Large-time behavior of bounded radial solutions of parabolic equations on $\mathbb{R}^N$: Part II—convergence for initial data with a linearly stable limit at infinity

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Dedicated to Giorgio Fusco

Abstract

We consider the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + f(u), && x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \mathbb{R}^N, \end{aligned}$$

where $N \geq 2$, f is a C^1 function satisfying minor nondegeneracy conditions, and u_0 is a radially symmetric function having a finite limit ζ as $|x| \to \infty$. We have previously proved that if ζ is a stable equilibrium of the equation $\dot{\xi} = f(\xi)$ and the solution u is bounded, then u is quasiconvergent: its ω-limit set with respect to the topology of $L^\infty_{loc}(\mathbb{R}^N)$ consists of steady states. In the present paper, we consider the case when ζ is linearly stable: $f(\zeta) = 0$ and $f'(\zeta) < 0$. Under this condition, we show that if the solution of the above Cauchy problem is bounded, then it converges, locally uniformly with respect to $x \in \mathbb{R}^N$, to a single steady state.

Key words: semilinear parabolic equations, Cauchy problem, radial solutions, convergence, normal hyperbolicity

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Contents

1 Introduction and the main theorem

This paper is a sequel to our previous work, [19], concerning bounded radial solutions of the semilinear heat equation

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{1.1}$$

Here, Δ in the Laplace operator in the spatial variable $x = (x_1, \dots, x_N)$ and f is a C^1 function on $\mathbb{R}$. Complementing (1.1) with the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$, we denote by $u(\cdot, t, u_0)$—or simply $u(\cdot, t)$ if u_0 is given and fixed—the unique classical solution of (1.1), (1.2), and by $T(u_0) \in (0, +\infty]$ its maximal existence time. We include the condition that $u(\cdot, t)$ is bounded for all $t \in (0, T(u_0))$ in the definition of a classical solution (so there are issues with nonuniqueness in larger classes of solutions). It will be one of our standing assumptions that u_0 is a radially symmetric (radial, for short), which implies that the corresponding solution u is radial in x.

If u is bounded on $\mathbb{R}^N \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is global. We then define its ω-limit set by

$$\omega(u) := \left\{\varphi \in C(\mathbb{R}^N) :\ u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\right\}, \tag{1.3}$$

where the convergence is in the topology of $L^\infty_{loc}(\mathbb{R}^N)$, that is, the locally uniform convergence. We recall the following well-known facts: the boundedness of u and parabolic regularity estimates imply that $\omega(u)$ is a nonempty compact connected set in $L^\infty_{loc}(\mathbb{R}^N)$. If $\omega(u)$ consists of a single element φ, then φ is a steady state of (1.1) and $u(\cdot, t) \to \varphi$ in $L^\infty_{loc}(\mathbb{R}^N)$. In this case, we say that u is *convergent*. We say that the solution u is *quasiconvergent* if all elements $\varphi \in \omega(u)$ are steady states of (1.1).

Many convergence and quasiconvergence results have been proved for (1.1), (1.2) under various conditions on f and u_0. On the other hand, it has also been proved that for a "robust" class of nonlinearities f, equation (1.1) possesses some bounded non-quasiconvergent solutions. We refer the reader to the introduction of [19] for a discussion of these results and some references. Here, we just recall contributions of [19] which are most relevant in the considerations below.

Assume that $N \geq 2$ and u_0 satisfies the following condition:

(IC) u_0 is a continuous radial function on $\mathbb{R}^N$ with $\zeta := \lim_{r\to\infty} u_0(r) \in \mathbb{R}$.

In (IC), we view the radial function u_0 as functions of the real variable $r = |x|$. This slight abuse of notation is repeated at several other places below.

As above, f is a C^1 function on $\mathbb{R}$. We further assume the following nondegeneracy conditions:

(ND) For each $\eta \in f^{-1}\{0\}$ there is $\epsilon > 0$ such that the function f is monotone (not necessarily strictly) in each of the intervals $(\eta - \epsilon, \eta]$, $[\eta, \eta + \epsilon)$.

(ND2) For $N = 2$ only: $f^{-1}\{0\}$ does not contain any (nonempty) open interval.

Essentially, (ND) just requires that f' not be oscillating in the one-sided neighborhoods of η. Obviously, (ND) and (ND2) are satisfied if all zeros of f are nondegenerate: $f'(\eta) \neq 0$ for each $\eta \in f^{-1}\{0\}$; but they are much weaker than the latter condition.

In the case when the limit ζ in (IC) is not a zero of f, the following convergence theorem is proved in [19]:

Theorem 1.1. *Assume* (IC), (ND), (ND2). *If* $f(\zeta) \neq 0$ *and the solution* u *of* (1.1), (1.2) *is bounded, then it is convergent: as* $t \to \infty$, $u(\cdot, t) \to \varphi$ *in* $L^\infty_{loc}(\mathbb{R}^N)$, *where* φ *is a (radial) steady state of* (1.1).

