Skating Through Math: An Introduction to Ice Models

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University of Minnesota

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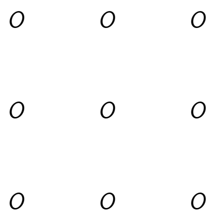
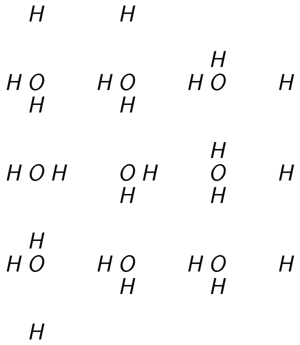


Figure: A diagram representing a sheet of square ice.

Ice Models



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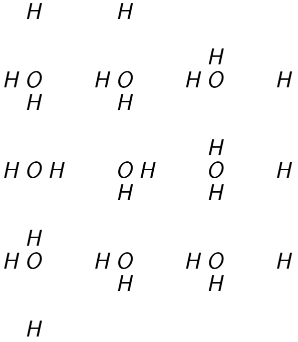

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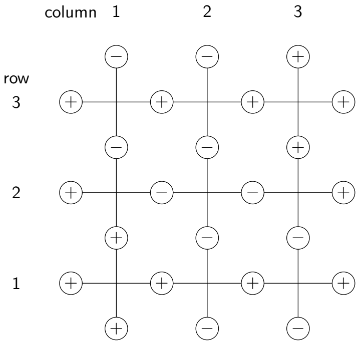

Figure: The corresponding lattice model state

Boltzmann Weights

Goal of Statistical Mechanics

Study *macroscopic* behavior (such as energy) from *microscopic* interactions (such as particle configurations).

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$\implies$ Every vertex in the lattice model state must be one of the following:

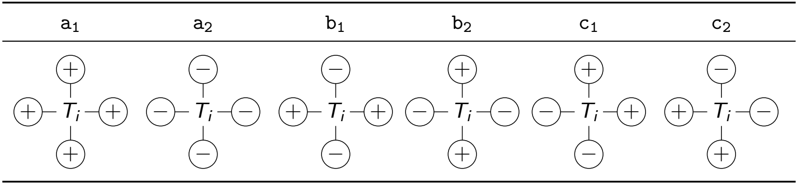

Figure: The admissible vertices for row i of the six-vertex model.

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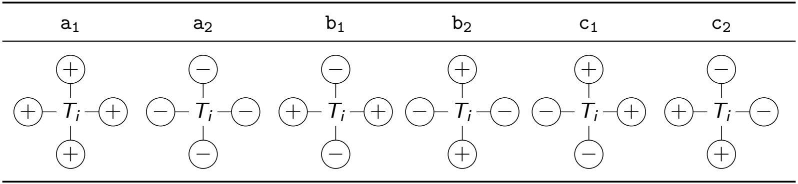

Figure: The admissible vertices for row *i* of the six-vertex model.

Assign each vertex v a *Boltzmann weight* $\mathrm{wt}(v)$ corresponding to its *probability*

The Partition Function

To compute the partition function:

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- Z is closely related to the energy of the system.

Examples of Lattice Models

Example 1: Actual Ice

a_1	a_2	b_1	b_2	c_1	c_2
$\begin{smallmatrix} & \oplus & \\ \oplus & T_i & \oplus \\ & \oplus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \ominus & T_i & \ominus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \oplus & T_i & \oplus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \oplus & \\ \ominus & T_i & \ominus \\ & \oplus & \end{smallmatrix}$	$\begin{smallmatrix} & \oplus & \\ \ominus & T_i & \oplus \\ & \ominus & \end{smallmatrix}$	$\begin{smallmatrix} & \ominus & \\ \oplus & T_i & \ominus \\ & \oplus & \end{smallmatrix}$
1	1	1	1	1	1

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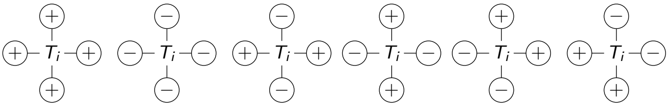

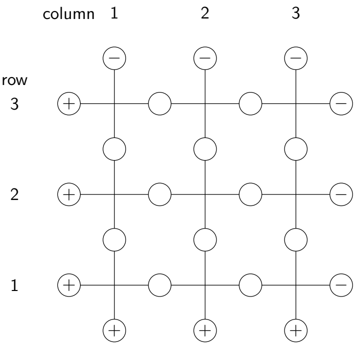

$Z =$

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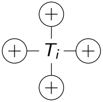	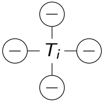	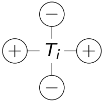	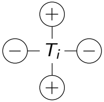	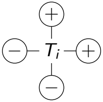	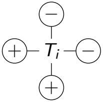
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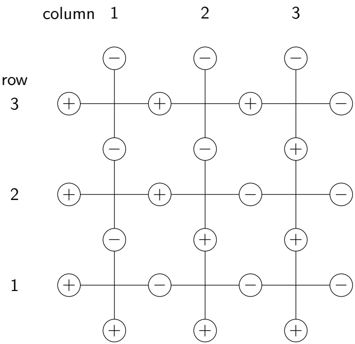

$$Z = 1+$$

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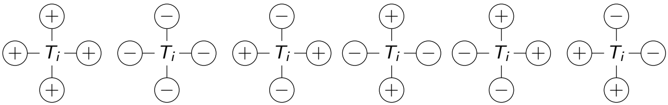

