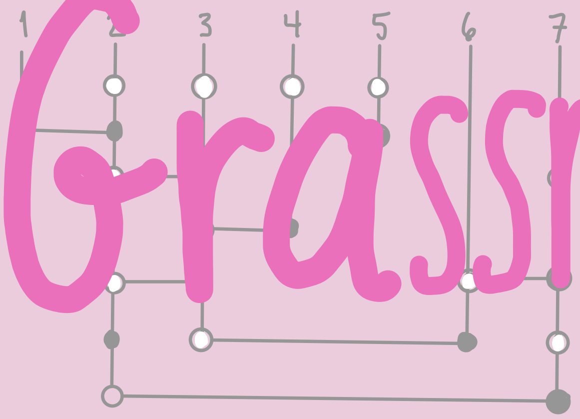
the Grassmannian



UMN Student Combinatorics
& Algebra Seminar

February 10th, 2022

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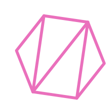
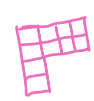
What's the Plan Stan?

Today, we'll explore the Grassmannian in a few different settings:

I. Some Enumerative Geometry
peak into Schubert calculus

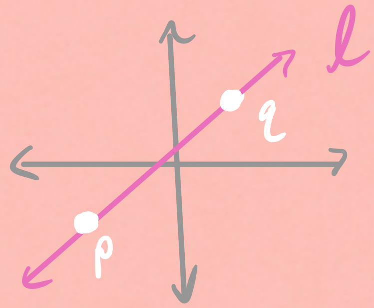
II. $Gr(2,n)$: Triangulations of n-gons 

III. $Gr(k,n)$: Tableaux Combinatorics 

Motivation: Enumerative Geometry

Q1: How many lines pass through two given points in the plane?

A: one line!



Q2: How many lines intersect four given lines in 3-dimensional space?

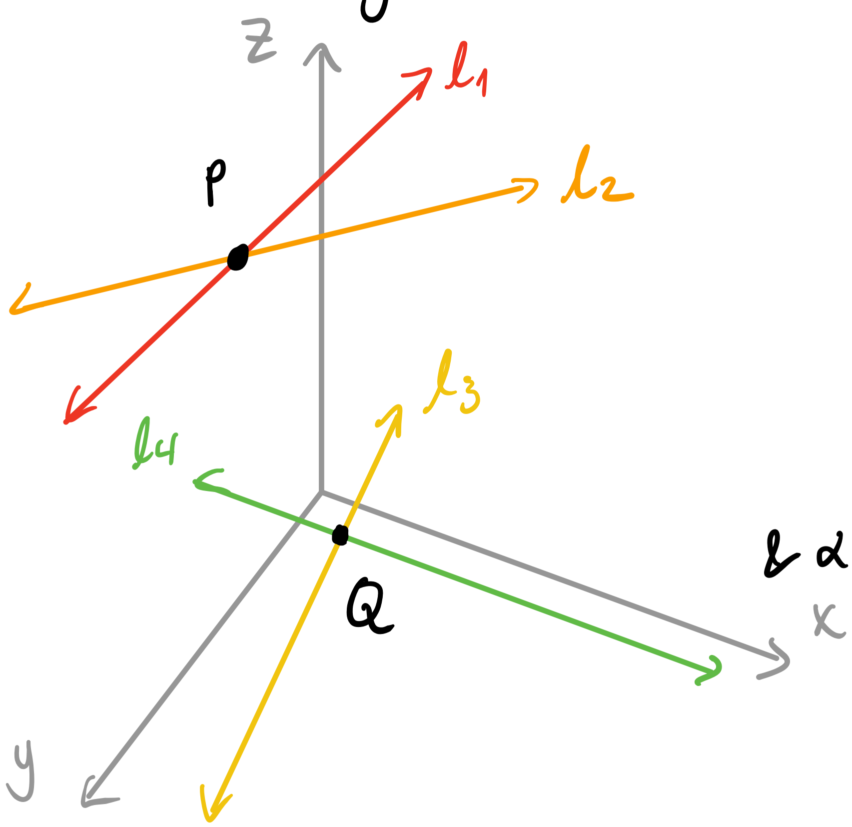
A: (generically) 2

See Schuberts "Principle of Conservation of Number"

Principle on Conservation of Number:

<u>Claim</u>: There are generically 2 lines that intersect 4 given lines in 3-dimensional space.

