}^{\infty} (\phi_t + \tfrac{1}{2}u^2)_B\, dx - gI_3.$$

But a solitary wave evidently has $I_4 = 0$ for all t by virtue of its symmetry (i.e. the upward vertical momentum forward of the wave crest is balanced by the downward momentum behind). Moreover, since $\phi_t = -c\phi_x$, the integral of $(\phi_t)_B$ in the last equation is just cC. Combining these conclusions, we have

$$\frac{dI_7}{dt} = 4K - 3V + chC - ghI_3. \tag{6.12}$$

According to the definition of I_7, the left-hand side of (6.12) is

$$\frac{d}{dt}\int_S \phi(y\,dx - x\,dy) = \frac{d}{dt}\int_{-\infty}^{\infty} \Phi\{\eta - (x' + ct)\,\eta_x\}\, dx',$$

where for a solitary wave Φ and η are functions of $x' = x - ct$ alone; and so it equals

$$-c\int_{-\infty}^{\infty} \Phi\eta_x\, dx = c\hat{I}_1 = c(I_1 - hC),$$

where $\hat{I}_1$ is the horizontal impulse and I_1 the horizontal momentum as was denoted by M in equation (5.7). Thus (6.12) gives

$$4K - 3V = -2chC + cI_1 + ghI_3. \tag{6.13}$$

Two other relationships among solitary-wave properties are found more easily. From (6.7) it is seen immediately that

$$I_1 = cI_3,$$

and we also have that

$$2K = \int_{-\infty}^{\infty} \Phi\eta_t\, dx = -c\int_{-\infty}^{\infty} \Phi\eta_x\, dx = c\hat{I}_1.$$

Hence (6.13) reduces to

$$3V = cI_1 - ghI_3 = (c^2 - gh)\, I_3.$$

The first and third of these last three relationships were discovered by Starr (1947), the second having already been established by McCowan (1891; see Longuet-Higgins 1974 for a more recent discussion).

As a final comment, it seems appropriate to warn against identifying I^7 with wave *action*, which too, like I^8, has the dimensions of energy times time. The following considerations indicate that I^7 is distantly related to action but does not carry its main attributes. Returning to a fundamental appraisal of the hydrodynamic problem as in §2, we note that for a solid or infinitely distant submerged boundary the motion is determined completely at each instant by the functions η and η_t, which together fix $\Phi = \phi(x, \eta(x, t), t)$ (except for arbitrary function of time alone) when the kinematic surface condition $\Phi_{(n)} = R^{-1}\eta_t$ completes the auxilliary Neumann problem for ϕ on D_η. Thus we may write $\Phi = \mathscr{B}_\eta \eta_t$, where for fixed η the operation $\mathscr{B}_\eta$ is *linear* and symmetric. The function Φ was treated in §2 as a generalized momentum density in the Hamiltonian formulation; but we may also regard

$$K = \int_{S_0} \tfrac{1}{2}\Phi\eta_t \, dx$$

as a symmetric quadratic functional in η_t for fixed η, so that the functional derivative $K_{\eta t} = \Phi$. It is also readily seen that

$$(K_\eta)_{\eta_t=\text{const.}} = -(K_\eta)_{\Phi=\text{const.}}.$$

Hence the dynamical boundary condition, hitherto expressed in Hamiltonian form by the second of (2.8), is alternatively representable by

$$\frac{d}{dt}(L_{\eta_t}) - L_\eta = 0,$$

where $L = K - V$ is thus the Lagrangian for the problem. Reproducing a common property of finite conservative systems, this conclusion means that trajectories of the water-wave system between arbitrary times t_0 and t_1 are extremals of the action integral

$$\mathscr{A} = \int_{t_0}^{t_1} L \, dt.$$

This is the correct expression for the action as a characterization of particular solutions, for which initial values of η and η_t (or Φ) are prescribed at $t = t_0$; and the connection with I^7 is illuminated by another standard concept from Hamiltonian theory. We may formally define an action density

$$A = A(x, t) = \int \Phi \, d\eta,$$

reckoning this as an integral in the phase space, that is, in the product of the function spaces to which the components η and Φ of solutions respectively belong. Thus A is not a property of single solutions, but rather characterizes a family of neighbouring solutions (cf. Hayes 1970). Considered in this special light, the one-form ω_7 shown in §5 to be generated by the symmetry $\mathbf{v}_7$ can be seen to define a function akin to a conserved density, with A as the component without explicit dependence on t. In fact, as the equivalent form of ω_7 given in §5 plainly promises, an examination on the lines of §6.2 reveals that if Γ is a solid boundary, then

$$d\mathscr{A}/dt = K - V, \quad \text{where} \quad \mathscr{A} = \int_{S_0} A \, dx.$$

This outcome is expectable by analogy with finite-dimensional conservative systems, and it shows the formal but physically inconsequential relationship between A and the conserved density T_7, hence between $\mathscr{A}$ and I^7. Namely, the one-form ω_7 was combined with another, ω_9 generated by the scaling symmetry $\mathbf{v}_9$, in order to make T_7 a proper conserved density.

6.4. *The case of a uniform ocean*

For completeness, the conservation laws are now expressed collectively in the simpler forms that apply to a fluid lying on a horizontal solid plane $y = -h$, where h is finite. Some of these expressions have already been implied in our discussion of the solitary-wave example, and they all follow more or less immediately from the respective equations introduced in explaining Theorem 6.1 [i.e. equations (6.3)–(6.10)]. We note that $v = 0$ on the bottom $y = -h$, and that according to the Bernoulli integral of the dynamical equations the pressure there is given by $p = gh - (\phi_t + \frac{1}{2}u^2)_{y=-h}$.

It is assumed that the solution η, Φ is smooth enough, and that $\eta \to 0$ and $|\nabla\phi| \to 0$ fast enough as $|x| \to \infty$, for the following integrals to converge. Here I_1, I_4 and I_8 representing momenta are complete contour integrals, just the respective versions of I^1, I^4 and I^8; but the rest are integrals over the free surface alone (e.g. $I_3 = I_S^3$), being expressed as integrals with respect to x over $S_0 = \mathbb{R}$ by use of the notation introduced in §2 and the relation $dy = \eta_x\,dx$, which is justified by our supposing as before that $|\eta_x|$ is bounded everywhere. Thus we consider

$$I_1 = -\oint \phi\,dy = (\phi_\infty - \phi_{-\infty})\,h - \int_{S_0} \Phi\eta_x\,dx \quad \text{(horizontal momentum } M\text{)},$$

$$I_2 = \int_{S_0} (\tfrac{1}{2}\Phi\Phi_{(n)}\,R + \tfrac{1}{2}g\eta^2)\,dx \quad \text{(energy } K+V\text{)},$$

$$I_3 = \int_{S_0} \eta\,dx \quad \text{(excess mass } m\text{)},$$

$$I_4 = \oint \phi\,dx \quad \text{(vertical momentum)},$$

$$I_5 = \int_{S_0} x\eta\,dx \quad (m\bar{x}),$$

$$I_6 = \int_{S_0} \tfrac{1}{2}\eta^2\,dx \quad (m\bar{y}),$$

$$I_7 = \int_{S_0} \Phi(\eta - x\eta_x)\,dx \quad \text{(virial)},$$

$$I_8 = \oint \phi(x\,dx + y\,dy) \quad \text{(angular momentum)}.$$

The term *localized* will be used to denote solutions such that $|\eta_x|$ is bounded and all these integrals converge. Since no existence and regularity theory is available for the full problem, we cannot ensure this property of solutions precisely by conditions on the initial values of η and Φ; however, it seems certain to hold for a finite time in a wide range of meaningful examples. Various more or less technical versions of the property can be stated in terms of function classes for η and Φ, but we pass over this aspect here.