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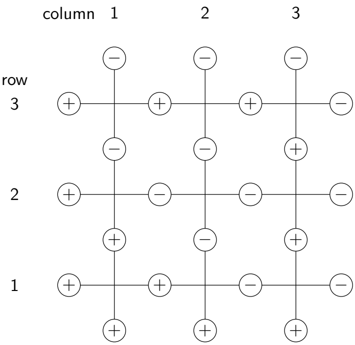

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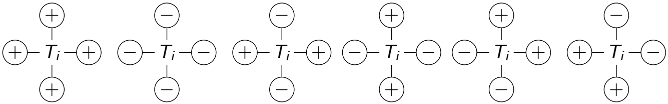

a_1	a_2	b_1	b_2	c_1	c_2
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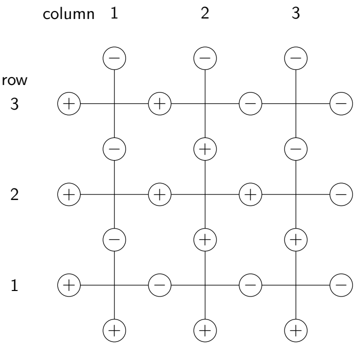

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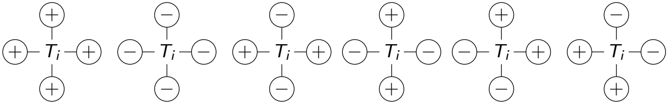

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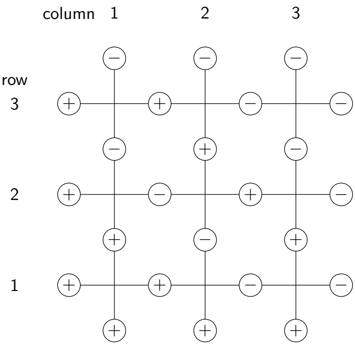

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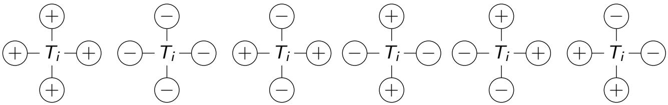

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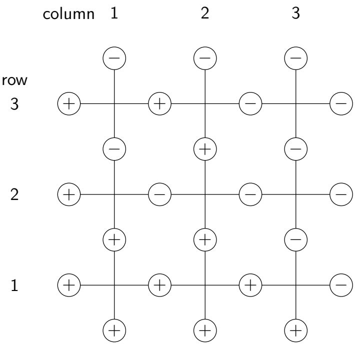

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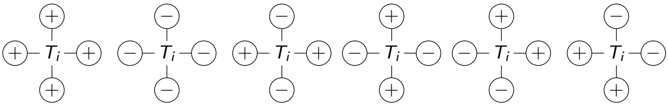

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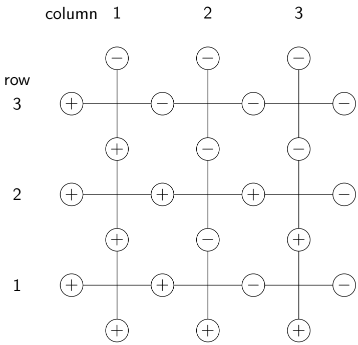

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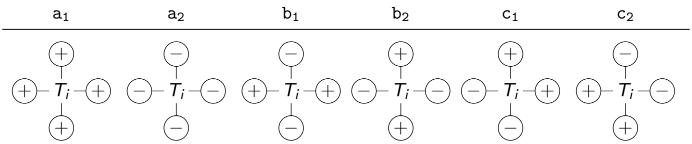

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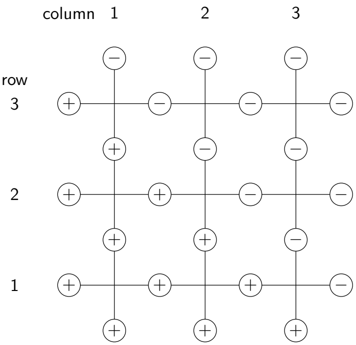

$$Z = 1+1+1+1+1+1+1 = 7$$

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Bijection between states of this lattice model and alternating sign matrices:

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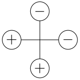
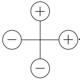

$\leftrightarrow 1,$ 
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all others $\leftrightarrow$ 0.

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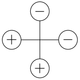
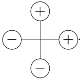
 $\leftrightarrow 1,$  $\leftrightarrow -1,$

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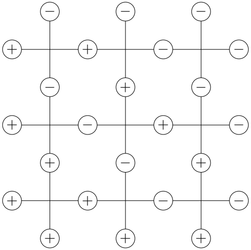

$$\leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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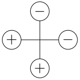
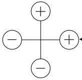
 $\leftrightarrow 1,$  $\leftrightarrow -1,$

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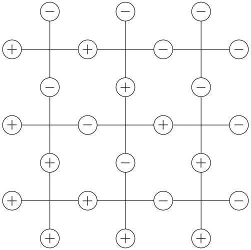 $\leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Theorem (Zeilberger (84 pages), Kuperberg (12 pages))

There are

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}$$

$n \times n$ *alternating sign matrices.*

Example 2: Schur Polynomials

λ, μ: strict partitions, N positive integer.