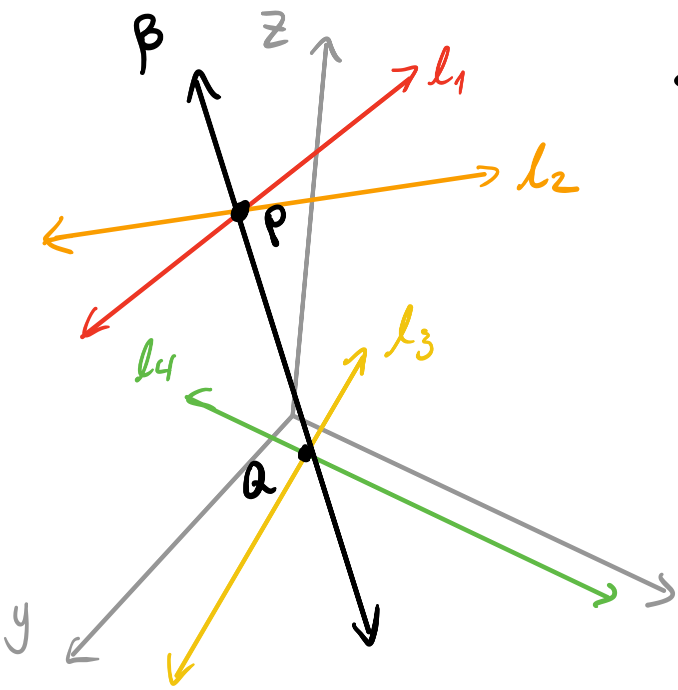
Suppose the four lines l_1, l_2, l_3, l_4 were arranged so that



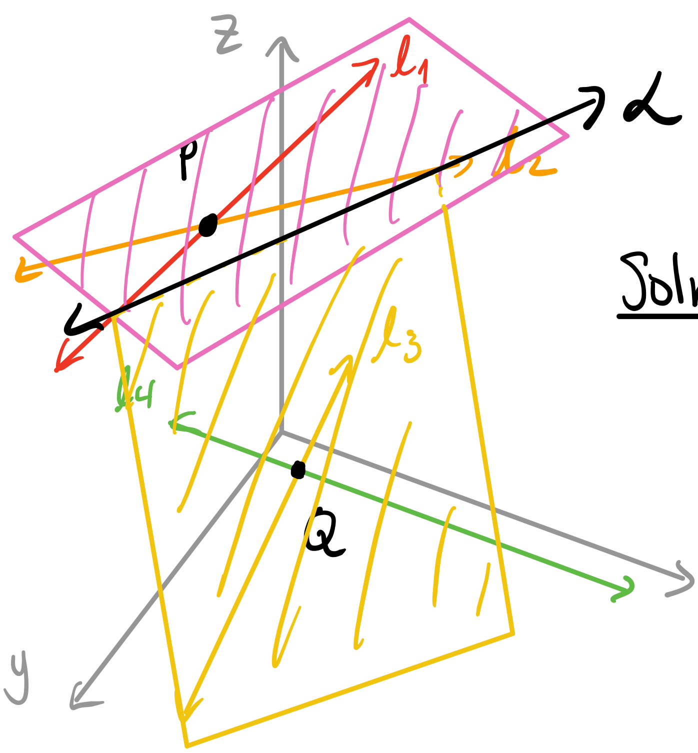
& none of the pairs of lines intersect but planes generated by $\{l_1, l_2\} =: \alpha_1$ & $\alpha_2 = \{l_3, l_4\}$ are not parallel

Then we can find exactly two lines that intersect all 4 given lines.

Principle of Conservation of Number Cont.



Soln ①: the line β that is determined by P & Q



Soln ②: the line that intersects planes α_1 & α_2

α

This Solution: Not Accepted

Schubert argued that this constituted a proof that there were only 2 solutions for any 4 lines in 3-dimensional space.

Where is the rigor? How can we generalize?

Q3: How many k-dimensional subspaces of $\mathbb{C}^n$ intersect each of $k \cdot (n-k)$ fixed subspaces of dimension $n-k$ non-trivially?