The following quasiconvergence theorem of [19] concerns the case when $f(\zeta) = 0$ and $f' \leq 0$ on a neighborhood of ζ. Note that the latter condition is equivalent to the stability of ζ as an equilibrium of the equation $\dot{\xi} = f(\xi)$ when (ND) is in effect.

Theorem 1.2. *Assume* (IC), (ND), (ND2). *Further assume that* $f(\zeta) = 0$ *and there is* $\delta > 0$ *such that* $f' \le 0$ *on* $(\zeta - \delta, \zeta + \delta)$. *If the solution* u *of* (1.1), (1.2) *is bounded, then it is quasiconvergent:* $\omega(u)$ *consists of (radial) steady states of* (1.1).

We remark that the quasiconvergence conclusion is not valid if ζ as an unstable equilibrium of the equation $\dot{\xi} = f(\xi)$. A counterexample with $N \geq 3$, $\zeta = 0$, and $f(u) = u^p$ with $p > (N + 2)/(N - 2)$ can be found in [20].

The investigation leading to Theorems 1.1, 1.2 was motivated by earlier quasiconvergence results for bounded solutions with convergent initial data in the case $N = 1$, that is, for equations (1.1) on the real line (see [16, 17, 18, 23]). Results similar to Theorems 1.1, 1.2 were independently proved in [22] in a more general setting of gradient systems of reaction-diffusion equations. The conclusions in [22] are stronger, stating the convergence rather than the quasiconvergence, but under the assumptions of [22], requiring in particular that the set of steady states with a given stable limit at infinity be discrete, the convergence and quasiconvergence are equivalent (see the next paragraph for more on this).

Under the assumptions of Theorem 1.2, the following additional information on the steady states in $\omega(u)$ is given in [19]. If $\omega(u)$ does not consist of a single steady state, then it contains a continuum of radial steady states which have the limit ζ, same as u_0, as $|x| \to \infty$. Since ζ is assumed to be a *stable* equilibrium of the equation $\dot{\xi} = f(\xi)$, it is a nontrivial question whether any such continuum may exist for some nonlinearities f. For generic nonlinearities it does not exist. This is a consequence of recent results of [21] concerning generic gradient systems of reaction diffusion equations. Thus, for generic nonlinearities, Theorem 1.2 yields the convergence of the solution u. We note that "generic nonlinearities" refers here to functions f forming a residual set (a countable union of open dense sets) in suitable functional spaces; it is not a set described by simple explicit conditions on f.

The previous remarks raise the following interesting question. Regardless of the existence of continua of steady states mentioned in the previous paragraph, is the convergence to single steady state guaranteed by some *explicit* nondegeneracy condition? A simple condition that comes to mind is the linear stability of the limit ζ (viewed as an equilibrium for the equation $\dot{\xi} = f(\xi)$): $f'(\zeta) < 0$. The main theorem of the present paper states that this condition is indeed sufficient for the convergence:

Theorem 1.3. *Assume* (IC), (ND), (ND2). *If* $f(\zeta) = 0 > f'(\zeta)$ *and the solution* u *of* (1.1), (1.2) *is bounded, then it is convergent: as* $t \to \infty$, $u(\cdot, t) \to \varphi$ *in* $L^\infty_{loc}(\mathbb{R}^N)$*, where* φ *is a (radial) steady state of* (1.1).

We make some comments on the proof of this results and outline its key ideas.

There are several standard ways of proving convergence of solutions of parabolic equations when their quasiconvergence is known. We can exclude techniques using Lojasiewicz inequality, which usually require the nonlinearities to be analytic. Zero number techniques, which are applicable in the radial setting (cp. Section 2), are useful in some steps of our proof, but do not lead to the convergence conclusion in any simple or familiar way. The fact that radial solutions of linear equations on unbounded domains may have infinite zero number at all times is a major detriment to zero number techniques. Instead, our proof relies on another common tool: an abstract convergence result based on normal hyperbolicity of a manifold of steady states. Two crucial ingredients are usually needed for successfully using this method for proving that a solution u converges to a steady state φ. The first one requires the set of all steady states near φ to form a manifold of some finite dimension m. The second one is the local normal hyperbolicity of this manifold, which mandates that the linearization of the right-hand side of equation at φ has 0 as an eigenvalue of multiplicity m, same as the dimension of the manifold, and it has no other eigenvalues on the imaginary axis.

