Convergence of anisotropically decaying solutions of a supercritical semilinear heat equation

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Abstract

We consider the Cauchy problem for a semilinear heat equation with a supercritical power nonlinearity. It is known that the asymptotic behavior of solutions in time is determined by the decay rate of their initial values in space. In particular, if an initial value decays like a radial steady state, then the corresponding solution converges to that steady state. In this paper we consider solutions whose initial values decay in an anisotropic way. We show that each such solution converges to a steady state which is explicitly determined by an average formula. For a proof, we first consider the linearized equation around a singular steady state, and find a self-similar solution with a specific asymptotic behavior. Then we construct suitable comparison functions by using the self-similar solution, and apply our previous results on global stability and quasi-convergence of solutions.

Keywords: semilinear parabolic equation; critical exponent; anisotropic decay; quasi-convergence; self-similar solution.

1 Introduction

Consider the Cauchy problem

$$u_t = \Delta u + u^p \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $p > 1$ and u_0 is a continuous nonnegative function on $\mathbb{R}^N$. We shall examine a class of global solutions for p in a certain supercritical range, as specified below.

Problem (1.1), (1.2) plays an important role in the theory of nonlinear parabolic equations. It has been widely studied as a model superlinear problem with the purpose of finding paradigms for blow up and other qualitative properties of solutions. Also it is an important canonical problem to which many more general superlinear equations can be reduced (after taking a scaling limit, for example). Given the importance and relatively simple appearance of (1.1), (1.2), it has been an attractive challenge to achieve as complete as possible an understanding of the behavior of its solutions, be it global or blowing up solutions. For an account of many achievements in this vein we refer the reader to the recent monograph [22] and the extensive list of references given there. The existing results as well as current research on this equation have quite different flavors for different ranges of p. We briefly summarize basic results on global positive solutions (that is, solutions defined for all $t \geq 0$).

First of all we note that there are no such solutions if $1 < p \leq p_F$, p_F being the Fujita exponent $1 + 2/N$ (see [5, 13, 22]). Global positive solutions do exist for $p > p_F$; for example, one can construct positive self-similar solutions decaying to zero as $t \to \infty$ (see [9]). It appears, although it has not been proved in full generality yet (see [22] for available results), that all global solutions decay to zero if p is Sobolev-subcritical, that is, $p < p_S$, where

$$p_S = \infty \text{ if } N = 1, 2, \text{ and } p_S := \frac{N+2}{N-2} \text{ if } N > 2.$$

A basic reason for this is the absence of positive steady states [6, 2] and, more generally, of entire positive solutions of (1.1) [1, 17, 18].

Positive steady states exist for all $p \geq p_S$. In fact, there is a one-parameter family of radial positive steady states $\{\varphi_\alpha : \alpha > 0\}$ given by

$$\varphi_\alpha(x) = \alpha\Phi(\alpha^{(p-1)/2}|x|), \tag{1.3}$$

where $\Phi = \Phi(r)$, $r = |x|$, is the (unique) radial steady state with $\Phi(0) = 1$. The structure of these steady states, which to a large extent determines the behavior of other solutions, depends of the relation of p to another critical exponent p_{JL}, frequently referred to as the Joseph-Lundgren exponent:

$$p_{JL} := \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \leq 10. \end{cases}$$

In the intermediate range, $p_S \leq p < p_{JL}$, the graphs of any two steady states intersect each other [23] and each steady state is unstable in any reasonable interpretation of stability [7, 12]. In this case it is not even known whether any nonstationary global positive solutions that stay away from 0 exist (some negative results - nonexistence of such solutions with an additional structure - can be found in [15]). Also it is not clear whether positive global unbounded solutions can exist (again, a negative result can be found in [15]).

On the other hand, for $p \geq p_{JL}$ the steady states $\{\varphi_\alpha : \alpha > 0\}$ form a simply ordered family (see [23, 7]). As a consequence, there are many other positive global solutions, for example, solutions trapped between two steady states. Also, the steady states have certain asymptotic stability properties [7, 8, 19], hence there is a large class of nonstationary solutions that converge to a steady state as $t \to \infty$. This also applies to a singular steady state φ_∞, hence there are global solutions that approach φ_∞ as $t \to \infty$ and are therefore unbounded [19]. Different classes of global positive solutions have been exhibited in [19, 20]. As shown in [19], there exist positive global solutions that approach a continuum of radial steady states not settling down to any single one of them. Even more complicated behavior has been observed for solutions not trapped between radial steady states. Examples of [20] show that such solutions may contain functions having different centers of symmetry in their ω-limit set.

In this paper, we continue our study of positive solutions bounded above by the radial singular steady state in the supercritical range $p > p_{JL}$ (thus we assume $N \geq 11$). We proved earlier (see [19]), that if the initial value of such a solution decays as a positive steady state, then the solution converges to that steady state as time approaches infinity. If the initial value does not decay as any particular steady state, then in general not much can be said about the asymptotic behavior of the solution, it may not even be convergent. We thus restrict our attention to solutions whose initial values decay as a steady state along in any fixed direction, but the steady state may depend on the direction. It is a curious problem whether the asymptotics of such a solution can be determined in some way. In analogy with the linear heat equation one might speculate, perhaps a little naively, that the solution should be convergent with the limit determined by an averaging formula of some sort. No matter how naive, this guess surprisingly turns out to be correct as we prove in the main theorem of this paper.