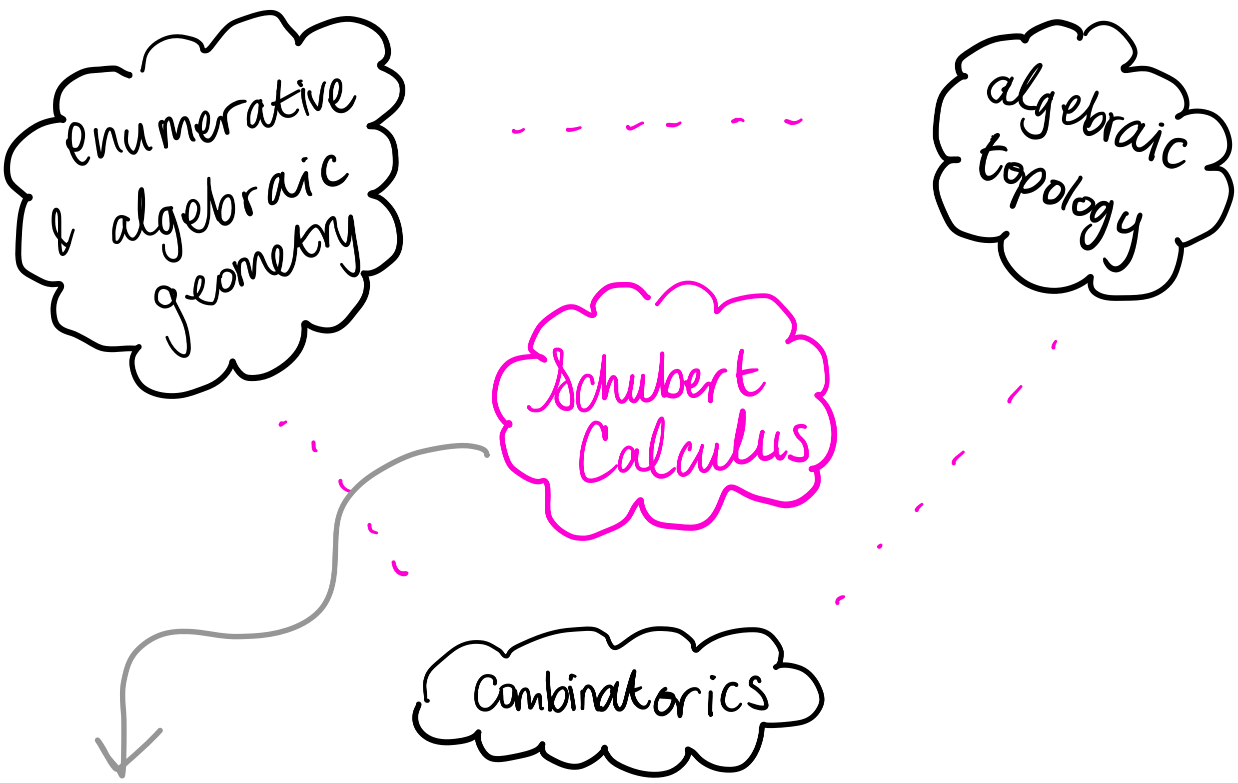
A: enter world of Schubert Calculus

Hilbert's 15^{th} Problem: rigorize Schubert's Principle of Conservation of Number to answer these questions

Schubert Calculus: Bird's Eye View

Q1, Q2, Q3 are all classical enumerative geometry questions – the study of counting the intersection points of geometric objects.

Schubert Calculus: a modern approach



c.f. "Variations on a Theme of Schubert Calculus"
Maria Gillespie

Formalizing Our General Question:

Q3: How many k-dimensional subspaces of $\mathbb{C}^n$ intersect each of $k\cdot(n-k)$ fixed subspaces of dimension $n-k$ non-trivially?

A: define the Grassmannian

The Grassmannian is

$$Gr(k,n) := \{k\text{-dim'l subspaces of } \mathbb{C}^n\}$$

A Hot Topic @UMN: The Grassmannian

Rebecca Patrias
PhD '16

Moriah Elkin
Current grad student

Chris Fraser
postdoc '21

Sunita Chepuri
PhD '20

numerous UMN Combinatorics REU Projects!

and more!!!

Understanding the Grassmannian: Two Levels during this Talk

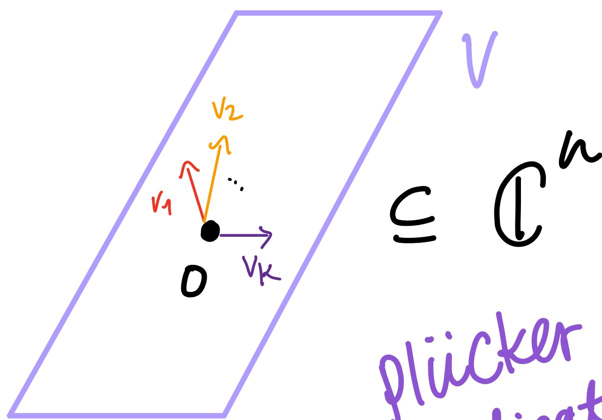
① Coordinate-wise

$$\begin{bmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & 2 \end{bmatrix}$$

$$\wr$$

$$\begin{bmatrix} 0 & 1 & 3 & 1 & -6 & 4 & -5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix}$$

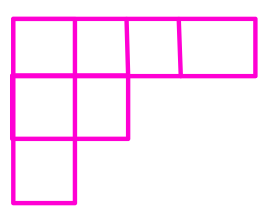
$\Delta_{i,j,k}$?