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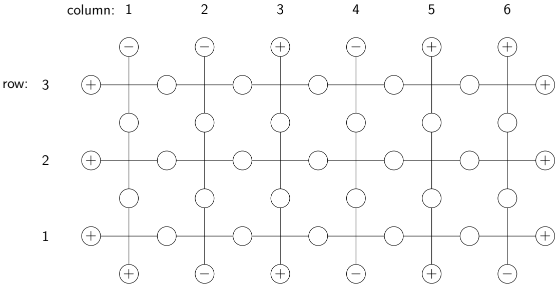

Figure: Boundary conditions for $\mathfrak{S}_{\lambda/\mu}$ with $\lambda = (6, 4, 2)$, $\mu = (4, 2, 1)$

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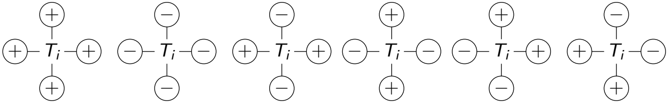

a_1	a_2	b_1	b_2	c_1	c_2
1	0	1	x_i	x_i	1

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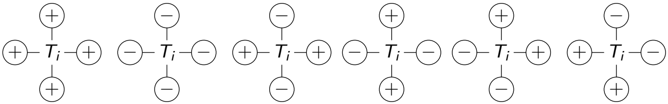

a_1	a_2	b_1	b_2	c_1	c_2
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Model is $\mathfrak{S}_{\lambda/\mu}$ with:

- $N = 2$
- $\lambda = (3)$
- $\mu = (1)$

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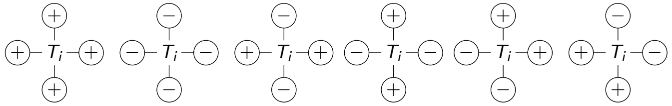

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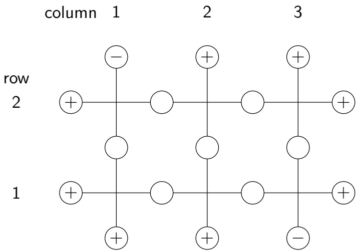

Example 2: Schur Polynomials (cont.)

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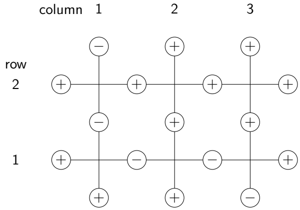
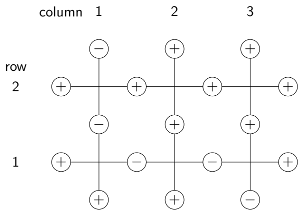


x_1^2

Example 2: Schur Polynomials (cont.)



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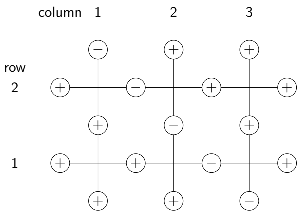
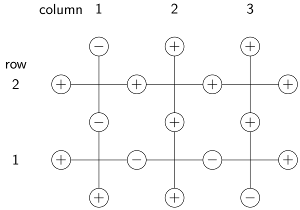


$x_1 x_2$

Example 2: Schur Polynomials (cont.)



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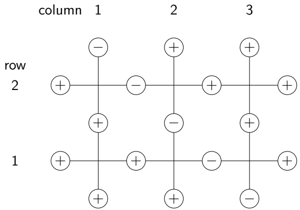

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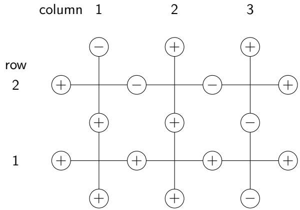
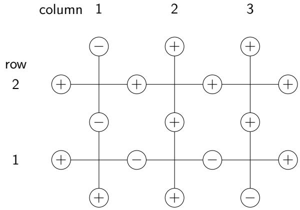


x_2^2

Example 2: Schur Polynomials (cont.)



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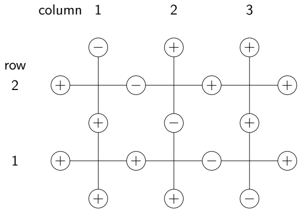

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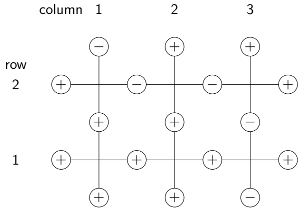
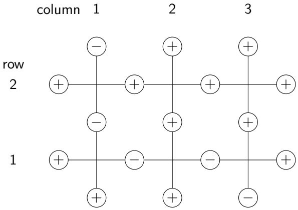


x_2^2

$$Z(\mathfrak{S}_{(3)/(1)}) = x_1^2 + x_1 x_2 + x_2^2$$
$$= s_{(2)}(x_1, x_2).$$

Example 2: Schur Polynomials (cont.)