To formulate our results we need to recall further properties of the radial steady states of (1.1). Let

$$\varphi_\infty(x) := L|x|^{-m} \quad (x \in \mathbb{R}^N \setminus \{0\}),$$

with

$$m := \frac{2}{p-1} \quad \text{and} \quad L := \{m\,(N-2-m)\}^{1/(p-1)}. \tag{1.4}$$

This is a singular radial steady state of (1.1). The regular radial steady state φ_α satisfies $\varphi_\alpha(0) = \alpha$ and for $p > p_{JL}$ it has the following expansion as $|x| \to \infty$ (see [14]):

$$\varphi_\alpha(x) = L|x|^{-m} - a(\alpha)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}), \tag{1.5}$$

where λ_1 is the positive constant given by

$$\lambda_1 := \frac{N-2-2m-\sqrt{(N-2-2m)^2-8(N-2-m)}}{2},$$

and $a(\alpha)$ is a positive number depending on α. We note that λ_1 is the smaller root of the quadratic equation

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0. \tag{1.6}$$

This equation has two positive roots if and only if $p > p_{JL}$.

By (1.3) and (1.5), we have

$$\varphi_\alpha(x) = \alpha\Phi(\alpha^{(p-1)/2}|x|) = L|x|^{-m} - \alpha^{-\lambda_1/m}a(1)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}),$$

so that $a(\alpha) = \alpha^{-\lambda_1/m} a(1)$. Hence $a(\alpha)$ is monotone decreasing and ranges over $(0, \infty)$ as α varies on $(0, \infty)$. Observe in particular that any two positive steady states (and hence all functions between them) have the same leading term in the expansion at $|x| = \infty$, they only differ in the second term.

For the purpose of the following discussion, assume for a while that $\varphi_\delta \leq u_0 \leq \varphi_\infty$ for some $\delta > 0$. As we showed in [21], under these assumptions the solution $u(\cdot, t)$ of (1.1), (1.2) approaches as $t \to \infty$ a connected subset of the family of steady states $\{\varphi_\alpha : \alpha \geq \delta\}$. We have also proved (see [19]) that if $|x|^{m+\lambda_1}(L|x|^{-m} - u_0(x))$ has a limit $b \in [0, \infty)$ as $|x| \to \infty$, then $u(\cdot, t)$ converges to the steady state φ_α, where α is determined from the relation $a(\alpha) = b$. On the other hand, as demonstrated by examples in [19], if $|x|^{m+\lambda_1}(L|x|^{-m} - u_0(x))$ does not have a definite limit, then the solution u may not converge to any particular steady state; its limit set may be a nontrivial continuum of steady states.

In this paper we consider the case when $|x|^{m+\lambda_1}(L|x|^{-m} - u_0(x))$ does have a limit along any ray in $\mathbb{R}^N$ emanating from the origin, but the limit may vary with the ray. More specifically, our assumption reads as follows:

$$u_0(r\omega) = L|x|^{-m} - b(\omega)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}), \tag{1.7}$$

where b is a positive continuous function on the unit sphere S^{N-1}. We will show that in this case again the solution $u(\cdot, t)$ of (1.1), (1.2) converges to a steady state φ_α. In addition, rather surprisingly in the context of nonlinear equations, the limit equilibrium is explicitly determined by the average formula

$$a(\alpha) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} b(\omega)\, d\sigma_\omega. \tag{1.8}$$

Here is the precise formulation of our main result.

Theorem 1.1. *Let $N \geq 11$ and $p > p_{JL}$. Assume that b is a positive continuous function on S^{N-1} and u_0 is a continuous function on $\mathbb{R}^N$ such that $0 \leq u_0 \leq \varphi_\infty$ and* (1.7) *holds uniformly in $\omega \in S^{N-1}$. Then the solution $u(\cdot, t)$ of* (1.1), (1.2) *satisfies*

$$\lim_{t \to \infty} \|u(\cdot, t) - \varphi_\alpha\|_{L^\infty(\mathbb{R}^N)} = 0,$$

where α is (uniquely) determined by (1.8).

The proof of this theorem can roughly be outlined as follows. Using our earlier results, we first observe that it is sufficient to prove the result for a particular solution u whose initial value u_0 satisfies (1.7). To construct such a particular solution, we prove the existence of a self-similar solution v of the linearization of (1.1) around the singular steady state φ_∞. Using a specific asymptotics of $v(x,t)$ as $|x| \to \infty$ and $|x| \to 0$, we are able to understand the behavior of u and thus prove the convergence result of Theorem 1.1.

The remainder of the paper is organized as follows. In Section 2 we consider self-similar solutions of the linearization of (1.1) around φ_∞. Using separation of variables we are lead to an ordinary differential equation which is analyzed in Section 4. The proof of Theorem 1.1 is given in Section 3.

2 Self-similar solutions of a linearized equation

Consider the linearization of (1.1) about its singular steady state φ_∞:

$$v_t = \Delta v + \frac{pL^{p-1}}{|x|^2} v \qquad (x \in \mathbb{R}^N \setminus \{0\}). \tag{2.1}$$

We look for a solution v which behaves in a self-similar way:

$$v(x,t) = (t+1)^{-\ell/2} w(y), \quad y = (t+1)^{-1/2} x, \tag{2.2}$$

where

$$\ell := m + \lambda_1.$$

A simple computation shows that v satisfies (2.1) if w solves the equation

$$\Delta w + \frac{1}{2} y \cdot \nabla w + \frac{\ell}{2} w + \frac{pL^{p-1}}{|y|^2} w = 0 \qquad (y \in \mathbb{R}^N \setminus \{0\}). \tag{2.3}$$