Plücker coordinates

② local

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{bmatrix} \leftrightarrow$$



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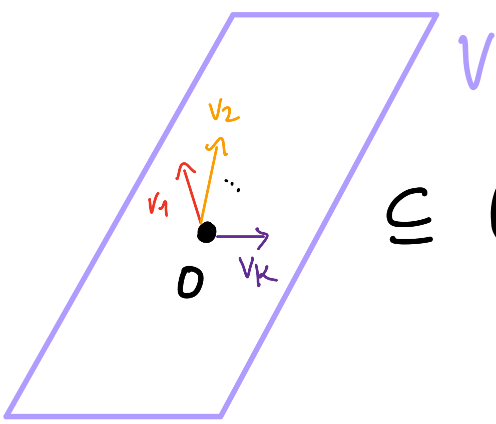
Schubert cells

Coordinatizing $Gr(k,n)$:

Q: What do elements in $Gr(k,n)$ look like?

Fix $0 \leq k \leq n$.

$$Gr(k,n) = \{ V \subseteq \mathbb{C}^n : \dim(V) = k \}$$



$\subseteq \mathbb{C}^n$

Choose a basis $v_1, v_2, \ldots, v_k$ & put them into a $k \times n$ matrix M

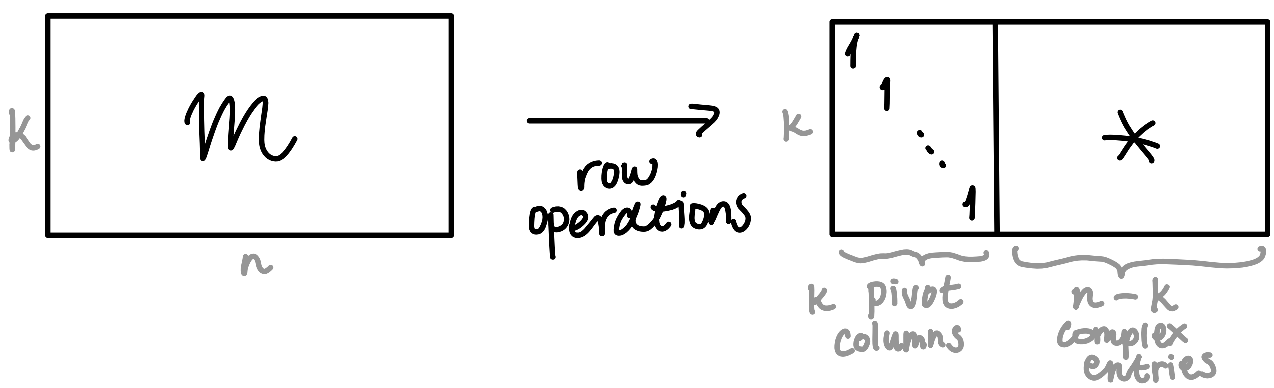
$$V \xleftrightarrow{\text{is the same data as}\ldots} M = \begin{array}{c} \\ k\left\{ \right. \end{array} \underbrace{\begin{pmatrix} & & & \\ & & & \end{pmatrix}}_{n}$$

will be full rank

Consider M up to change of basis or row operations!

Coordinatizing $Gr(k,n)$ Continued:

$$Gr(k,n) = \left\{ \begin{array}{c} \text{full rank } k \times n \\ \text{matrices} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{row} \\ \text{operations} \end{array} \right\}$$



Fact: Each point of $Gr(k,n)$ is the row span of a unique full rank $k \times n$ matrix in reduced row echelon form.

Ex $Gr(3,7)$

$$\begin{pmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Plücker Coordinates:

Entries $\forall \in \mathbb{C}$ are coordinates of $Gr(k,n)$

$[n] := \{1, 2, \ldots, n\}$

$\binom{[n]}{k} = \{ J \subseteq [n] : |J| = k \}$

k-elt subsets of $\{1, \ldots, n\}$

For any $J \in \binom{[n]}{k}$, the Plücker coordinate $\Delta_J(m)$ = det of submatrix of m with column set J

Ex| $Gr(2,4)$, take point $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & -3 & 0 & 3 \end{pmatrix}$.