On reference to Theorem 6.1 or to (6.3)–(6.10), the only non-zero boundary functions given by integrals along the bottom are found to be

$$B_4 = -\int_b (\phi_t + \tfrac{1}{2}u^2)\,dx, \quad B_7 = h\int_b \tfrac{1}{2}u^2\,dx,$$

and

$$B_8 = -\int_b (\phi_t + \tfrac{1}{2}u^2)\,x\,dx.$$

Here b against the integral sign denotes evaluation of the integrand at $y = -h$, and integration over $\mathbb{R}$ in the sense of x increasing. Accordingly, the results in question can be stated as follows:

THEOREM 6.2. *For an infinite ocean lying on a horizontal solid bottom at finite depth, suppose that the solution* η, Φ *of the two-dimensional problem without surface tension remains localized during a time interval* $[0, T]$. *Then for all* $t \in [0, T]$ *the eight quantities* I_j *satisfy*

$$I_1 = \text{const.}, \quad I_2 = \text{const.}, \quad I_3 = \text{const.},$$
$$dI_4/dt = -gI_3 + B_4,$$
$$dI_5/dt = I_1, \quad dI_6/dt = I_4$$
$$dI_7/dt = 4I_2 - 7gI_6 + B_7 = 4K - 3V + B_7,$$
$$dI_8/dt = -gI_5 + B_8.$$

The proof of this theorem consists merely in applying Theorem 6.1 to a bounded domain $\{(x, y): -X < x < X, \ -h \leqslant y \leqslant \eta(x, t)\}$, and then confirming that the stated identities are obtained in the limit $X \to \infty$. Except for the component $h(\phi_\infty - \phi_{-\infty})$ in the definition of I_1 and hence in the identities for $I_5(t)$ and $I_8(t)$, all contributions from integrals over the vertical sides of the domain are seen to cancel in the limit. Note that according to the Bernoulli integral both ϕ_∞ and $\phi_{-\infty}$ are independent of time, although not necessarily the same constants.

The steady horizontal motion of the mass centroid, as expressed by the identity for I_5, has been discussed in great generality by Benjamin & Mahony (1971). Albeit only in the case $I_3 = 0$, the identity for I_6 was first recognized by Longuet-Higgins (1950) in the differentiated form

$$\frac{d^2 I_6}{dt^2} = -gm - \int_b (\phi_t + \tfrac{1}{2}u^2)\,dx = -gm + \int_b (p - gh)\,dx, \tag{6.14}$$

and he used it to explain the generation of microseisms on the ocean floor. This result shows that on the floor $y = -h$ the excess force due to disturbances of the free surface is (for unit density) equal to d^2I_6/dt^2 plus, as would be expected, the weight gm of excess fluid. An integral like I_8 expressing angular momentum has been considered by Longuet-Higgins (1980) in the case of *periodic* progressive waves, but the form of the identity given here appears to be new, as also do those for I_4 and I_7.

6.5. *The case of infinite depth*

This case calls for careful interpretation as regards the linear momenta I_1 and I_4, also the angular momentum I_8. It has been seen that these quantities are unequivocally determinate, and have obvious dynamical connotations, in the case of a uniform ocean with finite depth h. But they are no longer so when the preceding results are taken to the limit $h \to \infty$, or when the ocean is otherwise reckoned to be unbounded below. On the other hand, it appears that I_1 and I_4 may determine or be determined by the temporal variations of I_5 and I_6, which are properties of the free surface alone.

A point deserving emphasis in the present connection has been noted in §5, at the start of the paragraph including equation (5.7). Namely, the constituent $\hat{I}_1 \equiv I_S^1$ of I_1 ($\equiv M$) is the horizontal Kelvin impulse which is the more significant quantity dynamically. Correspondingly, $\hat{I}_4 \equiv I_S^4$ is the vertical Kelvin impulse. The cardinal attribute of the dynamical problem in the present case is that when an external force is somehow applied to a finite part of the system, $\hat{I}_1$ and $\hat{I}_4$ rather than I_1 and I_4 change at rates equal respectively to the horizontal and vertical components of the force, say $F_{[x]}$ and $F_{[y]}$. Thus $\hat{I}_1$ and $\hat{I}_4$ take roles like the momenta of an isolated finite system.

For example, suppose that in an infinitely deep ocean a wave motion is generated from rest by the application of a pressure $P(x, t)$ to the free surface. Let $P(x, t)$ be an L^1 function of x for each t, and *vice versa*, which is non-zero only while $0 \leqslant t \leqslant t_0$. On the assumption that $|\eta_x|$ remains bounded during this interval, the components of the total force exerted against the surface are given by

$$F_{[x]}(t) = \int_{S_0} P\eta_x \, dx, \quad F_{[y]}(t) = -\int_{S_0} P \, dx.$$

At large distances r from the centre of the (compact) support of P, the asymptotic behaviour of ϕ cannot have more than dipole strength, i.e. $\phi = O(r^{-1})$ and $|\nabla\phi| = O(r^{-2})$. Hence, for an infinitely distant submerged boundary Γ (e.g. a semicircle of infinite radius), we have

$$\int_\Gamma (\partial_n \phi) \, ds = 0,$$

which, by virtue of the fact that $\Delta\phi = 0$ everywhere in the fluid, implies that $dI_3/dt = 0$ and therefore

$$I_3 = \int_{S_0} \eta \, dx = 0 \quad \text{for all} \quad t. \tag{6.15}$$

The asymptotic property of ϕ also implies that

$$\int_\Gamma |\nabla\phi|^2 \, dx = \int_\Gamma |\nabla\phi|^2 \, dy = 0,$$

and that $\eta \to 0$ as $x \to \pm\infty$. Accordingly, retracing the steps that led to (6.3) and (6.6), but allowing now that $p = P$ in Bernoulli's equation evaluated at the free surface, we find as expected that

$$\frac{d\hat{I}_1}{dt} = F_{[x]}, \quad \frac{d\hat{I}_4}{dt} = F_{[y]}. \tag{6.16}$$

Thus $\hat{I}_1$ and $\hat{I}_4$ change in the interval $[0, t_0]$ from their initial values (which are zero for a state of rest), and thereafter they have the constant values

$$\hat{I}_1 = \int_0^{t_0} F_{[x]}(t) \, dt, \quad \hat{I}_4 = \int_0^{t_0} F_{[y]}(t) \, dt. \tag{6.17}$$

By the same argument, the expressions (6.17) give the changes in $\hat{I}_1$ and $\hat{I}_4$ caused by the application of an external force to a pre-existing wave motion with $I_3 = 0$. In the case that $I_3 = m \neq 0$, the conclusion about $\hat{I}_1$ is unchanged, but the weight gm of excess fluid adds to the downward component of force, so that

$$\hat{I}_4(t) = \hat{I}_4(0) - gm + \int_0^t F_{[y]}(t') \, dt'.$$

[To clarify the significance of the results for $\hat{I}_4$, it should perhaps be stressed that the identification of $d\hat{I}_4/dt$ with the net upwards force is by no means inconsistent with the simpler case where a *uniform* P is applied to the free surface of still water in a finite closed basin, obviously causing no motion at all. In this case the uniform increase in pressure everywhere in the fluid corresponds to $\phi_t = -P$ in the residual form of Bernoulli's equation, and therefore $I^4 = I^4_S + I^4_\Gamma = 0$ for all t even though I^4_S changes.]