We want to find a solution of this equation with a certain prescribed asymptotics at 0 and at ∞, the latter being determined by a positive continuous function b on S^{N-1}. It will be convenient to work with the Fourier expansion of b with respect to spherical harmonics. To define it, let Δ_S denote the Laplace-Beltrami operator on S^{N-1} and let $\mu_0 = 0 < \mu_1 < \cdots < \mu_k < \cdots$ be the eigenvalues of $-\Delta_S$. Then $\mu_k \to \infty$ as $k \to \infty$ and the eigenspace of μ_k equals the space of spherical harmonics of degree k. In particular, μ_0 is a simple eigenvalue with a constant eigenfunction. For each k, let n_k denote the multiplicity of μ_k and let ψ_{kj}, $j = 1, \dots, n_k$, be an orthonormal basis of the eigenspace of μ_k, with respect to the $L^2(S^{N-1})$-inner product

$$\langle \psi, \tilde{\psi} \rangle = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \psi(\omega) \tilde{\psi}(\omega) \, d\sigma_\omega.$$

For $k = 0$, we choose $\psi_{01} = 1$. Then ψ_{kj}, $j = 1, \dots, n_k$, $k = 0, 1, \dots$ is an orthogonal basis of $L^2(S^{N-1})$. Given a continuous function b on S^{N-1}, we set

$$c_{kj} := \langle b, \psi_{kj} \rangle, \tag{2.4}$$

so that

$$b = \sum_{k=0}^{\infty} \sum_{j=1}^{n_k} c_{kj} \psi_{kj}$$

with convergence in $L^2(S^{N-1})$ and with uniform absolute convergence if b is smooth. Note that

$$c_{01} = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} b(\omega) \, d\sigma_\omega. \tag{2.5}$$

Lemma 2.1. *Given any positive continuous function b on S^{N-1} and defining c_{01} by (2.5), there exists a positive (classical) solution w of (2.3) such that*

$$\begin{aligned} \rho^\ell w(\rho\omega) &\to b(\omega) \quad (\rho \to \infty), \\ \rho^\ell w(\rho\omega) &\to c_{01} \quad (\rho \to 0), \end{aligned} \tag{2.6}$$

weakly in $L^2(S^{N-1})$. Moreover, if b is smooth, then the convergence in (2.6) is uniform in $\omega \in S^{N-1}$.

The proof of the lemma will use separation of variables which we now introduce. Given an integer $k \geq 0$, we want to find z such that a solution of (2.3) is given by $w(y) = \psi(\omega) z(\rho)$, $y = \rho\omega$, $\omega \in S^{N-1}$, $\rho > 0$, where $\psi \in \operatorname{span}\{\psi_{kj} : j = 1, \dots, n_k\}$. This is the case if z satisfies the ordinary differential equation (ODE)

$$z_{\rho\rho} + \frac{N-1}{\rho} z_\rho + \frac{\rho}{2} z_\rho + \frac{\ell}{2} z + \frac{pL^{p-1} - \mu_k}{\rho^2} z = 0 \qquad (\rho > 0). \tag{2.7}$$

The equation is derived in the usual way, separating the variables ρ and ω in (2.3) after expressing Δ in spherical coordinates:

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{N-1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_S.$$

The analysis of (2.7) is carried out in a slightly more general context in Section 4. The result we prove there (see Lemma 4.1) gives the following.

Lemma 2.2. *For each $k = 0, 1\dots,$ there is a positive solution g_k of* (2.7) *with the following properties:*

$$\begin{aligned} &g_0(\rho) = \rho^{-\ell}, \qquad (\textit{hence } \lim_{\rho\to 0} \rho^\ell g_0(\rho) = \lim_{\rho\to\infty} \rho^\ell g_0(\rho) = 1), \\ &\textit{if } k \geq 1 \textit{ then } \lim_{\rho\to 0} \rho^\ell g_k(\rho) = 0, \ \lim_{\rho\to\infty} \rho^\ell g_k(\rho) = 1. \end{aligned} \tag{2.8}$$

Moreover, for $k \geq 1$ the function $\rho^\ell g_k(\rho)$ is increasing on $(0,\infty)$, in particular

$$0 < \rho^\ell g_k(\rho) < 1 \qquad (0 < \rho < \infty). \tag{2.9}$$

Proof of Lemma 2.1. With g_k as in Lemma 2.2 and c_{kj} as in (2.4), define

$$w(y) := \tilde{w}(\rho,\omega) := \sum_{k=0}^{\infty}\sum_{j=1}^{n_k} c_{kj} g_k(\rho)\psi_{kj}(\omega) \quad (y = \rho\omega,\, \omega \in S^{N-1},\, \rho > 0). \tag{2.10}$$

By (2.9), for each fixed $\rho > 0$ the sequence $\{g_k(\rho)\}_{k=0,1,\dots}$ is bounded, hence the series in (2.10) converges in $L^2(S^{N-1})$. Moreover, if b is smooth then the series is uniformly absolutely convergent on S^{N-1} (as is the Fourier series of b). Multiplying the series by ρ^ℓ and taking the limits of each term as $\rho \to \infty$ and $\rho \to 0$, using (2.8), we formally obtain (2.6). Although this computation is formal, it does show that (2.6) holds with weak convergence in $L^2(S^{N-1})$. To prove the uniform convergence, if b is smooth, let w_m and $\tilde{w}_m$ be defined as w and $\tilde{w}$ with c_{kj} set equal zero for $k > m$. Clearly, (2.6) holds with the uniform convergence if w is replaced by w_m and b is replaced by its corresponding finite Fourier series b_m. In view of (2.9) (and the smoothness of b), we can make the remainders $\rho^\ell|\tilde{w}(\rho,\omega) - \tilde{w}_m(\rho,\omega)|$, $|b(\omega) - b_m(\omega)|$ as small as we wish, uniformly in ρ and ω, by taking m large. The desired uniform convergence properties now follow readily.