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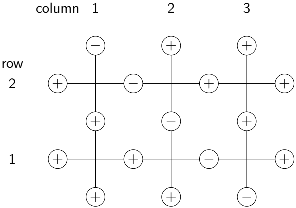
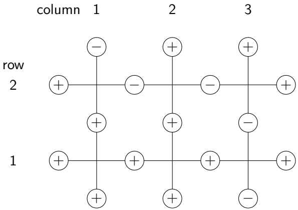


$x_1 x_2$



x_2^2

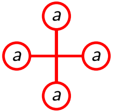
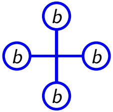
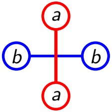
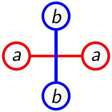
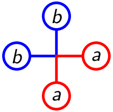
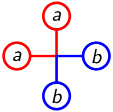
$$Z(\mathfrak{S}_{(3)/(1)}) = x_1^2 + x_1 x_2 + x_2^2$$
$$= s_{(2)}(x_1, x_2).$$

In general,

$$Z(\mathfrak{S}_{\lambda+\rho/\mu+\rho}) = s_{\lambda/\mu},$$

where $\rho = (N-1, \dots, 2, 1)$.

Example 3: Lattice Models with Color

a_1	a_2	b_1	b_2	c_1	c_2
					
1	1	x_i	0	1	1

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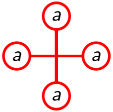	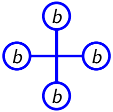	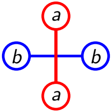	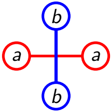	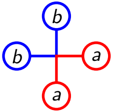	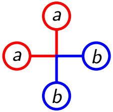
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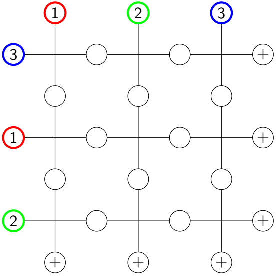

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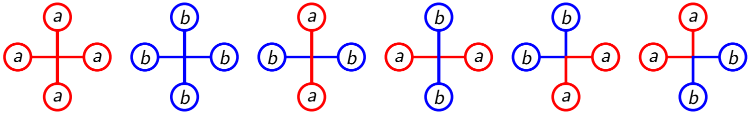

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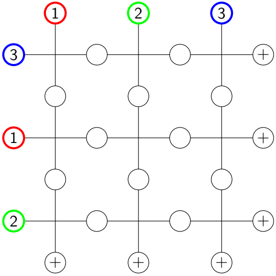

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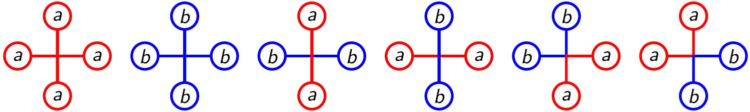					
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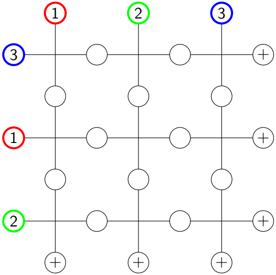

This is a lattice model for...

...Schubert polynomials!

Solvability and the Yang-Baxter Equation

The Yang-Baxter Equation

Introduce a set of "diagonal" vertices called *R-vertices*:

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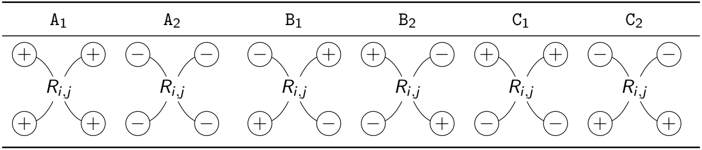

Figure: A set of R-vertices for the six-vertex model.

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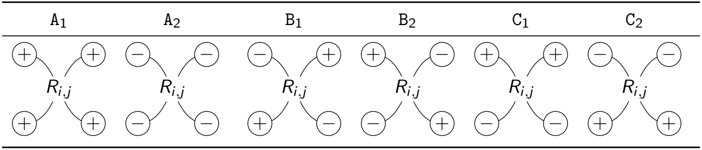

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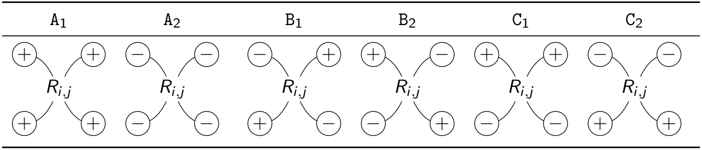

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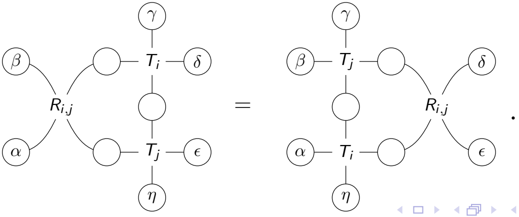