It remains to reconcile this comparatively straightforward interpretation with the possibility that the identities

$$\frac{dI_5}{dt} = I_1, \quad \frac{dI_6}{dt} = I_4 \tag{6.18}$$

included in Theorem 6.2 carry over to the case of infinite depth. Unlike $\hat{I}_1$ and $\hat{I}_4$ whose significance is unequivocal, I_1 and I_4 in the present case are found not to be calculable from the Hamiltonian conjugate variables η and Φ *unless* conditions more precise than needed hitherto are imposed at infinite distances in the fluid. So the present meanings of (6.18) are not immediately evident. They become clear when one appreciates, amplifying a point made earlier (after (6.7) and (6.8)), that (6.18) are just kinematic identities whose validity depends on a particular choice of conditions at infinity. The forms of the identities change when other such conditions are arbitrarily chosen, but the difference is merely one of incidental interpretation to be attached to the quantities dI_5/dt and dI_6/dt which are intrinsic properties of the evolutionary process.

To pinpoint the issue, let us carefully retrace the arguments that demonstrate (6.18), considering now that the fluid has a submerged *solid* boundary Γ in the form of an infinite semicircle. The steps whereby dI_5/dt is reduced are

$$\frac{d}{dt}\int_{S_0} x\eta\,dx = \int_{S_0} x\eta_t\,dx = \int_S x(v\,dx-u\,dy) = \oint_{S\,\cup\,\Gamma} x(v\,dx-u\,dy),$$

where the last equality is evident because Γ is solid. As the final step, the contour integral around $S \cup \Gamma$ is reduced by Green's theorem to equal an integral over the enclosed domain D_η, and so it is concluded that

$$\frac{dI_5}{dt} = \int_{D_\eta} \{(xv)_y + (xu)_x\}\,dx\,dy = \int_{D_\eta} u\,dx\,dy = M \equiv I_1,$$

since $\operatorname{div}\mathbf{u} = u_x + v_y = 0$ everywhere in D_η. The second of (6.18) is demonstrable in a precisely similar way, with recourse to the incompressibility of the fluid and to the assumed condition on Γ but without reference to any dynamical condition. The kinematic meaning of (6.18), tied to the particular model for the fluid at infinity, is thus made clear; but it can also be appreciated that the physical quantities to be identified with dI_5/dt and dI_6/dt are to an extent arbitrary, depending on delicate specifications about infinity in the fluid that have no effect on the main dynamical equations (2.8) or results such as (6.15) and (6.16). (It will be shown in due course that, irrespective of the precise conditions at infinity, $I_1 = \hat{I}_1$ for all initial conditions that are realistic in a certain sense; however, for the time being we proceed generally without recourse to this simplification.)

For example, an alternative model for the submerged boundary Γ at infinity is a compliant surface such that hydrostatic pressure is exactly maintained upon it. This specification implies that $\phi = 0$ exactly on Γ, and consequently $I_1 = \hat{I}_1$. But the first of (6.18) becomes

$$\frac{dI_5}{dt} = \hat{I}_1 - \int_\Gamma x(v\,dx-u\,dy), \tag{6.19}$$

the second term of which is generally not zero. The sum on the right-hand side turns out, of course, to have the same value as I_1 in the previous case (*vide infra* for proof), and only the physical interpretation of this quantity has changed.

The matter is made transparent by working out the explicit asymptotic form of the velocity potential ϕ. For any finite point $\mathbf{x}$ inside the fluid, a standard construction of potential theory (cf. Lamb 1932, p. 60) shows that

$$\phi(\mathbf{x}) = -\int_S \{\phi(\hat{\mathbf{x}}) - \phi'(\hat{\mathbf{x}})\}\partial_n\,\psi(\mathbf{x}-\hat{\mathbf{x}})\,d\hat{s}, \tag{6.20}$$

where

$$\psi(\mathbf{x}-\hat{\mathbf{x}}) = \frac{1}{2\pi}\ln\frac{1}{|\mathbf{x}-\hat{\mathbf{x}}|}$$

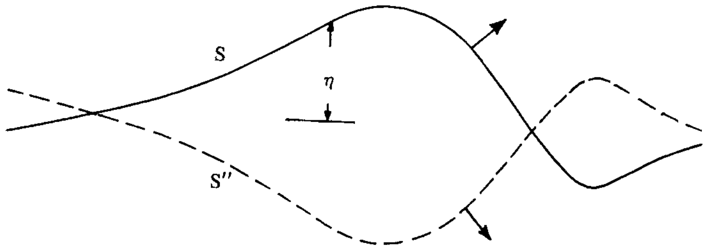

FIGURE 3. Illustration of the fictitious boundary S'', the reflection of the free surface S about the horizontal line $y = 0$.

is Green's function for the whole plane, and ϕ' is the potential defined in $\mathbb{R}^2 \backslash D_\eta$ satisfying $\partial_n \phi' = \partial_n \phi$ (both normal derivatives in direction out of D_η) on the lower boundary S and $\phi' = O(r^{-1})$ like $\phi = O(r^{-1})$ as $r \to \infty$ (along rays respectively in $\mathbb{R}^2 \backslash D_\eta$ and D_η). While depending on the basic supposition that $\phi = O(r^{-1})$ as $r \to \infty$, the representation (6.20) is independent of $\lim_{r\to\infty} r\phi$, which we shall therefore be able to specify separately later. Writing $x = r\cos\theta$, $y = -r\sin\theta$ and taking $(\hat{x}, \hat{y}) \in S$, we find that

$$\partial_n \psi(\mathbf{x}-\hat{\mathbf{x}}) = -\frac{1}{2\pi r}\left(\cos\theta \frac{d\hat{y}}{d\hat{s}} + \sin\theta \frac{d\hat{x}}{d\hat{s}}\right) + O(r^{-2})$$

as $r \to \infty$. Hence an asymptotic approximation to $\phi(\mathbf{x})$ correct to $O(r^{-1})$ is seen from (6.20) to be

$$\phi = \frac{1}{2\pi r}\left\{\cos\theta \int_S (\phi - \phi')\, dy + \sin\theta \int_S (\phi - \phi')\, dx\right\}. \tag{6.21}$$

Now, referring to figure 3 for illustration of the ideas, consider the potential $\phi''(x, y, t) = \phi'(x, -y, t)$ which is defined in $y \leqslant -\eta(x, t)$ and satisfies $\partial_n \phi'' = -R\eta_t$ on the fictitious boundary $y = -\eta(x, t)$ labelled S'' in the figure. Note that S'' translates horizontally in step with S, but moves down where S moves up and *vice versa*. The representation of ϕ''corresponding to (6.20) is a line integral over S'', and plainly the factor in the integrand expressible as $\phi''(\hat{x}, -\eta(\hat{x}, t), t) - \phi'''(\hat{x}, -\eta(\hat{x}, t), t)$ is the same as $-\{\phi(\hat{x}, \eta(\hat{x}, t), t) - \phi'(\hat{x}, \eta(\hat{x}, t), t)\}$.