We next show that w is a solution of (2.3). This is clearly true for w_m. By (2.9), the continuous functions $g_k(\rho)$, $k = 0, 1, \dots,$ are uniformly bounded on $(\epsilon, \epsilon^{-1})$ for each $\epsilon > 0$. Using this, one shows easily that $w \in L^2_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$w = \lim_{m\to\infty} w_m \ \text{ in } L^2_{loc}(\mathbb{R}^N \setminus \{0\}). \tag{2.11}$$

Fix any bounded domains Ω_1, Ω_2 with $\bar{\Omega}_1 \subset \Omega_2$ and $\bar{\Omega}_2 \subset \mathbb{R}^N \setminus \{0\}$. In view of (2.11), the norms $\|w_m\|_{L^2(\Omega_2)}$ are uniformly bounded, hence by local L^2-estimates for (2.3), $\|w_m\|_{H^2(\Omega_1)}$ are uniformly bounded. Therefore, w_m converges to w weakly in $H^2(\Omega_1)$ and strongly in $H^1(\Omega_1)$). This implies that $w \in H^2(\Omega_1)$ and it is a weak solution of (2.3) on Ω_1. Consequently, by elliptic regularity, it is a classical solution on Ω_1 and since Ω_1 was arbitrary, w is a solution on $\mathbb{R}^N \setminus \{0\}$.

Finally we verify that w is positive. It is sufficient to prove that the functions w_m are all positive. Then, by (2.11), w is nonnegative and, since it is nontrivial, it is positive by the maximum principle.

We can thus proceed assuming, without loss of generality, that $w = w_m$ for some m. This implies in particular that the convergence in (2.6) is uniform in ω and, consequently, $w(y)$ is positive if $|y|$ is sufficiently large, say if $|y| \geq R_0$. Consider now the function $\tilde{w}_\beta := w + \beta g_0$ with $\beta \in [0, \infty)$. It is a solution of (2.3) (g_0 is the radial solution of (2.3) corresponding to the function $b \equiv 1$). Of course, $\tilde{w}_\beta > 0$ if $|y| \geq R_0$. From (2.8) we further infer that for each $\beta \in [0, \infty)$ one has $\tilde{w}_\beta > 0$ near $|y| = 0$. Hence, taking β sufficiently large we make $\tilde{w}_\beta$ positive on $\mathbb{R}^N \setminus \{0\}$. Let $\beta_0 \geq 0$ be the infimum of all values β for which $\tilde{w}_\beta > 0$ on $\mathbb{R}^N \setminus \{0\}$. Then $w_{\beta_0} \geq 0$. Moreover, if $\beta_0 > 0$ then the positivity of $\tilde{w}_\beta$ for $|y| \approx 0$ and $|y| \approx \infty$ and the definition of β_0 imply that w_{β_0} vanishes somewhere. But this would contradict the the maximum principle, hence $\beta_0 = 0$ proving $w > 0$. $\square$

Remark 2.3. The fact that the radial function $g_0(|y|) = |y|^{-\ell}$ is a solution of (2.3) will be used below one more time.

3 Proof of the main result

We use two results from our earlier papers. As in the previous section, let $\ell = m + \lambda_1$. The first result is a special case of Theorem 4.2 of [19].

Proposition 3.1. *Assume $u_0, \tilde{u}_0 \in C(\mathbb{R}^N)$ satisfy $0 \leq u_0(x), \tilde{u}_0(x) \leq \varphi_\infty(x)$ ($x \in \mathbb{R}^N \setminus \{0\}$) and*

$$|u_0(x) - \tilde{u}_0(x)| = o(|x|^{-\ell}) \quad (|x| \to \infty).$$

Let u, $\tilde{u}$ be the solutions of (1.1) having the initial values u_0, $\tilde{u}_0$, respectively. Then

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad (t \to \infty).$$

The second result concerns the ω-limit set of a solution $u(x,t)$ of (1.1) with the initial value $u_0(x)$, which is defined by

$$\omega(u_0) := \{\phi : \lim_{n\to\infty} \|u(\cdot,t_n) - \phi\|_{L^\infty(\mathbb{R}^N)} = 0 \text{ for some sequence } t_n \to \infty\}.$$

We are especially interested in the ω-limit set of a solution whose initial value $u_0 \in C(\mathbb{R}^N)$ satisfies

$$\varphi_\delta \le u_0(x) \le \varphi_\beta(x) \quad (x \in \mathbb{R}^N) \tag{3.1}$$

for some $0 < \delta < \beta < \infty$. By the comparison principle, we then have $\varphi_\delta \le u(\cdot,t) \le \varphi_\beta$ for all $t \ge 0$ and parabolic estimates imply that the trajectory $\{u(\cdot,t) : t \ge 0\}$ is relatively compact in $C_0(\mathbb{R}^N)$, the space of continuous functions on $\mathbb{R}^N$ decaying to zero at $|x| = \infty$ (it is equipped with the supremum norm). In particular, the statement $u(\cdot,t) \to \varphi_\alpha$ is then equivalent to $\omega(u_0) = \{\varphi_\alpha\}$.