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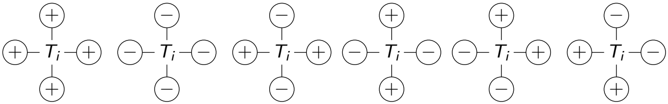
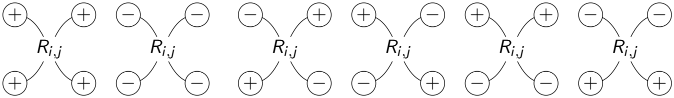
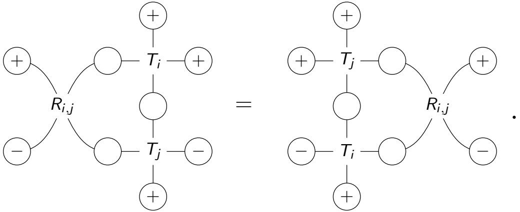


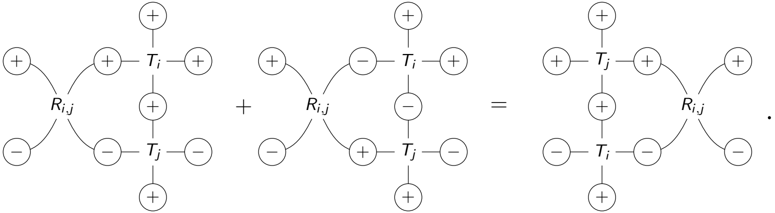
$\mathtt{a}_1$	$\mathtt{a}_2$	$\mathtt{b}_1$	$\mathtt{b}_2$	$\mathtt{c}_1$	$\mathtt{c}_2$
1	0	1	x_i	x_i	1



$\mathtt{A}_1$	$\mathtt{A}_2$	$\mathtt{B}_1$	$\mathtt{B}_2$	$\mathtt{C}_1$	$\mathtt{C}_2$
x_i	x_j	0	$x_i - x_j$	x_i	x_j

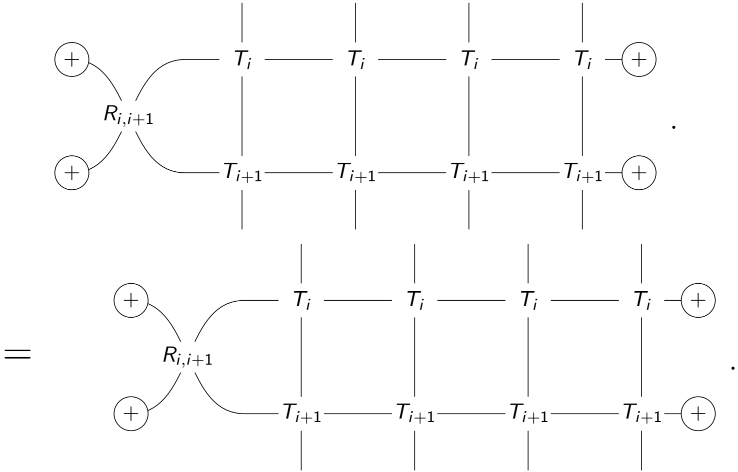
Example: Schur Polynomials (cont.)





$$x_i \cdot 1 \cdot x_j \quad + \quad (x_i - x_j) \cdot x_i \cdot 1 \quad = \quad 1 \cdot x_i \cdot x_i$$

The Train Argument (for Schur Polynomials)



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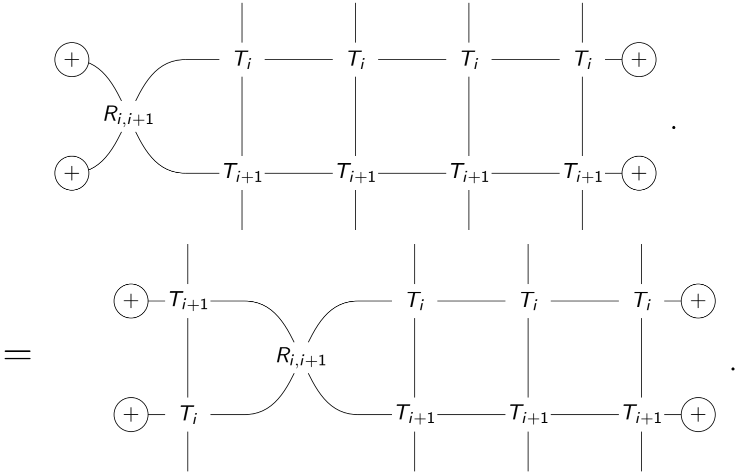

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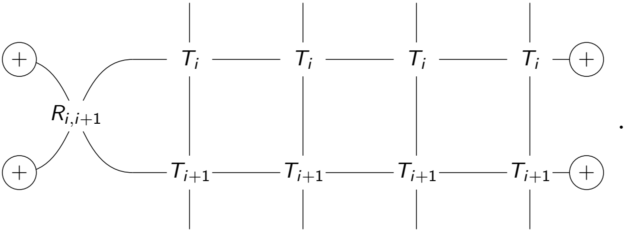
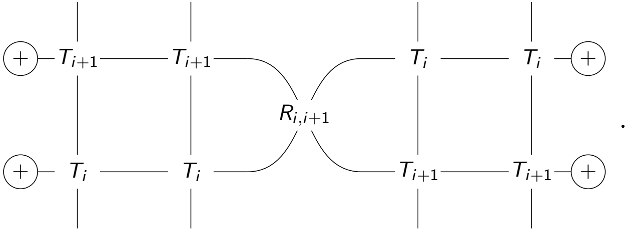