According to (6.20), the asymptotic behaviour of ϕ'' therefore reflects that of ϕ in the sense that if

$$\phi = r^{-1}(A\cos\theta + B\sin\theta) + O(r^{-2}),$$

then

$$\phi'' = r^{-1}(A\cos\theta - B\sin\theta) + O(r^{-2}).$$

Hence arguments precisely corresponding to those used above to verify (6.18) show that

$$\int_S (\phi + \phi')\, dy \equiv \int_S \phi\, dy - \int_{S''} \phi'' dy = -2\int_{S_0} x\eta_t\, dx,$$

$$\int_S (\phi + \phi')\, dx \equiv \int_S \phi\, dx + \int_{S''} \phi''\, dx = 2\int_{S_0} \eta\eta_t\, dx,$$

irrespective of the precise conditions at infinity. That is, contributions from integrals of ϕ and ϕ'' over an infinite semicircle Γ cancel in these identities. It follows that the asymptotic expression (6.21) for ϕ is the same as

$$\phi = \frac{1}{\pi r}\left\{\cos\theta \int_S (x\eta_t\, dx + \phi\, dy) - \sin\theta \int_S (\eta\eta_t - \phi)\, dx\right\}. \tag{6.22}$$

This result makes the interpretation entirely clear. In all cases it describes the dipole field that the wave motion produces at large distances, but this description can be modified as follows at infinity to represent the exact conditions arbitrarily imposed there. If the infinite semicircle Γ is taken to be a solid boundary, this dipole field

$$\phi = \frac{1}{r}(A\cos\theta + B\sin\theta), \quad \text{say,} \tag{6.23}$$

is modified to

$$\phi^* = \left(\frac{1}{r} + \frac{r}{\rho^2}\right)(A\cos\theta + B\sin\theta),$$

which is a potential satisfying $\partial_n \phi^* = 0$ on $r = \rho$. Then $\phi^* = (2/\rho)(A\cos\theta + B\sin\theta)$ on $r = \rho$; and after integral properties of ϕ^* on this semicircle have been evaluated, it is unambiguous to take the limit $\rho \to \infty$. In the limit, of course, the difference between ϕ and ϕ^* disappears at all finite distances r, however large, but the exact behaviour at infinity has been accommodated. Since

$$\int_\Gamma \rho^{-1}\cos\theta\, dy = -\int_0^\pi \cos^2\theta\, d\theta = -\tfrac{1}{2}\pi,$$

$$\int_\Gamma \rho^{-1}\sin\theta\, dx = -\int_0^\pi \sin^2\theta\, d\theta = -\tfrac{1}{2}\pi,$$

the argument proceeding from (6.22) thus recovers (6.18) in the case of a solid boundary at infinity. In the alternative case that $r\phi = 0$ at infinity, the appropriate modification of the dipole field is

$$\phi^* = \left(\frac{1}{r} - \frac{r}{\rho^2}\right)(A\cos\theta + B\sin\theta),$$

from which and from (6.22) it appears that (6.19) is satisfied identically irrespective of the value of $\hat{I}_1$. Similarly the identity for dI_6/dt becomes vacuous in this case. Needless to say, between these two extreme models for the boundary Γ at infinity, there is a continuous range of other models that can be arbitrarily imposed, with corresponding incidental interpretations in physical terms but without effect on the evolutionary process at the free surface.

[It is noteworthy that the foregoing analysis recovers some features exemplified in the comprehensive review of the *linearized* deep-water problem by Lamb (1932, §§238–241), who abstracted the important early contributions to the subject by Cauchy, Poisson, Rayleigh and others. In particular, part of the expression (6.22) for the dipole field at large r recovers the leading terms of asymptotic expansions of ϕ according to the linearized theory. But the terms with coefficients $\int \phi\, dy$ and $\int \eta\eta_t\, dx$ do not arise in that theory, being of second-order smallness. The classic treatments of the linearized problem are also helpful here as precedents for our view that the case $I_3 \neq 0$ is entirely compatible with the theoretical model comprising an infinitely deep ocean of incompressible fluid. At first sight, according to standard results, this case may seem pathological in that the wave motion has Fourier components whose speed of propagation is unbounded. The apparent difficulty is illusory, however; or rather it is outweighed by another, tractable feature that remains even when the speeds of all Fourier components are finite. Namely, at any finite distance, however, large, from the centre of a localized initial disturbance, some effect is manifested instantaneously (cf. Lamb, p. 394). This feature is accountable, of course, to the specification that the fluid is strictly incompressible, so that pressure changes in it propagate at infinite speed. The compensating attribute of the theoretical model which makes it amenable to decisive treatment is that in all cases, whether I_3 is zero or not, ϕ has only dipole strength at large distances and so $\eta_t = O(|x|^{-2})$.]

We come at last to the crucial aspect as regards realistic applications, which turns on the evaluation of the contour integral I_8 expressing angular momentum. The contribution to I_8 from the infinite semicircle Γ is evidently zero, irrespective of the values of ϕ there, but the contribution from the free surface S is indeterminate in the case that the coefficient A in (6.23) is non-zero. This conclusion is unaffected by the precise condition at infinity, and indeed an attempt to calculate the angular momentum directly (i.e. by integrating $xv - yu$ over D_η) shows it to be infinite like $\lim_{\rho\to\infty} \ln \rho$ if $A \neq 0$. We should duly appreciate that examples admitting this unrealistic feature do not conflict with any of the arguments given so far,† and in other physical respects they are fully determinate when conditions at infinity are chosen. On practical grounds, however, it is reasonable to exclude such examples from the account of the water-wave problem, and this proviso simplifies the interpretation considerably. Accordingly, we now assume the additional condition $x\Phi \in L^1(\mathbb{R})$, which ensures the existence of I_8 and concomitantly makes $A = 0$ always. This condition is evidently satisfied, for instance, when $xP \in L^1(\mathbb{R} \times (0, t_0))$ in addition to P having this attribution, where $P(x, t)$ is the external pressure considered earlier to generate a wave motion from rest.

To sum up from this new standpoint, we have that $I_1 = \hat{I}_1$ irrespective of the precise boundary conditions at infinity, and according to (6.22) we have that

$$dI_5/dt = \hat{I}_1, \tag{6.24}$$

which is known to be a constant of any free wave motion. It is appropriate to recall that dI_5/dt has the equivalent expressions

$$\int_{S_0} x\eta_t \, dx = \int_S x\Phi_{(n)} \, ds,$$

the first of which shows the prescription of dI_5/dt to be immediate according to the Lagrangian view of the initial-value problem. But, as the original discussion in §2 made clear, the second expression also determines dI_5/dt from the Hamiltonian conjugate variables η and Φ. It is noteworthy that I_5 has a status comparable with that of *ignorable* coordinates in finite-dimensional Hamiltonian systems, for which the corresponding momenta (like $\hat{I}_1$ here) are always constants.

The significance of the non-negative quantity I_6 is less conspicuous, no invariant property of the motion being indicated. On the supposition that the infinitely remote lower boundary is solid, dI_6/dt can be identified with the total vertical momentum I_4 which, unlike $\hat{I}_4$, is not a constant. In consequence, as indicated by (6.6) which carries over unambiguously to the case of infinite depth, a varying downward force $d^2I_6/dt^2 + gm$ is exerted against Γ additionally to the hydrostatic force. The same

† Take, for instance, the quite legitimate initial-value problem in which at $t = 0$ the free surface S has a semicircular depression of radius a (so that $I_3 = -\frac{1}{2}\pi a^2$) and $\eta_t = -c\eta_x$ with $c > 0$. Thus the initial motion of S is a horizontal translation at velocity c, and accordingly the initial form of the velocity potential is $\phi_0 = -ca^2 r^{-1} \cos\theta$ in $r \geqslant a$. Evaluating this on S, we have $-\Phi_0 = ca^2 x^{-1}$ for $|x| \geqslant a$ and $-\Phi_0 = cx$ for $|x| \leqslant a$, which describes the impulsive pressure on S needed to generate the motion from rest. The distribution of starting impulse amounts to an infinite (anticlockwise) impulsive couple exerted against the system, so imparting to it an infinite angular momentum when $t > 0$. Note that ϕ_0 continued analytically to the whole of the region $r > a$ in $\mathbb{R}^2$ describes the motion of an unbounded incompressible fluid caused by the displacement of a circular cylinder perpendicular to its length (Lamb 1932, §68). Equal and opposite but infinite angular momenta are then generated in the upper and lower half-spaces.

conclusion has already been demonstrated by (6.14) in the case of finite depth. It deserves re-emphasis, however, that the specification justifying this interpretation is arbitrary and can be changed without affecting the dynamical problem.