The following result is obtained in Theorem 1.1 of [21].

Proposition 3.2. *Assume $u_0 \in C(\mathbb{R}^N)$ satisfies (3.1) for some $0 < \delta < \beta < \infty$. Then*

$$\omega(u_0) \subset \{\varphi_\gamma : \delta \le \gamma \le \beta\}. \tag{3.2}$$

The following two lemmas are our final preparations for the proof of Theorem 1.1.

Lemma 3.3. *Assume that the hypotheses of Theorem 1.1 are satisfied. Then (3.2) holds for some $\delta, \beta \in (0,\infty)$.*

Proof. We may assume, without loss of generality, that $\varphi_\delta \le u_0 \le \varphi_\beta$ for some $0 < \delta \le \beta < \infty$. Indeed, assumption (1.7) on u_0 and the expansion (1.5) imply that for sufficiently large β and sufficiently small $\delta > 0$ we have $\varphi_\delta(x) \le u_0(x) \le \varphi_\beta(x)$ for all x outside a ball. Modifying u_0 in that ball we achieve the condition $u_\delta \le u_0 \le \varphi_\beta$, while the other hypotheses on u_0 are preserved. As such a modification has no effect on $\omega(u_0)$, by Proposition 3.1, we lose no generality by assuming $u_\delta \le u_0 \le \varphi_\beta$ from the start. The result now follows immediately from Proposition 3.2. □

Lemma 3.4. *Assume that the hypotheses of Theorem 1.1 are satisfied. In addition assume that b is smooth. Let α be as in (1.8). Then there exist $u_0^1, u_0^2 \in C(\mathbb{R}^N)$ with the following properties:*

(i) *Condition* (1.7) *holds, uniformly with respect to* $\omega \in S^{N-1}$, *if* u_0 *is replaced by any of the functions* u_0^1, u_0^2.

(ii) $0 \le u_0^1(x),\ u_0^2(x) \le \varphi_\infty(x) \quad (x \in \mathbb{R}^N \setminus \{0\})$,

(iii) *For each* $\phi_1 \in \omega(u_0^1)$

$$\liminf_{|x|\to\infty} |x|^\ell(\phi_1(x) - \varphi_\infty(x)) \ge -a(\alpha), \tag{3.3}$$

and for each $\phi_2 \in \omega(u_0^2)$

$$\limsup_{|x|\to\infty} |x|^\ell(\phi_2(x) - \varphi_\infty(x)) \le -a(\alpha). \tag{3.4}$$

Before giving the proof of this lemma, let us prove Theorem 1.1.

Proof of Theorem 1.1. Assume that the hypotheses of Theorem 1.1 are satisfied and for now also assume that b is smooth. As in the proof of Lemma 3.3, we may further assume, without loss of generality, that $\varphi_\delta \le u_0 \le \varphi_\beta$ for some $0 < \delta < \beta < \infty$. Let u_0^1, u_0^2 be as in Lemma 3.4. First we note that the statement (i) of the lemma gives

$$|u_0(x) - u_0^1(x)|,\ |u_0(x) - u_0^2(x)| = o(|x|^{-\ell}) \quad (|x| \to \infty).$$

Therefore, by Proposition 3.1, we have $\omega(u_0^1) = \omega(u_0^2) = \omega(u_0)$. By the assumptions on u_0 and Lemma 3.3, these identical sets consist of steady states φ_γ. Take any $\varphi_\gamma \in \omega(u_0)$. Statement (iii) yields

$$\lim_{|x|\to\infty} |x|^\ell(\varphi_\gamma(x) - \varphi_\infty(x)) = -a(\alpha).$$

Therefore, by the expansion (1.5) for φ_γ, we necessarily have $\gamma = \alpha$. This proves the desired result $\omega(u_0) = \{\varphi_\alpha\}$.

Now we remove the extra smoothness assumption on b (b is now assumed to be merely continuous). Choose sequences of smooth functions $\{b_k^+\}$, $\{b_k^-\}$ uniformly converging to b such that $0 < b_k^+ \le b \le b_k^-$. Then one easily finds continuous functions u_0^{k-}, u_0^{k+}, such that

$$0 \le u_0^{k-}(x) \le u_0(x) \le u_0^{k+}(x) \le \varphi_\infty(x) \quad (x \in \mathbb{R}^N)$$

and

$$u_0^{k-}(r\omega) = L|x|^{-m} - b_k^-(\omega)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}),$$

$$u_0^{k+}(r\omega) = L|x|^{-m} - b_k^+(\omega)|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}).$$

Applying the result proved above to $u_0^{k\pm}$ we obtain, by comparison, that each $\phi \in \omega(u_0)$ satisfies

$$\varphi_{\alpha_k^-} \le \phi \le \varphi_{\alpha_k^+},$$

where $\alpha_k^\pm$ is determined from (1.8) with b replaced by $b_k^\pm$. Taking the limits as $k \to \infty$, we obtain the general conclusion of Theorem 1.1. □

It remains to prove Lemma 3.4. We shall use comparison arguments with a suitable supersolution and subsolution. Here supersolution and subsolution are understood in a general sense, as in [19]. For example, a supersolution of (1.1) on $(\mathbb{R}^N \setminus \{0\}) \times (0,\infty)$ refers to a continuous function $\tilde{u}$ on $(\mathbb{R}^N \setminus \{0\}) \times (0,\infty)$ such that any solution u of (1.1) on $(\mathbb{R}^N \setminus \{0\}) \times [s,\infty) \subset \mathbb{R}^N \times (0,\infty)$ satisfying $\tilde{u}(\cdot,s) \ge u(\cdot,s)$ also satisfies $\tilde{u}(\cdot,t) \ge u(\cdot,t)$ for each $t \ge s$.