$=$



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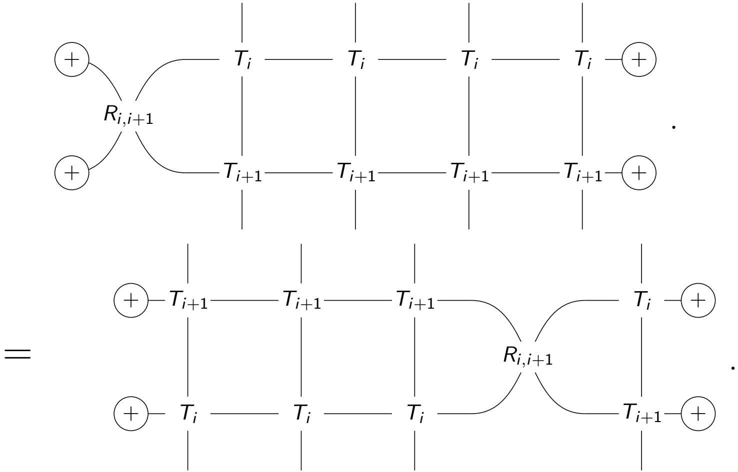

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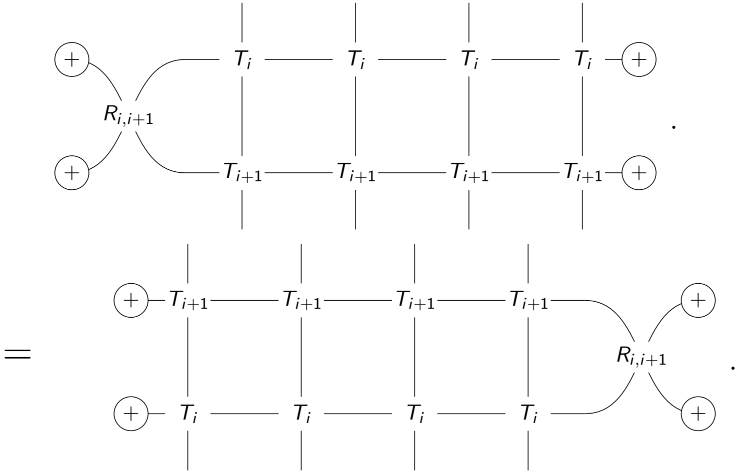

The Train Argument (cont.)

Upshot:

$$Z(\mathfrak{S}_{\lambda/\mu}(x_1, \dots x_n)) = Z(\mathfrak{S}_{\lambda/\mu}(x_1, \dots x_{i+1}, x_i, \dots, x_n)),$$

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Variations of this argument can allow us to:

- Find divided-difference operators for nonsymmetric “atoms”.
- Prove Cauchy identities for the partition function

Six-Vertex Model Solvability

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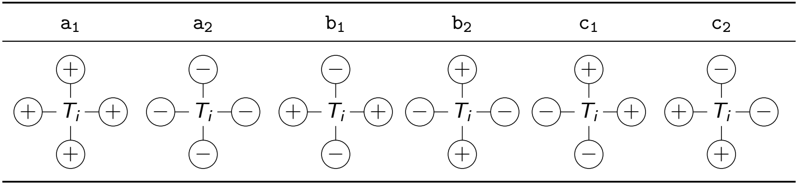

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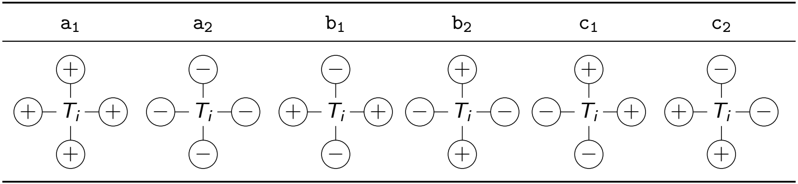

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Theorem (Baxter, Brubaker-Bump-Friedberg)

$\mathfrak{S}$ is solvable via the Yang-Baxter equation if and only if it satisfies the free fermion condition or a related, slightly expanded condition.

Quantum Groups

Quantum group: *quasitriangular Hopf algebra A*

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Main example: $U_q(\widehat{\mathfrak{sl}_n})$.

The Yang-Baxter equation from quantum groups

$V(z)$: *standard evaluation representation* of $U_q(\widehat{\mathfrak{sl}_n})$.