The present counterparts of the other identities in Theorem 6.2 are simpler. Unambiguously, I_2 still represents total energy which is constant. As already noted, the excess mass m is also constant. The quantity I_7, which has been named virial after the discussion in §6.3, is a convergent integral over S_0 by virtue of the dipole behaviour of ϕ at large r, even if $A \neq 0$. Expressing dI_7/dt, one encounters an integral over the infinite boundary Γ that reduces to

$$B_7 = -\int_\Gamma \tfrac{1}{2}(u^2+v^2)(y\,dx - x\,dy)$$

if Γ is supposed to be solid. With the appropriately corrected dipole form ϕ^* substituted, the integrand is non-zero in the limit $\rho \to \infty$, but the conclusion is that

$$B_7 = -\frac{1}{2\pi}(A^2+B^2)\int_0^\pi \cos 2\theta\, d\theta = 0.$$

Moreover, the same conclusion holds if any one of the alternative conditions on Γ is imposed. Hence it is concluded that

$$\frac{dI_7}{dt} = 4I_2 - 7gI_6 = 4K - 3V \tag{6.25}$$

in the case of infinite depth.

Finally, as I_8 is bounded under the assumption introduced above, there is no difficulty in concluding from (6.10′) and then (6.24) that

$$\frac{dI_8}{dt} = -gI_5 = -g[I_5(0) + \hat{I}_1 t]. \tag{6.26}$$

As might be expected, the angular momentum I_8 is thus shown to vary at a rate equal to the moment of the weight attributable to the displaced fluid, which moment varies linearly with time.

7. Conclusion

Apart from throwing some new light on the water-wave problem, our investigation may be of interest in exemplifying a general line of attack on nonlinear boundary-value problems that model evolutionary processes. Other prospective applications in fluid mechanics can be envisaged. As the first step, explained here in §4 and Appendix 2, the symmetry groups for the simplest, unrestricted form of the problem are identified systematically by use of infinitesimal-transformation and prolongation theory. Then, as exemplified in §5, an appeal to Hamiltonian structure or perhaps other variational characterization of the problem enables the corresponding conserved densities to be worked out. Having thus been disclosed, the set of conservation laws may yet be unclear as regards all their physical meanings, and our painstaking discussion in §6 showed how much further study may be needed to illuminate the significance of the formal mathematical results.

Two deliberate limitations of the present treatment should again be acknowledged. First, for the sake of simplicity, we have based most analytical developments on the assumption that the elevation of the free surface remains a single-valued function of horizontal position. Upon re-examination in the way indicated by Appendix 1, however, it is readily seen that all the results obtained have extensions to the case where the free surface becomes folded and so must be described parametrically. Second, and more important, there was an absence of rigorous justification for our claim that the derived lists of symmetries and conservation laws for the water-wave problem are exhaustive. The adopted method of systematic derivation points strongly to the truth of this claim, but full proof has been deferred to a separate study (Olver 1982).

We are indebted to the Science and Engineering Research Council for support of the programme that has included the investigation here reported. The paper was in part written during an extended visit by T. B. B. to the Mathematics Research Center, University of Wisconsin–Madison.

Appendix 1. Parametric representation of free surface

Here we reformulate the dynamical problem in order to cover the class of situations that was reviewed in §3, being precursory to the breaking of water waves. The free surface S becomes folded as illustrated in figure 1 (b), so that its elevation η is no longer a single-valued function of position in the horizontal plane and a parametric representation of S is then necessary.

To take advantage of the summation convention of Cartesian-tensor notation, the axes used in the main text are now relabelled $(x_1, x_2, x_3) \equiv (x, z, y)$. Accordingly, S is supposed to be described parametrically by

$$x_i = X_i(\mu, \nu, t), \quad i = 1, 2, 3, \tag{A 1}$$

where (μ, ν) ranges over a fixed two-dimensional domain Ω. With the same meaning as before, we again write

$$\Phi = \phi_S = \phi(X_1, X_2, X_3, t);$$

and, further compressing the notation for spatial derivatives of ϕ evaluated on S, we write

$$\Phi_{(i)} = (\partial\phi/\partial x_i)_S.$$

Note that in terms of

$$\gamma_1 = \frac{\partial(X_2, X_3)}{\partial(\mu, \nu)}, \quad \gamma_2 = \frac{\partial(X_3, X_1)}{\partial(\mu, \nu)}, \quad \gamma_3 = \frac{\partial(X_1, X_2)}{\partial(\mu, \nu)}, \quad J = \surd(\gamma_i^2),$$

the components of the outward unit normal to S are γ_i/J, and the element of surface area on S is $J\,d\mu\,d\nu$. It can be assumed without significant loss of generality that $J > 0$ everywhere.

Needless to say, the representation (A 1) is not unique. One of the possibilities, which is evidently suitable for certain other purposes, is to let (μ, ν) be Lagrangian coordinates specifying particular fluid particles in S, in which case we would have $\partial_t X_i = \Phi_{(i)}$. This choice has no special advantage at present, however, and moreover it is inconsistent with the simpler description used previously, which is, of course, recoverable from (A 1) by taking $(\mu, \nu) = (x_1, x_2)$, $X_1 = \mu$, $X_2 = \nu$, $X_3 = \eta(\mu, \nu, t)$. We therefore proceed from (A 1) on a general basis, leaving open the choice of parameter system. It is immediately plain that on this basis the dynamical problem cannot be reduced to the standard Hamiltonian form (1.2), with time derivatives of the functions X_i given *explicitly* in terms of the instantaneous state (X_i, Φ) of the system. But an appropriately modified Hamiltonian form (1.4) may be recognized as follows.

The kinematical boundary condition (2.4), expressing the velocity of S normal to itself in terms of the velocity potential for the fluid motion, is now replaced by its generalization

$$\gamma_i \partial_t X_i = J\Phi_{(n)} = \gamma_i \Phi_{(i)}. \tag{A 2}$$

Correspondingly, with the effect of surface tension included, the dynamical boundary condition (2.5′) becomes

$$\partial_t \Phi = -(\tfrac{1}{2}q^2 + gX_3 - 2\sigma H) + \Phi_{(i)} \partial_t X_i, \tag{A 3}$$

in which the mean curvature H is expressible in the standard way for parametric representations. Just as for their simpler versions, these generalized equations, coupled with the remaining, linear boundary conditions that determine ϕ from its boundary values Φ on S, complete the specifications of the evolutionary system. As before, we consider the solution as a vector-valued function, in the present case

$$\mathbf{U} = (X_1, X_2, X_3, \Phi) = \mathbf{U}(\mu, \nu, t), \tag{A 4}$$

and seek a quasi-Hamiltonian formulation in terms of $\mathbf{U}$.

Now, the kinetic energy of the system is given by

$$K = \frac{1}{2}\int_D |\nabla\phi|^2 \, dx_1 \, dx_2 \, dx_3 = \frac{1}{2}\int_\Omega \Phi_{(n)} \Phi J \, d\mu \, d\nu,$$

and the potential energy by

$$V = \tfrac{1}{2}g \int_\Omega X_3^2 \gamma_3 \, d\mu \, d\nu + \sigma \int_\Omega (J - \gamma_3) \, d\mu \, d\nu.$$

In expressing the first variation of the total energy $E = K + V$, we use the facts that

$$\dot{\Phi} = (\dot{\phi})_S + \Phi_{(i)} \dot{X}_i$$

and

$$J\dot{n} = \gamma_i \dot{X}_i.$$

In the reduction of $\dot{K}$, Green's theorem is again used as in §2; and the reduction of $\dot{V}$ is made through integrations by parts. The details in respect of the superficial energy proportional to σ are the same as are familiar from the theory of minimal surfaces. Thus, defining the gradient of E by the inner product corresponding to integration with respect to (μ, ν) over Ω, we find the four components of grad E to be

$$\left.\begin{aligned} \operatorname{grad}_{X_i} E &= (\tfrac{1}{2}q^2 + gX_3 - 2\sigma H)\gamma_i - J\Phi_{(n)}\Phi_{(i)}, \\ \operatorname{grad}_\Phi E &= J\Phi_{(n)} = \gamma_i \Phi_{(i)}. \end{aligned}\right\} \tag{A 5}$$