Let w be as in Lemma 2.1 and v be the self-similar solution of (2.1) given by (2.2). Following an idea of [19], we use these solutions of the linearized problems to construct a subsolution and a supersolution of the nonlinear equation (1.1). We set

$$\underline{u}(x,t) := \max\{0,\ \varphi_\infty(x) - v(x,t)\} \quad (x \in \mathbb{R}^N \setminus \{0\},\ t \ge 0), \tag{3.5}$$

$$\overline{u}(x,t) := \min\{\varphi_\infty(x),\ \varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - v(x,t)\} \quad (x \in \mathbb{R}^N \setminus \{0\},\ t \ge 0), \tag{3.6}$$

where $\epsilon > 0$ is sufficiently small.

Lemma 3.5.

(i) $\underline{u}(x,t)$ *defined by* (3.5) *is a subsolution of* (1.1) *on* $\mathbb{R}^N \setminus \{0\}$.

(ii) *If* $\epsilon > 0$ *is sufficiently small, then* $\overline{u}(x,t)$ *defined by* (3.6) *is a supersolution of* (1.1) *on* $\mathbb{R}^N \setminus \{0\}$.

Proof. For (x,t) with $\underline{u}(x,t) > 0$, we have

$$\begin{aligned} \underline{u}_t - \Delta\underline{u} - (\underline{u})^p &< -v_t - \Delta(\varphi_\infty - v) - (\varphi_\infty)^p + p(\varphi_\infty)^{p-1}v \\ &= -\{\Delta\varphi_\infty + (\varphi_\infty)^p\} - v_t + \Delta v + \frac{pL^{p-1}}{|x|^2}v \\ &= 0. \end{aligned}$$

This implies that $\underline{u}$ is a subsolution of (1.1) on $\mathbb{R}^N \setminus \{0\}$.

Taking $\epsilon > 0$ so small that $a(\epsilon)|y|^{-\ell} \geq w(y)$ for all $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$a(\epsilon)|x|^{-\ell} - v(x,t) = (t+1)^{-\ell/2}\{a(\epsilon)|y|^{-\ell} - w(y)\} \geq 0 \quad (y = (t+1)^{-1/2}x)$$

for all $t > 0$. Then for (x,t) with $\varphi_\infty(x) > \overline{u}(x,t)$, we obtain

$$\begin{aligned}
\overline{u}_t - \Delta\overline{u} - (\overline{u})^p > &-v_t - \Delta\{\varphi_\epsilon + a(\epsilon)|x|^{-\ell} - v\} \\
&- (\varphi_\epsilon)^p - p(\varphi_\infty)^{p-1}\{a(\epsilon)|x|^{-\ell} - v\} \\
= &-\{\Delta\varphi_\epsilon + (\varphi_\epsilon)^p\} - a(\epsilon)\Big\{\Delta|x|^{-\ell} + \frac{pL^{p-1}}{|x|^2}|x|^{-\ell}\Big\} \\
&- v_t + \Delta v + \frac{pL^{p-1}}{|x|^2}v \\
= &\ 0.
\end{aligned}$$

Consequently $\overline{u}$ is a supersolution of (1.1) on $\mathbb{R}^N \setminus \{0\}$. □

Now, let us complete the proof of Lemma 3.4.

Proof of Lemma 3.4. By (2.6) (where the convergence is uniform due to the smoothness assumption on b), we have for each fixed $t > 0$

$$|x|^\ell v(x,t) = \Big(\frac{x}{\sqrt{t+1}}\Big)^\ell w\Big(\frac{x}{\sqrt{t+1}}\Big) \to c_{01} = a(\alpha) > 0 \quad \text{ as } x \to 0.$$

Thus $v(x,t) > L|x|^{-m} = \varphi_\infty(x)$ for $|x|$ sufficiently small (depending on t). Also, we can fix $\epsilon > 0$ so small that statement (ii) of Lemma 3.5 applies and in addition $\varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - v(x,t) > \varphi_\infty(x)$ for $|x|$ sufficiently small (depending on t). Then for each fixed t

$$\underline{u}(x,t) = 0 \ \text{ for } |x| \approx 0 \ \text{ and } \ \overline{u}(x,t) = \varphi_\infty(x) \ \text{ for } |x| \approx 0. \tag{3.7}$$

Let now $u_0^1(x) := \underline{u}(x,0)$ and let u_0^2 be any continuous nonnegative function on $\mathbb{R}^N$ such that $u_0^2(x) \leq \overline{u}(x,0)$ for all $x \in \mathbb{R}^N$ and $u_0^2(x) = \overline{u}(x,0)$ for all $x \in \mathbb{R}^N$ with $|x| > 1$. Then u_0^1, u_0^2 are continuous functions satisfying statement (ii) of Lemma 3.4. We claim that the other two statements, (i) and (iii), are satisfied as well. To prove (i), we note that for large $|x|$

$$\begin{aligned}
u_0^1(x) &= \underline{u}(x,0) = \varphi_\infty(x) - v(x,0) = \varphi_\infty(x) - w(x), \\
u_0^2(x) &= \overline{u}(x,0) = \varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - w(x) = \varphi_\infty(x) - w(x) + o(|x|^{-\ell}).
\end{aligned}$$

(We have used (1.5) in the last equality). Thus statement (i) follows from (2.6) and the fact that we are assuming that b is smooth (so the convergence in (2.6) is uniform in ω).