The Plücker coordinates are:

$$\left(\underset{\Delta_{12}}{\begin{vmatrix} 0 & 0 \\ 1 & -3 \end{vmatrix}} : \underset{\Delta_{13}}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} : \underset{\Delta_{14}}{\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}} : \underset{\Delta_{23}}{\begin{vmatrix} 0 & 1 \\ -3 & 0 \end{vmatrix}} : \underset{\Delta_{24}}{\begin{vmatrix} 0 & 2 \\ -3 & 3 \end{vmatrix}} : \underset{\Delta_{34}}{\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}} \right)$$

which parameterizes the point $(0 : -1 : -2 : 3 : 6 : 3) \in \mathbb{P}^{\binom{4}{2}-1}$

Plücker Relations:

Ex $Gr(2,4) \ni \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$

Then $\Delta_{12} = 1$, $\Delta_{13} = c$, $\Delta_{14} = d$,
$\Delta_{23} = -a$, $\Delta_{24} = -b$, $\Delta_{34} = ad-bc$

Plücker coordinates Δ_J satisfy algebraic relations:

$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

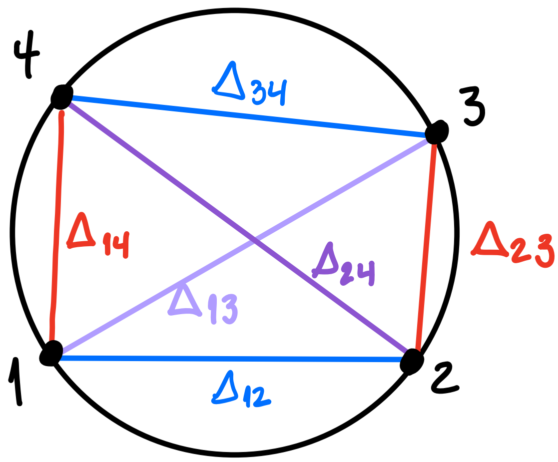
$$c \cdot (-b) = 1 \cdot (ad-bc) + d \cdot (-a)$$

These algebraic relations are precisely the equations whose zero loci give the projective variety structure on $Gr(k,n)$.

Connection to Ptolemy's Theorem:

In $Gr(2,n)$, the Plücker relations suggest a connection to an old theorem from geometry.

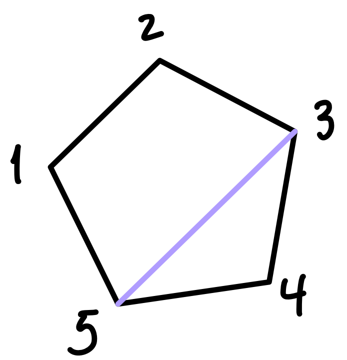
Let $\Delta_{\{i,j\}}$ = segment connecting i to j



$$\Delta_{13} \cdot \Delta_{24} = \Delta_{12} \cdot \Delta_{34} + \Delta_{14} \cdot \Delta_{23}$$

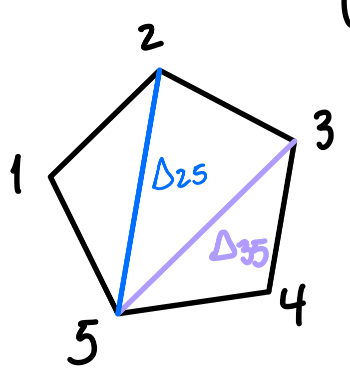
$Gr(2,n)$: Triangulations of n-gons

Each Plücker coordinate $\Delta_{\{i,j\}}$ is associated to a diagonal of an n-gon



$\longleftrightarrow \quad \Delta_{35}$

Taking maximal set of non-crossing diagonals gives a triangulation:

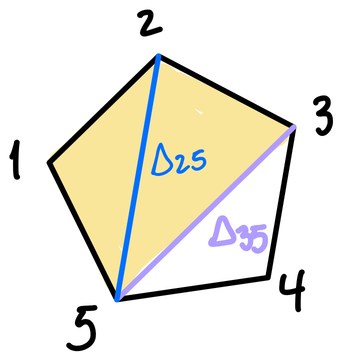


$\longleftrightarrow \quad T = \{\Delta_{25}, \Delta_{35}\}$

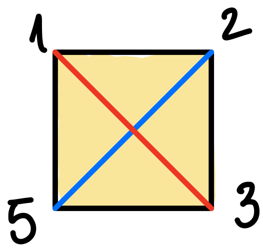
<u>Q</u>: How can we compute the remaining $\Delta_{\{i,j\}}$ diagonals?

<u>A</u>: iterated Ptomely Theorem or Plücker relations!

Triangulations & the Associahedron:



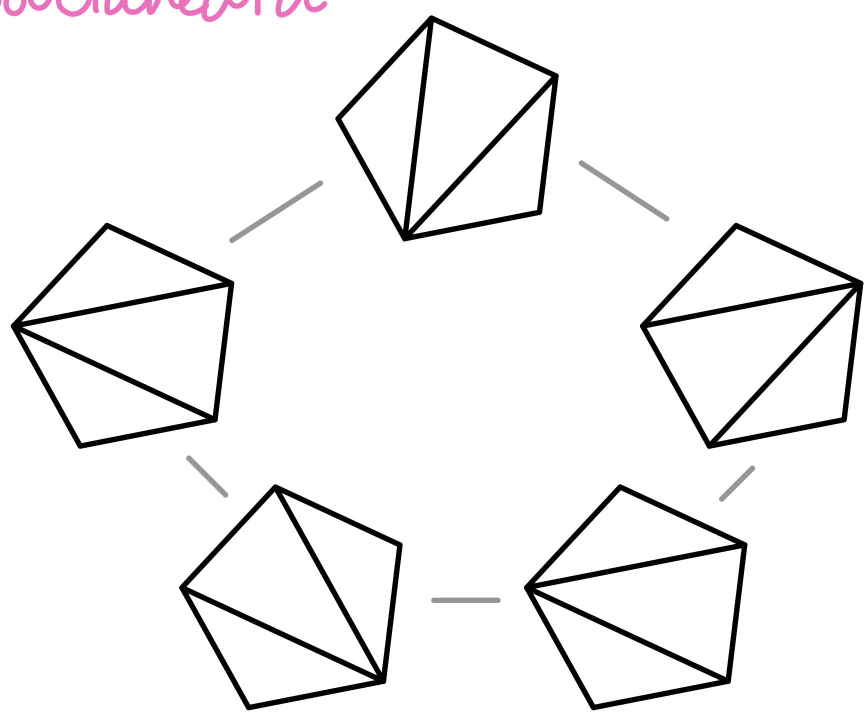
Our initial seed...
then apply Plücker
relation on :



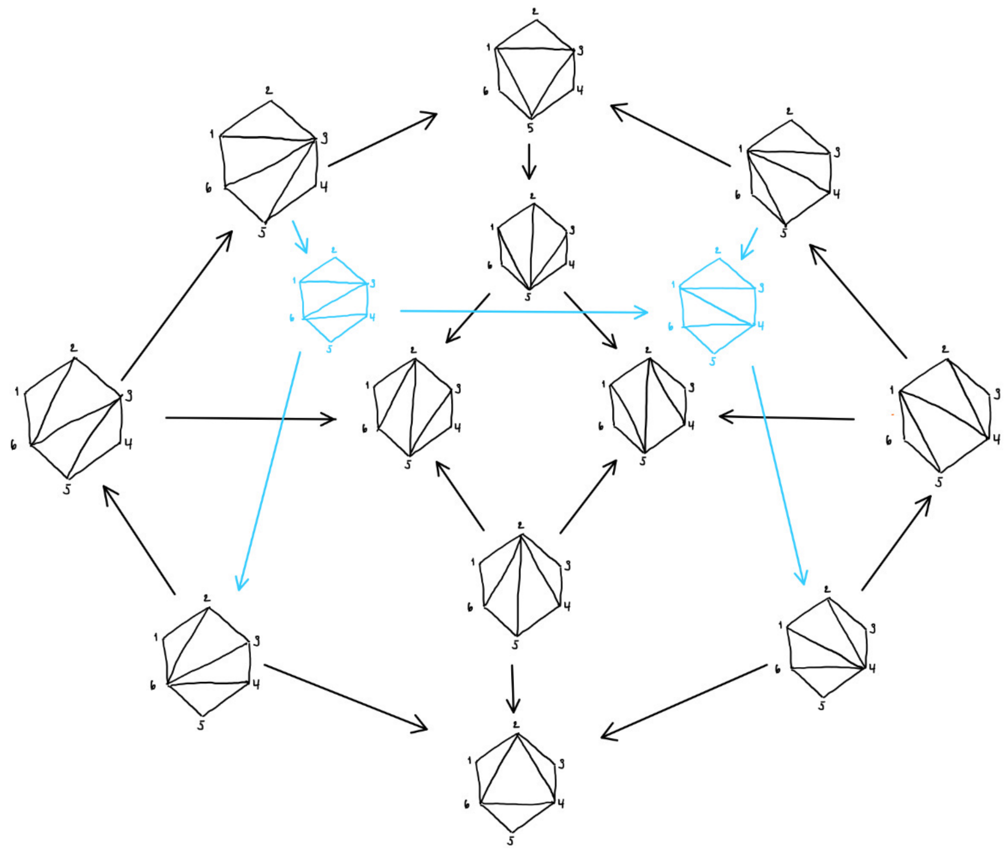
$$\Delta_{25}\Delta_{13} = \Delta_{12}\Delta_{35} + \Delta_{15}\Delta_{23}$$