Hence, with $\mathbf{U}$ defined by (A 4), equations (A 2) and (A 3) are seen to be equivalent to

$$\mathscr{K}(\partial_t \mathbf{U}) = \operatorname{grad} E(\mathbf{U}), \tag{A 6}$$

where $\mathscr{K}$ is the skew-symmetric matrix defined as follows. In terms of

$$c_{ij} = \gamma_i \Phi_{(j)} - \gamma_j \Phi_{(i)} = -c_{ji},$$

we have

$$\mathscr{K} = \begin{pmatrix} 0 & c_{12} & c_{13} & -\gamma_1 \\ c_{21} & 0 & c_{23} & -\gamma_2 \\ c_{31} & c_{32} & 0 & -\gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{pmatrix}.$$

Given (A 2) and (A 3), substitution for $\partial_t X_i$ and $\partial_t \Phi$ confirms that the four components of (A 6) recover (A 5). Conversely, given (A 5) and (A 6), equation (A 2) is at once recovered as the fourth component of (A 6), and the first three components provide

$$(\tfrac{1}{2}q^2+gX_3-2\sigma H)\gamma_i - J\Phi_{(n)}\Phi_{(i)} = c_{ij}\partial_t X_j - \gamma_i \partial_t \Phi.$$

Multiplying by γ_i, summing over i, and then substituting for $J\Phi_{(n)}$ from the already implied result (A 2), we obtain

$$J^2\{\partial_t \Phi + \tfrac{1}{2}q^2+gX_3-2\sigma H\} = \gamma_i c_{ij}\partial_t X_j + \gamma_i \gamma_j \Phi_{(i)}\partial_t X_j = J^2\Phi_{(i)}\partial_t X_i,$$

which recovers (A 3) upon division by $J^2 > 0$.

Equation (A 6) exemplifies the quasi-Hamiltonian form (1.4), which is plainly the one concomitant with parametric representations of the free surface. It is readily confirmed that $\det[\mathscr{K}] = 0$, so that, as expected, $\mathscr{K}$ is generally not invertible.

Note that a two-dimensional version of the problem is included in the preceding account. To obtain it, the description (A 1) of S is simplified to

$$x_1 = X_1(\mu), \quad x_2 = \nu, \quad x_3 = X_3(\mu),$$

corresponding to which we have

$$\gamma_1 = -\partial_\mu X_3, \quad \gamma_2 = 0, \quad \gamma_3 = \partial_\mu X_1.$$

In (A 2), (A 3) and succeeding equations, the summations are then over $i = 1, 3$ only.

It may be of interest to note also how, in the case that S is not folded and so a non-parametric description is possible, the present formulation collapses into the simpler one used in the main text. Putting $X_1 = \mu = x_1$, $X_2 = \nu = x_2$ and $X_3 = \eta(\mu, \nu, t)$, we have $\gamma_1 = -\eta_\mu$, $\gamma_2 = -\eta_\nu$ and $\gamma_3 = 1$. Although none of the entries in the matrix $\mathscr{K}$ is cancelled in this case, the first two components of (A 5) and the left-hand side of (A 6) are vacuous since X_1 and X_2 are invariable. Thus (A 6) reduces to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_t \begin{pmatrix} \eta \\ \Phi \end{pmatrix} = \operatorname{grad} E(\eta, \Phi),$$

which is equivalent to (2.8).

Appendix 2. Symmetry groups for free-boundary problems

The study of symmetry groups associated with partial differential equations, a subject pioneered by Sophus Lie, has been significantly advanced in recent years, notably by L. V. Ovsiannikov. There is now an extensive literature on the subject, and for basic concepts bearing on what follows reference may be made to the books by Bluman & Cole (1974) and by Ovsiannikov (1982) (see also Olver 1979*b*, 1980*b*). It appears, however, that no general discussion of free-boundary problems from this standpoint is yet available. Accordingly, a brief treatment of such problems in abstract is here presented, outlining a procedure whereby the symmetry groups can be identified systematically.

For any free-boundary problem in the general class to be considered, the independent variables are written $(\mathbf{x}, y) = (x^1, \ldots, x^p, y) \in \mathbb{R}^{p+1}$. Time may be included as one of the variables x^i with label $i \leqslant p$, and y $(\equiv x^{p+1})$ is a distinguished coordinate such as that with the vertical direction in the water-wave problem. The dependent variables are written $\boldsymbol{\phi} = (\phi^1, \ldots, \phi^q) \in \mathbb{R}^q$. The problem is supposed to comprise a system of partial differential equations

$$\mathbf{\Lambda}(\mathbf{x}, y, \boldsymbol{\phi}) = 0 \tag{A 7}$$

satisfied in a domain

$$D_\eta = \{(\mathbf{x}, y) \colon y \leqslant \eta = h(\mathbf{x})\} \subset \mathbb{R}^{p+1},$$

together with boundary conditions holding on the free surface

$$S = \{(\mathbf{x}, y): y = \eta = h(\mathbf{x})\}.$$

(The need to introduce a separate notation η for the dependent variable describing the free surface, and to distinguish it from any *particular* function $\eta = h(\mathbf{x})$, will become apparent in the treatment that follows.) This representation is somewhat restrictive, implying S to be definable as the graph of a single-valued function of $\mathbf{x}$. More generally, S could be any surface described parametrically as in Appendix 1. For simplicity, however, the case where $h(\mathbf{x})$ is single-valued will be discussed first, and the modifications needed for the more general case will be indicated at the end.

The boundary conditions at S are represented by

$$\mathbf{\Gamma}(\mathbf{x}, \eta, \boldsymbol{\phi}_S) = 0, \tag{A 8}$$

being equations in the p-component independent variable $\mathbf{x}$. These will usually involve partial derivatives of η with respect to the components of $\mathbf{x}$, as well as η itself. They will also usually involve both the evaluation $\boldsymbol{\phi}_S \equiv \boldsymbol{\phi}(\mathbf{x}, \eta(\mathbf{x}))$ of the set of dependent variables at S and, as is left implicit in (A 8), the corresponding evaluations of derivatives of $\boldsymbol{\phi}$ with respect to y as well as $\mathbf{x}$. The system of equations (A 7) and (A 8), supplemented by conditions that can be left unwritten determining suitable asymptotic behaviour of $\boldsymbol{\phi}$ far from S in D_η, constitute the general form of free-boundary problem now in question. Its solution should be understood as a pair of functions $h: \mathbb{R}^p \to \mathbb{R}$ and $\mathbf{f}: D_\eta \to \mathbb{R}^q$, such that $\eta = h(\mathbf{x})$ and $\boldsymbol{\phi} = \mathbf{f}(\mathbf{x}, y)$ satisfy the system (A 7, 8). Clearly, the problem of gravity waves on water of unbounded depth has this form.

(Note that this form of problem, respective to a perturbed half-space D_η in $\mathbb{R}^{p+1}$, is the most fertile for any inquiry into symmetry groups. Further delimitations of D_η as may be required for various practical models, such as the introduction of fixed spatial boundaries at finite distances or the imposition of initial conditions with regard to a time variable included in $\mathbf{x}$, obviously cannot result in any increase of symmetry. Note also that for present purposes it is wholly justified to treat h and $\mathbf{f}$ as C^∞ functions in the stated senses.)