Now let u^1, u^2 be the solutions of (1.1) with initial values u_0^1, u_0^2. Using comparison arguments with the supersolution $\underline{u}$ and supersolution $\overline{u}$ (which is justified by (3.7) and the definitions of u_0^1, u_0^2), we have

$$u^1(x,t) \geq \underline{u}(x,t), \quad u^2(x,t) \leq \overline{u}(x,t) \quad (x \in \mathbb{R}^N \setminus \{0\},\ t > 0).$$

Therefore, for each $\phi_1 \in \omega(u_1)$

$$\phi_1(x) \geq \liminf_{t\to\infty} \underline{u}(x,t) \quad (x \in \mathbb{R}^N \setminus \{0\})$$

and for each $\phi_2 \in \omega(u_2)$

$$\phi_2(x) \leq \limsup_{t\to\infty} \overline{u}(x,t) \quad (x \in \mathbb{R}^N \setminus \{0\}).$$

Now, for any fixed $x \in \mathbb{R}^N \setminus \{0\}$

$$\underline{u}(x,t) \geq \varphi_\infty(x) - v(x,t) = L|x|^{-m} - |y|^\ell w(y)|x|^{-\ell}, \tag{3.8}$$

with $y = x(t+1)^{-1/2} \to 0$ as $t \to \infty$. Hence, by (2.6), the right-hand side of the equality in (3.8) converges to $L|x|^{-m} - a(\alpha)|x|^{-\ell}$. Since this is a lower bound on $\phi_1(x)$ for each $x \neq 0$, we obtain (3.3).

By a similar computation,

$$\begin{aligned}
\overline{u}(x,t) &\leq \varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - v(x,t) \\
&= \varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - |y|^\ell w(y)|x|^{-\ell} \\
&\to \varphi_\epsilon(x) + a(\epsilon)|x|^{-\ell} - a(\alpha)|x|^{-\ell}
\end{aligned}$$

as $t \to \infty$. Combining this with the expansion for φ_ϵ (see (1.5)), we find an upper bound on $\phi_2(x)$ of the form $L|x|^{-m} - a(\alpha)|x|^{-\ell} + o(|x|^{-\ell})$. This proves (3.4). $\square$

4 ODE analysis

Consider the ordinary differential equation

$$z_{\rho\rho} + \frac{N-1}{\rho} z_\rho + \frac{\rho}{2} z_\rho + \frac{\ell}{2} z + \frac{pL^{p-1} - \mu}{\rho^2} z = 0 \tag{4.1}$$

with a parameter μ. For Lemma 2.2, it suffices to show the following properties (see Lemma 3.1 of [3] and Lemma 2.1 of [16] for related results).

Lemma 4.1. *Assume $p > p_{JL}$. Then the following statements hold.*

(i) *For $\mu = 0$, $z = \rho^{-\ell}$ and $z = \rho^{N-2-\ell}$ satisfy* (4.1).

(ii) *For every $\mu > 0$, there is a unique (up to a constant multiple) solution of* (4.1) *such that $\rho^\ell z(\rho)$ is strictly increasing in ρ and*

$$\lim_{\rho\to 0} \rho^\ell z(\rho) = 0, \quad \lim_{\rho\to\infty} \rho^\ell z(\rho) < \infty.$$

Proof. Transforming (4.1) by $z = \rho^{-\gamma} Z$, we have

$$Z_{\rho\rho} + \frac{N-1-2\gamma}{\rho} Z_\rho + \frac{\rho}{2} Z_\rho + \frac{\ell-\gamma}{2} Z + \frac{\gamma^2 - (N-2)\gamma + pL^{p-1} - \mu}{\rho^2} Z = 0. \tag{4.2}$$

Using the equality

$$\ell^2 - (N-2)\ell + pL^{p-1} = 0 \tag{4.3}$$

(cf. (1.4), (1.6) and recall that $\ell = m + \lambda_1$), we have

$$\gamma^2 - (N-2)\gamma + pL^{p-1} = (\gamma - \ell)(\gamma - (N-2-\ell)).$$

Hence the quadratic equation

$$\gamma^2 - (N-2)\gamma + pL^{p-1} = \mu$$

has two real roots $\gamma_1(\mu) < \gamma_2(\mu)$ for each $\mu \ge 0$. We note that $\gamma_1(0) = \ell$ and $\gamma_2(0) = N-2-\ell$. Further, $\gamma_1(\mu)$ is decreasing in μ, $\gamma_2(\mu)$ is increasing in μ, and

$$N - 2\gamma_1(\mu) > 2 > N - 2\gamma_2(\mu).$$

We first take $\mu = 0$ and $\gamma = \ell$ in (4.2). By (4.3), we simplify (4.2) to

$$Z_{\rho\rho} + \frac{N-1-2\ell}{\rho} Z_\rho + \frac{\rho}{2} Z_\rho = 0.$$

Clearly $Z \equiv 1$ is a solution of this equation, which implies that $z = r^{-\ell}$ satisfies (4.1). Similarly, if we take $\mu = 0$ and $\gamma = N-2-\ell$ in (4.2), we see that $z(\rho) = \rho^{N-2-\ell}$ satisfies (4.1).