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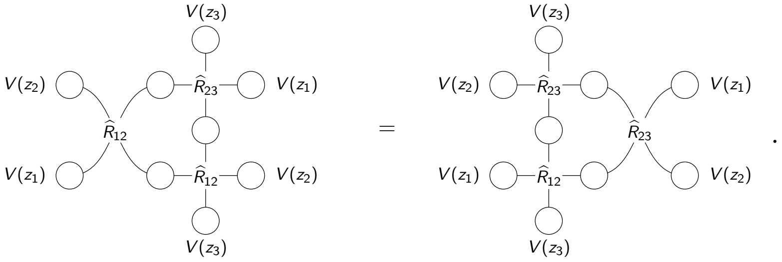

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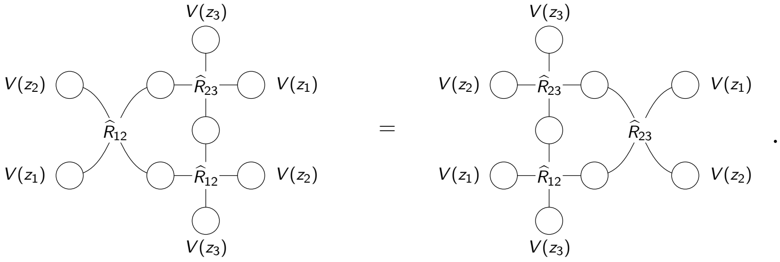

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The map $T_i \mapsto \widehat{R}_{i,i+1}(z_1, z_2)$ forms a module of the Hecke algebra

Identities from Lattice Models

Identities and Properties

Can prove several types of identities and properties:

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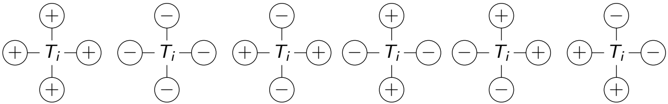
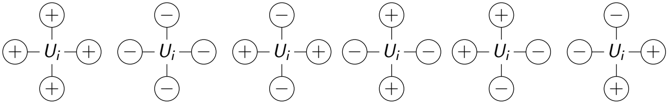
- Littlewood-Richardson rule (rare!): crazy stuff with puzzles

Example 1: Dual Cauchy Identity for Schur Polynomials

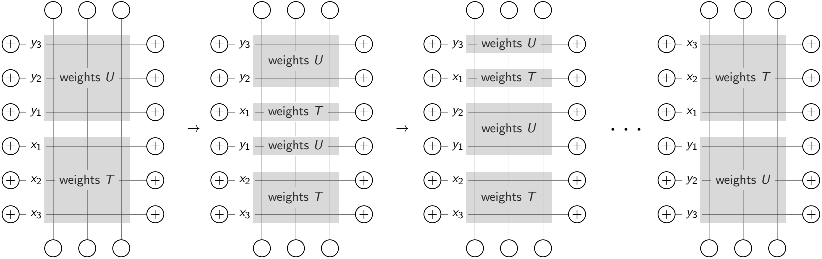


a_1	a_2	b_1	b_2	c_1	c_2
1	0	1	x_i	x_i	1

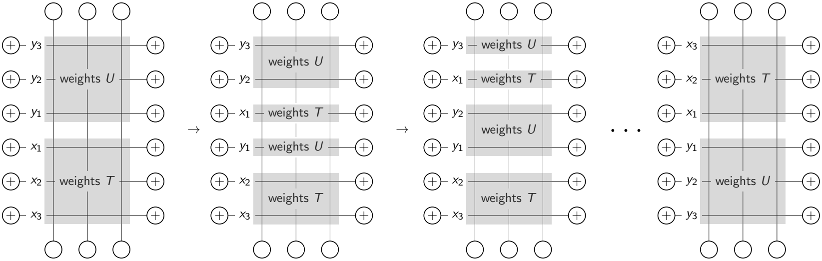


a_1	a_2	b_1	b_2	c_1	c_2
1	y_i	1	0	y_i	1

Example 1 (cont.)

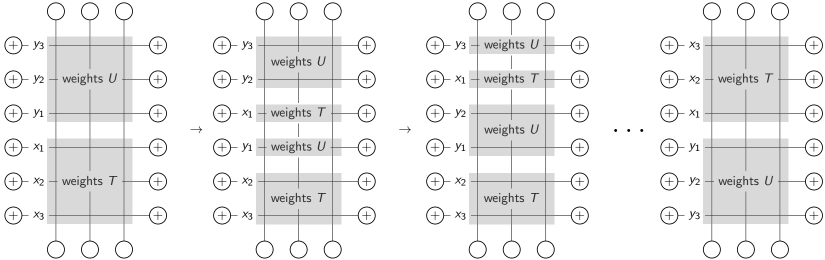


Example 1 (cont.)



$LHS : \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y)$, $RHS : 1$, $T_i \leftrightarrow U_j$ gives a factor of $1 + x_i y_j$.

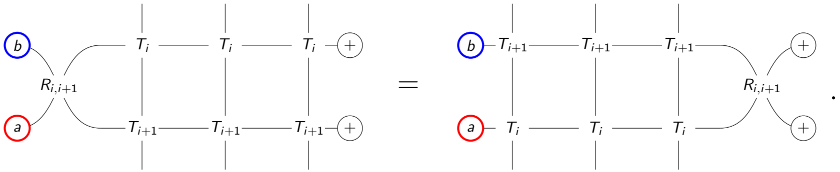
Example 1 (cont.)