We consider a diffeomorphism of the whole space $\mathbb{R}^{p+1} \times \mathbb{R}^q$ to be given by

$$\tilde{\mathbf{x}} = \mathbf{X}(\mathbf{x}, y, \boldsymbol{\phi}), \quad \tilde{\mathbf{y}} = Y(\mathbf{x}, y, \boldsymbol{\phi}), \quad \tilde{\boldsymbol{\phi}} = \mathbf{P}(\mathbf{x}, y, \boldsymbol{\phi}).$$

If this is sufficiently close to the identity map, a domain D_η defined by $\eta = h(\mathbf{x})$ and a function $\boldsymbol{\phi} = \mathbf{f}(\mathbf{x}, y): D_\eta \to \mathbb{R}^q$ will be transformed one-to-one into a new domain $D_{\tilde{\eta}}$, with free boundary $\tilde{\eta} = \tilde{h}(\tilde{\mathbf{x}})$, and a new function $\tilde{\mathbf{f}}: D_{\tilde{\eta}} \to \mathbb{R}^q$. These are defined implicitly by the identities

$$\left.\begin{aligned} Y\{\mathbf{x}, h(\mathbf{x}), \mathbf{f}(\mathbf{x}, h(\mathbf{x}))\} &= \tilde{\mathbf{h}}\{\mathbf{X}(\mathbf{x}, h(\mathbf{x}), \mathbf{f}(\mathbf{x}, h(\mathbf{x}))\}, \\ \mathbf{P}\{\mathbf{x}, y, \mathbf{f}(\mathbf{x}, y)\} &= \tilde{\mathbf{f}}\{\mathbf{X}(\mathbf{x}, y, \mathbf{f}(\mathbf{x}, y)), \; Y(\mathbf{x}, y, \mathbf{f}(\mathbf{x}, y))\}. \end{aligned}\right\} \tag{A 9}$$

(Note that if the restriction to a neighbourhood of the identity is relaxed, then $\tilde{\eta}$, $\tilde{\boldsymbol{\phi}}$ may not be defined as single-valued functions, so that in general the class of transformations in question is only locally well-defined. It will be seen, however, that this proviso is admissible without loss of scope for present purposes.) Such a diffeomorphism is to be called a *symmetry* of the free-boundary problem (A 7, 8) if $\tilde{\eta}$, $\tilde{\boldsymbol{\phi}}$ is a (local) solution whenever η, $\boldsymbol{\phi}$ is a solution.

Henceforth indices i and j will denote labelling numbers in $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$

respectively, and repeated indices will imply summation. According to standard terminology of differential geometry, a *vector field* $\mathbf{v}$ on $\mathbb{R}^{p+1}\times\mathbb{R}^q$ is a first-order differential operator acting on smooth functions $\mathbb{R}^{p+1}\times\mathbb{R}^q\rightarrow\mathbb{R}$, thus

$$\mathbf{v}=\alpha^i(\mathbf{x},y,\boldsymbol{\phi})\frac{\partial}{\partial x^i}+\beta(\mathbf{x},y,\boldsymbol{\phi})\frac{\partial}{\partial y}+\gamma^j(\mathbf{x},y,\boldsymbol{\phi})\frac{\partial}{\partial\phi^j}. \tag{A 10}$$

Such a vector field is the *infinitesimal generator* of a *one-parameter group* of diffeomorphisms, obtainable by integrating the system of ordinary differential equations

$$\dot{x}^i=\alpha^i,\quad \dot{y}=\beta,\quad \dot{\phi}^j=\gamma^j, \tag{A 11}$$

where $\dot{x}^i$, $\dot{y}$, $\dot{\phi}^j$ denote derivatives with respect to the group parameter ϵ. (The value $\epsilon=0$ is taken to correspond to the identity element of the group.) The method used to establish symmetry groups for the present problem, just as for systems of partial differential equations without free-boundary conditions, consists in finding their infinitesimal generators as follows.

First we consider the vector field $\mathbf{v}$ prolonged to the space of derivatives of the dependent variables, up to some requisite order. In terms of the prolongation $\mathrm{pr}\,\mathbf{v}$, to be defined (cf. (A 14)), the infinitesimal criterion

$$(\mathrm{pr}\,\mathbf{v})\,\Lambda=0\quad\text{whenever}\quad\Lambda(\mathbf{x},y,\boldsymbol{\phi})=0, \tag{A 12}$$

then gives a number of elementary differential equations in the coefficient functions of $\mathbf{v}$, and their general solution defines the (infinitesimal) symmetry group for the system $\Lambda=0$.

To extend this notion to the free-boundary problem, we need to consider the prolongation of $\mathbf{v}$ on the boundary itself. This can most readily be done after adopting a perturbational description of the symmetries, which is now outlined.

For small values of the group parameter ϵ, the action of the group on a particular function $\boldsymbol{\phi}=\mathbf{f}(\mathbf{x},y)$ is

$$\tilde{f}^j(\mathbf{x},y)=f^j(\mathbf{x},y)+\epsilon\delta f^j(\mathbf{x},y)+O(\epsilon^2),$$

in which according to (A 9)

$$\delta f^j(\mathbf{x},y)=\gamma^j-\alpha^i\frac{\partial f^j}{\partial x^i}-\beta\frac{\partial f^j}{\partial y},$$

with the right-hand side evaluated at $(\mathbf{x},y,\mathbf{f}(\mathbf{x},y))$. This may be written in the concise form

$$\delta f^j=\delta\phi^j-\mathbf{v}_1(f^j), \tag{A 13}$$

where $\delta\phi^j\equiv\gamma^j\{\mathbf{x},y,\mathbf{f}(\mathbf{x},y)\}$ and

$$\mathbf{v}_1=\alpha^i\{\mathbf{x},y,\mathbf{f}(\mathbf{x},y)\}\frac{\partial}{\partial x^i}+\beta\{\mathbf{x},y,\mathbf{f}(\mathbf{x},y)\}\frac{\partial}{\partial y}$$

is the projection of $\mathbf{v}$, as determined by $\mathbf{f}$, onto the space of independent variables. Now the prolonged vector field to any required order $k\geqslant 1$ can be expressed by

$$\mathrm{pr}\,\mathbf{v}=\mathbf{v}+\delta\phi^j_k\frac{\partial}{\partial\phi^j_k}, \tag{A 14}$$

where ϕ^j_k with $j=1,\ldots,q$ denotes all the k-th and lower-order derivatives of ϕ^j with respect to the $p+1$ independent variables. Thus, at order κ $(1\leqslant\kappa\leqslant k)$, there are included $qp_\kappa=q(p+\kappa)!/p!\kappa!$ components

$$\frac{\partial^\kappa\phi^j}{\partial(x^1)^{\lambda_1}\ldots(\partial x^{p+1})^{\lambda_{p+1}}},\quad\lambda_1+\ldots+\lambda_{p+1}=\kappa,$$

and the total number of components is $q(p_1+\ldots+p_k)$. In the same sense we shall also use the notation $\partial_k \phi^j = \phi^j_k$. Since evidently

$$\delta(\partial f^j/\partial x^i) = \partial(\delta f^j)/\partial x^i, \quad i = 1,\ldots,p+1,$$

equation (A 13) implies that

$$\begin{aligned}\delta(\partial_k \phi^j) &= \delta(\partial_k f^j)+\mathbf{v}_1(\partial_k f^j)\\ &= \partial_k\{\delta\phi^j-\mathbf{v}_1(f^j)\}+\mathbf{v}_1(\partial_k f^j).\end{aligned}$$

Moreover, since this formula holds for all possible particular choices of the function $\boldsymbol{\phi} = \mathbf{f}(\mathbf{x}, y)$, we may replace the derivatives $\partial_k f^j$ by the derivatives ϕ^j_k of the dependent variables ϕ^j wherever they occur. This final step recovers the known formula

$$\left.\begin{aligned}\delta(\phi^j_k) &= \partial_k\{\gamma^j-\mathbf{v}_1(\phi^j)\}+\mathbf{v}_1(\phi^j_k),\\ \mathbf{v}_1 &= \alpha^i\partial/\partial x^i+\beta\partial/\partial y,\end{aligned}\right\} \tag{A 15}$$

which expresses the infinitesimal variation of ϕ^j_k (cf. Olver 1979*b*, 1980*b*; also Eisenhart 1933, p. 106, eqn. (28.12), for the same result in a recursive form). Note that the coefficients α^i, β, γ^j in (A 15) are now all in their original form, depending on $\mathbf{x}$, y, $\boldsymbol{\phi}$.