Next we assume $\mu > 0$ and set $\gamma = \gamma_i(\mu)$ $(i = 1, 2)$ in (4.2). This gives

$$Z_{\rho\rho} + \frac{N-1-2\gamma_i(\mu)}{\rho} Z_\rho + \frac{\rho}{2} Z_\rho + \frac{\ell - \gamma_i(\mu)}{2} Z = 0.$$

Let $Z = G_i(\rho;\mu)$ be the solution of this equation subject to the initial conditions $Z(0) = 1$ and $Z_\rho(0) = 0$, and set

$$g_i(\rho;\mu) := \rho^{-\gamma_i(\mu)} G_i(\rho;\mu), \qquad i = 1, 2.$$

Then $z = g_i(\rho;\mu)$ satisfies (4.1). Thus we obtain two linearly independent solutions of (4.1). Moreover, since $\gamma_1(\mu) < \ell < \gamma_2(\mu)$ for $\mu > 0$, $z = g_1(\rho;\mu)$ is the only solution (up to a constant multiple) of (4.1) satisfying $\rho^\ell z(\rho) \to 0$ as $\rho \to 0$.

Next we show that $g_1(\rho;\mu)$ is positive for all ρ and that $\rho^\ell g_1(\rho;\mu)$ is strictly increasing in ρ. Setting $\gamma = \ell$ in (4.2), we have

$$Z_{\rho\rho} + \frac{N-1-2\ell}{\rho} Z_\rho + \frac{\rho}{2} Z_\rho - \frac{\mu}{\rho^2} Z = 0. \tag{4.4}$$

Then

$$Z = h(\rho;\mu) := \rho^\ell g_1(\rho;\mu) = \rho^{\ell-\gamma_1(\mu)} G_1(\rho;\mu)$$

satisfies this equation. Since $\ell - \gamma_1(\mu) > 0$, $h(\rho;\mu)$ is positive and strictly increasing for small $\rho > 0$. Rewriting (4.4) as

$$(\rho^{N-1-2\ell} e^{\rho^2/4} h_\rho)_\rho = \mu \rho^{N-3-2\ell} e^{\rho^2/4} h \tag{4.5}$$

and integrating this on $[\delta, \rho]$ with a sufficiently small $\delta > 0$, we obtain

$$\rho^{N-1-2\ell} e^{\rho^2/4} h_\rho(\rho;\mu) > \delta^{N-1-2\ell} e^{\delta^2/4} h_\rho(\delta;\mu) > 0,$$

provided $h \geq 0$ on $[\delta, \rho]$. This implies that h is positive for all $\rho > 0$ and consequently also $h_\rho > 0$ for all $\rho > 0$.

Finally, we prove that $h(\rho;\mu)$ converges to a positive constant as $\rho \to \infty$. Fix any $\beta \in (0, 2)$, and set $H(\rho) = 1 - \theta^\beta \rho^{-\beta}$, where $\theta > 0$ is sufficiently large (as specified below). Then for $\rho > \theta$ we have

$$\begin{aligned}
H_{\rho\rho} &+ \frac{N-1-2\ell}{\rho} H_\rho + \frac{\rho}{2} H_\rho - \frac{\mu}{\rho^2} H \\
&= -\frac{\mu}{\rho^2} - \theta^\beta \Big[\big\{\beta^2 - \beta(N-2-2\ell) - \mu\big\} \rho^{-\beta-2} - \frac{\beta}{2}\rho^{-\beta} \Big] \\
&= -\mu \rho^{\beta-2} \rho^{-\beta} - \theta^\beta \Big[\big\{\beta^2 - \beta(N-2-2\ell) - \mu\big\} \rho^{-2} \rho^{-\beta} - \frac{\beta}{2}\rho^{-\beta} \Big] \\
&> -\mu \theta^{\beta-2} \rho^{-\beta} - \theta^\beta \Big[\big|\beta^2 - \beta(N-2-2\ell) - \mu\big| \theta^{-2} \rho^{-\beta} - \frac{\beta}{2}\rho^{-\beta} \Big] \\
&= \theta^{\beta-2} \rho^{-\beta} \Big[\frac{\beta}{2}\theta^2 - \mu - \big|\beta^2 - \beta(N-2-2\ell) - \mu\big| \Big].
\end{aligned}$$

Therefore, if $\theta > 0$ is sufficiently large, then

$$H_{\rho\rho} + \frac{N-1-2\ell}{\rho} H_\rho + \frac{\rho}{2} H_\rho - \frac{\mu}{\rho^2} H > 0 \quad (\rho > \theta),$$

or equivalently,

$$(\rho^{N-1-2\ell} e^{\rho^2/4} H_\rho)_\rho > \mu \rho^{N-3-2\ell} e^{\rho^2/4} H \quad (\rho > \theta). \tag{4.6}$$

Using (4.5) and (4.6), we obtain

$$\begin{aligned} 0 &< \int_\theta^\rho \Big[h(r)(r^{N-1-2\ell} e^{r^2/4} H_r(r))_r - H(r)(r^{N-1-2\ell} e^{r^2/4} h_r(r))_r \Big] dr \\ &= \Big[h(r)(r^{N-1-2\ell} e^{r^2/4} H_r(r)) - H(r)(r^{N-1-2\ell} e^{r^2/4} h_r(r)) \Big]_{r=\theta}^{\rho}. \end{aligned}$$

Hence, using $h(\theta) > 0$, $H(\theta) = 0$ and $H_\rho(\theta) > 0$, we obtain $hH_\rho - h_\rho H > 0$, that is, $(h/H)_\rho < 0$ on (θ, ∞). Since H converges to 1 as $\rho \to \infty$, h must be bounded as $\rho \to \infty$. As it is increasing in ρ, it has a finite positive limit at $\rho = \infty$. This completes the proof. $\square$

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