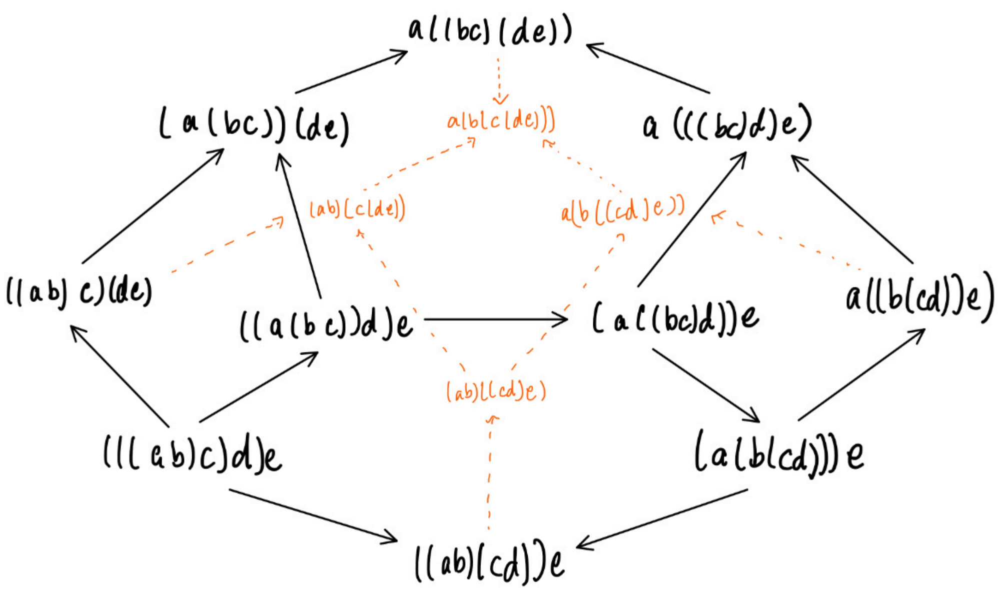
$$\rightsquigarrow \Delta_{13} = \frac{\Delta_{12}\Delta_{35} + \Delta_{15}\Delta_{23}}{\Delta_{25}}$$

The Associahedra:
(n=2)



The Associahedra ($n = 3$):

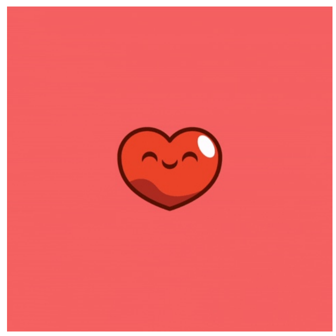




Connection to Cluster Algebras:

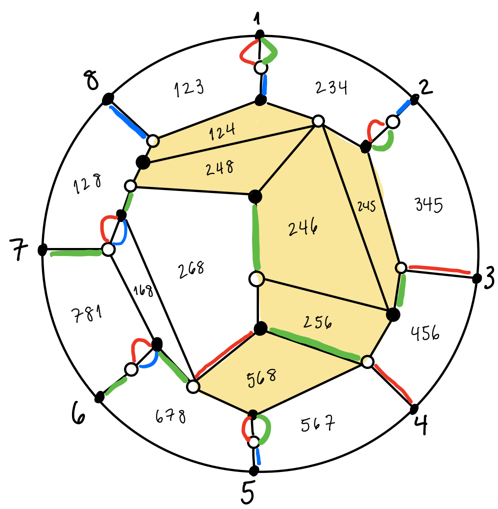
It turns out you can endow a cluster algebra structure on $Gr(2,n)$ using this model of triangulations of n-gon.

Cluster algebras are amazing & you should learn about them



Last Cluster Remark: For $k \geq 3$, $Gr(k,n)$'s cluster algebra is not well understood...

Shout out to Gregg & Moriah for investigating $Gr(3,8)$ with me :)



Local Structure of Gr(k,n):

Now, we explore local topological structure of Gr(k,n)!

Recall: points in the Grassmannian can be represented as $k \times n$ matrices in reduced row echelon form!

Ex Gr(3, 7)

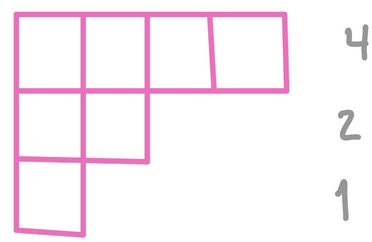
$$\begin{pmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The set of all such matrices in a particular reduced row echelon form is called a Schubert cell!

Schubert Cells & Schubert Varieties:

Schubert cells are indexed by partitions!

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \lambda = (4,2,1)$$



The Schubert cell Ω_λ° is essentially

$$\Omega_\lambda^\circ = \{ V \in Gr(k,n) : \text{matrix associated to } V \text{ gives } \lambda \}$$

$$= \{ V \in Gr(k,n) : \dim(V \cap \langle e_1, \dots, e_r \rangle) = i \}$$

w/ $n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}$

$$\cong \mathbb{C}^{\# * s}$$

Ex| $\mathbb{C}^5$ as we have 5 $*$'s

The closure of $\overline{\Omega_\lambda^\circ} = \Omega_\lambda$ is a Schubert variety!