To investigate behaviour at the free boundary, we write $f^j_S(\mathbf{x}) = f^j(\mathbf{x}, h(\mathbf{x}))$ as before and note that

$$\begin{aligned}\tilde{f}^j_{\tilde{S}}(\mathbf{x}) &= \tilde{f}^j(\mathbf{x}, \tilde{h}(\mathbf{x}))\\ &= f^j(\mathbf{x}, \tilde{h}(\mathbf{x}))+\epsilon\delta f^j(\mathbf{x}, \tilde{h}(\mathbf{x}))+O(\epsilon^2)\\ &= f^j(\mathbf{x}, h(\mathbf{x}))+\epsilon\{f^j_y(\mathbf{x}, h(\mathbf{x}))\,\delta h(\mathbf{x})+\delta f^j(\mathbf{x}, h(\mathbf{x}))\}+O(\epsilon^2).\end{aligned}$$

Thus

$$\delta(f^j_S) = (\delta f^j)_S+(f^j_y)_S\,\delta h. \tag{A 16}$$

It will be helpful to introduce the notation $\Phi^j = \phi^j_S$, $\Phi^j_{(k)} = (\phi^j_k)_S$, in terms of which (A 16) evaluated on S becomes

$$\delta\Phi^j = \delta(f^j_S)+\mathbf{v}_0(f^j_S),$$

where

$$\mathbf{v}_0 = \alpha^i\{\mathbf{x}, h(\mathbf{x}), \mathbf{f}_S(\mathbf{x})\}\frac{\partial}{\partial x^i}$$

is the restriction of $\mathbf{v}$ to S. Moreover,

$$\frac{\partial f^j_S}{\partial x^i} = \left(\frac{\partial f^j}{\partial x^i}+\frac{\partial f^j}{\partial y}\frac{\partial h}{\partial x^i}\right)_S,$$

and therefore

$$\delta\Phi^j = (\delta f^j_S)+(f^j_y)_S\,\delta h+[\mathbf{v}_0(f^j)+f^j_y\,\mathbf{v}_0(h)]_S.$$

Using the counterpart of (A 13) for δh, namely

$$\delta h = \beta-\mathbf{v}_0(h), \tag{A 17}$$

we hence have

$$\begin{aligned}\delta\Phi &\equiv \delta(\phi^j_S) = [\gamma^j-\mathbf{v}_1(f^j)+\beta f^j_y+\mathbf{v}_0(f^j)]_S\\ &= (\gamma^j)_S = (\delta\phi^j)_S.\end{aligned} \tag{A 18}$$

The operations δ and restriction to S are thus shown formally to commute. The same argument plainly extends to derivatives of ϕ^j, and so we also have

$$\delta(\Phi^j_{(k)}) \equiv \delta[(\phi^j_k)_S] = [\delta(\phi^j_k)]_S, \tag{A 18'}$$

in which $\delta(\phi^j_k)$ is given by (A 15).

Finally, we must discuss the prolongation of $\mathbf{v}$ to the derivatives of η. It is clear that

$$\delta\eta = \beta_S = \delta h + \mathbf{v}_0(h).$$

Moreover, since $\partial(\delta h)/\partial x^i = \delta(\partial h/\partial x^i)$, we have

$$\begin{aligned}\delta(\eta_k) &= \delta(\partial h/\partial x^k) + \mathbf{v}_0(\partial h/\partial x^k)\\ &= (\partial/\partial x^k)\{\beta_S - \mathbf{v}_0(h)\} + \mathbf{v}_0(\partial h/\partial x^k).\end{aligned}$$

As before, since this formula holds for all particular functions $\eta = h(\mathbf{x})$ and $\boldsymbol{\phi} = \mathbf{f}(\mathbf{x}, y)$, we may replace $\partial h/\partial x^k$ by η_k *and* $(\partial_k f^j)_S$ by $(\phi^j_k)_S = \Phi^j_{(k)}$ wherever they occur. The result complementing (A 15) is therefore

$$\left.\begin{aligned}\delta(\eta_k) &= \partial_k\{\beta_S - \mathbf{v}_0(\eta)\} + \mathbf{v}_0(\eta_k),\\ \mathbf{v}_0 &= \alpha^i_S\,\partial/\partial x^i,\end{aligned}\right\} \tag{A 19}$$

in which β_S and all the α^i_S are evaluated at $\mathbf{x}, \eta, \boldsymbol{\phi}_S$. This is just the standard prolongation for the vector field

$$\mathbf{v}_S = \alpha^i(\mathbf{x}, \eta, \boldsymbol{\phi})\,\partial/\partial x^i + \beta(\mathbf{x}, \eta, \boldsymbol{\phi})\,\partial/\partial\eta,$$

with η considered as a function of $\mathbf{x}$ and ϕ as an arbitrary function of $\mathbf{x}$ and η.

Accordingly, the infinitesimal criterion of invariance gives the following result:

THEOREM A 1. *Take the free-boundary problem* (A 7), (A 8). *Let* $\mathbf{v}$ *be a vector field with prolongation* pr $\mathbf{v}$ *defined by* (A 14), *and with boundary prolongation defined by*

$$\mathrm{pr}\,\mathbf{v}_S = \alpha^i_S\,\partial/\partial x^i + (\delta\phi^j_k)_S\,\partial/\partial\Phi^j_{(k)} + \delta(\eta_k)\,\partial/\partial\eta_k, \tag{A 20}$$

in which $\delta(\phi^j_k)_S$ *is given by* (A 15) *and* $\delta(\eta_k)$ *by* (A 19). *Then* $\mathbf{v}$ *is a one-parameter symmetry group for the problem if and only if*

$$\mathrm{pr}\,\mathbf{v}(\boldsymbol{\Lambda}) = 0 \quad \text{whenever} \quad \boldsymbol{\Lambda} = 0, \tag{A 21}$$

and

$$\mathrm{pr}\,\mathbf{v}_S(\boldsymbol{\Gamma}) = 0 \quad \text{whenever} \quad \boldsymbol{\Gamma} = 0. \tag{A 22}$$

This theorem can readily be generalized to the case that the free surface S is defined parametrically, being represented in the form $\mathbf{x} = \mathbf{h}(\boldsymbol{\mu})$, where the parameters $\boldsymbol{\mu}$ range over $\mathbb{R}^p$ or some subdomain of $\mathbb{R}^p$. Clearly the first two sums of terms in (A 20) are unchanged, with the restrictions to S given their appropriate meanings which are obvious. To modify the third sum of terms appropriately, one needs to work out $\delta(x^j_k)_S$ with $x^j_k = \partial x^j/\partial\mu^k$. By reasoning similar to that leading to (A 19), it is found that

$$\delta(x^j_k) = \partial\alpha^j_S/\partial\mu^k. \tag{A 19'}$$

Accordingly, Theorem A 1 holds as before, except that (A 20) is replaced by

$$\mathrm{pr}\,\mathbf{v}_S = \alpha^i_S\,\partial/\partial x^i + (\delta\phi^j_k)_S\,\partial/\partial\Theta^j_{(k)} + \delta(x^j_k)\,\partial/\partial x^j_k,$$

with $\delta(x^j_k)$ given by (A 19′).

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