$LHS : \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y)$, $RHS : 1$, $T_i \leftrightarrow U_j$ gives a factor of $1 + x_i y_j$.

$$\text{Dual Cauchy: } \sum_{\lambda} s_\lambda(x) s_{\lambda'}(y) = \prod_{i,j} (1 + x_i y_j).$$

Example 2: Demazure Operators for Schubert Polynomials



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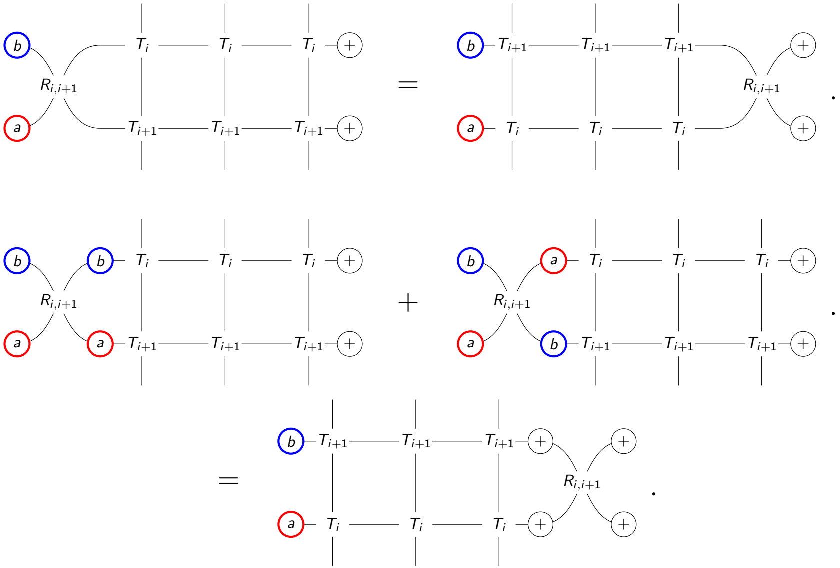
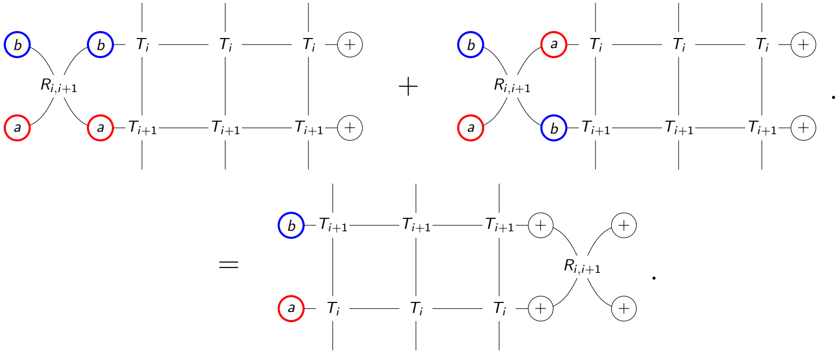
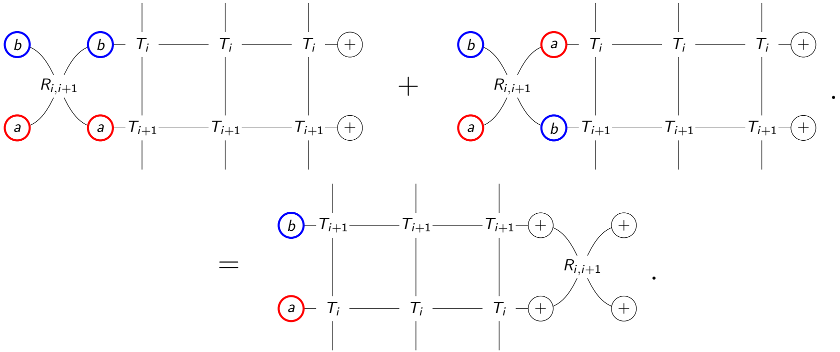


Example 2 (cont.)



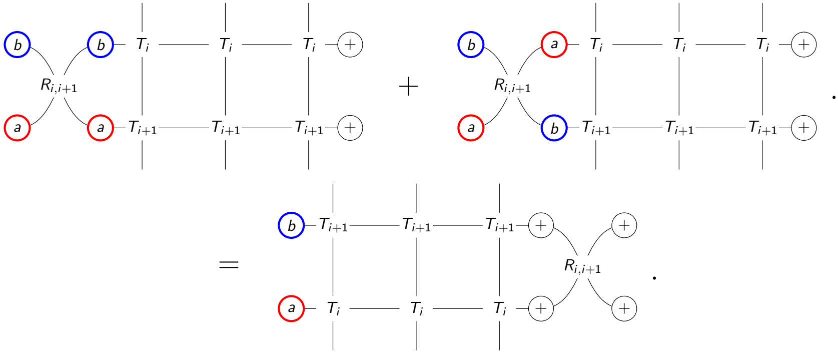
Example 2 (cont.)



Upshot: if $s_i w < w$,

$$S_w(x_1, \ldots, x_n) + (x_{i+1} - x_i) S_{s_i w}(x_1, \ldots, x_n) = S_w(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$

Example 2 (cont.)



Upshot: if $s_i w < w$,

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$$S_{s_i w} = \partial_i S_w, \qquad \text{where} \qquad \partial_i = \frac{1 - s_i}{x_i - x_{i+1}}.$$

Useful References

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