Schubert Decomposition of $Gr(k,n)$:

Schubert Decomposition:

"Schubert cells"

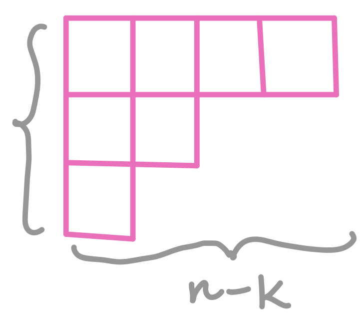
$$Gr(k,n) = \bigsqcup_{\lambda} \Omega_{\lambda}$$

where λ is a partition of n w/ at most $(n-k)$ parts (ie) λ is a tableaux that fits inside a $k \times (n-k)$ rectangle.

Ex| $Gr(3,7)$

$\lambda = (4,2,1)$

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \longleftrightarrow$$



Topological Consequences to Schubert Decomposition:

<u>BIG IDEA</u>: Schubert cells give a cell complex structure on $Gr(k,n)$ which can be used to compute $H^*(Gr(k,n))$

Ex| $Gr(2,4)$

0-skeleton: $\Omega^\circ_{(2,2)}$

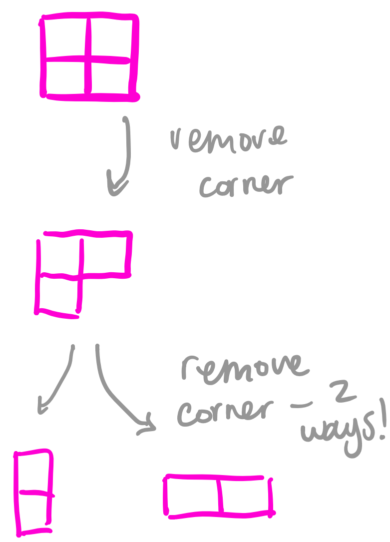
attach 2-cell

2-skeleton: $\Omega^\circ_{(2,1)}$

attach two 4-cells

4-skeleton: $\Omega^\circ_{(1,1)} \sqcup \Omega^\circ_{(2)}$

⋮



$$Gr(2,4) = \Omega^\circ_{(2,2)} \sqcup \Omega^\circ_{(2,1)} \sqcup \Omega^\circ_{(1,1)} \sqcup \Omega^\circ_{(2)} \sqcup \Omega^\circ_{(1)} \sqcup \Omega^\circ_{\emptyset}$$

Intense Results:

① $H^*(Gr(k,n);\mathbb{Z}) = \bigoplus_{\lambda} \mathbb{Z}\cdot\sigma_\lambda$

where $\sigma_\lambda = [\Omega_\lambda^\circ]$ is a Schubert class.

② Multiplication in the ring $H^*(Gr(k,n))$ is given by intersecting Schubert varieties.

③ This multiplication table is completely determined using tableaux combinatorics

e.g. Littlewood Richardson, Pieri, etc.

④ Since Schur functions are also indexed by partitions, there is a connection to the ring of symmetric functions

$$H^*(Gr(k,n)) \cong \Lambda(x_1, x_2, \ldots)/(\lambda \notin B)$$

$$\sigma_\lambda \longmapsto s_\lambda$$

Schubert $\qquad$ Schur

Recap of the Gr(k,n) Whirlwind:

The Grassmannian is a natural object to aim to understand!

↳ dating back to Hilbert's 15^{th} problem

Gr(k,n) has rich structure & people have developed beautiful maths to understand it as a ...

topological space

projective variety

Cluster algebra

etc.

Thank you for listening

Questions?

hit me up for references & further readings

kaylaw@umn.edu

extra
Example of Schubert Decomposition:

Take $\lambda = \square\square$, look at $\Omega_{\square\square}$ in $Gr(1,6)$. $(n=6, k=1)$

$(n-k+i-\lambda_i$
$= 6-1+1-2 = 4)$

By defn,

$$\Omega_{\square\square} = \{ V \in Gr(1,6) : \dim(V \cap \langle e_1, \ldots, e_4 \rangle) \geq 1 \}$$

$\Rightarrow$ any $V \in \Omega_{\square\square}$ has $e_5, e_6 = 0$. So any point in this Schubert variety is of the form:

$$[0,0,1,*,*,*], \quad [0,0,0,1,*,*]$$
$$[0,0,0,0,1,*], \quad [0,0,0,0,0,1]$$

So:

$$\Omega_{\square\square} = \Omega^{\circ}_{\square\square} \sqcup \Omega^{\circ}_{\square\square\square} \sqcup \Omega^{\circ}_{\square\square\square\square} \sqcup \Omega^{\circ}_{\square\square\square\square\square}$$