In our setting, the manifold structure of steady states is easy to establish. The radial steady states of (1.1) are uniquely determined by their value at the origin, which gives a simple one-dimensional parametrization of continua of steady states, should they exist. With the condition $f'(\zeta) < 0$, it is also rather easy to understand the spectrum of the linearization of the right-hand side of (1.1) at any radial steady state φ with $\varphi(\infty) = \zeta$. Namely, $\Delta + f'(\varphi(r))$ is a Schrödinger operator with a radial potential whose essential spectrum is contained in $[-f'(\zeta), \infty)$ and whose eigenvalues below $-f'(\zeta)$ are all real and simple in the radial setting.

The problem with this approach is that we are considering the ω-limit set of the solution, as well as its convergence, in $L^\infty_{loc}(\mathbb{R}^N)$, which is a metrizable locally convex space, whereas convergence results based on normal hyperbolicity require a Banach space setup (so concepts such as submanifold, spectrum, and normal hyperbolicity have a standard meaning). To reconcile these stipulations, we prove that if $\omega(u)$ is not a single equilibrium, then there is $\varphi \in \omega(u)$ with the following properties:

(i) For some sequence $t_n \to \infty$ one has $u(\cdot, t_n) \to \varphi$ in $L^\infty(\mathbb{R}^N)$ (not just in $L^\infty_{loc}(\mathbb{R}^N)$).

(ii) There is a one-dimensional Lipschitz submanifold of $L^\infty(\mathbb{R}^N)$ containing φ and consisting of steady states.

It is perhaps worth emphasizing at this point that statement (i) is shown to hold assuming that $\omega(u)$ is not a singleton, which is an assumption which we want to rule out in the end. And when we do rule it out, (i) never applies, so our theorem only gives the convergence of the solution in $L^\infty_{loc}(\mathbb{R}^N)$. We derive (i) from this assumption using zero number techniques.

Once (i), (ii) are derived, we are able to apply an abstract convergence result of [2] in a Banach space setting. This leads to the conclusion that $u(\cdot, t) \to \varphi$ in $L^\infty(\mathbb{R}^N)$ as $t \to \infty$. Consequently, the ω-limit set of u (with respect to the topology of $L^\infty(\mathbb{R}^N)$, hence also with respect to the topology of $L^\infty_{loc}(\mathbb{R}^N)$) consists of the single element φ. This way we reach a desired contradiction to the assumption that $\omega(u)$ is not a singleton.

We add a few remarks concerning the nondegeneracy conditions (ND), (ND2). As shown in [19, Lemma 2.2], under conditions (ND), (ND2), results of [9] yield the convergence property of steady states: if ψ is a bounded steady state of (1.1), then $\psi(r)$ has a (finite) limit as $r \to \infty$ (see also [11, 14] for such convergence results). This convergence property is not guaranteed in general (see [13] for counterexamples with $N = 2$), but when it does hold for a specific equation, the above theorems, including the new Theorem 1.3, are valid for such an equation with hypotheses (ND), (ND2) removed.

In the rest of the paper, we assume as our **standing hypotheses** that $N \geq 2$, u_0 satisfies condition (IC) above, and f satisfies the following condition

(F) $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative.

We make the extra assumption that the derivative of f is bounded just for convenience and with no loss generality. Indeed, given any bounded solution, we can modify f outside a large interval $(-R, R)$, with no effect on the solution, so as to achieve the boundedness of f'. In addition, we can make the modification in such a way that the new function f is strictly monotone in each of the intervals $(-\infty, R) \cup (R, \infty)$. This monotonicity ensures that conditions (ND), (ND2) continue to be satisfied if they were satisfied by the original nonlinearity.

The remainder of the paper is organized as follows. The proof of Theorem 1.3 is given in Section 3. Section 2 contains preliminary results concerning limits, as $r \to \infty$, of radial solutions of (1.1) and nodal properties of radial solutions of linear parabolic equations.

2 Preliminaries

This section contains preliminary results concerning spatial limits of radial solutions of (1.1) and the zero number of radial solutions of linear parabolic equations.

Recall that assumption (IC) implies that the solution $u(\cdot, t, u_0)$ is radial. Viewed as a function of t and r, it satisfies the following equation and boundary condition at $r = 0$:

$$u_t = u_{rr} + \frac{N-1}{r} u_r + f(u), \qquad r > 0,\ t > 0, \tag{2.1}$$

$$u_r(0, t) = 0, \qquad t > 0. \tag{2.2}$$

2.1 Limits at spatial infinity

The following result is proved in [19, Propoposition 3.2].

Proposition 2.1. *Let u be the radial solution of* (1.1)*,* (1.2) *(with u_0 satisfying the standing hypotheses* (IC)*.) Then for any $t \in [0, T(u_0))$ the limit $\xi(t) := \lim_{r\to\infty} u(r, t)$ exists, and the function $\xi(t)$ is the solution of the differential equation $\dot{\xi} = f(\xi)$ with $\xi(0) = \zeta$.*

In particular, when $f(\zeta) = 0$, as in Theorem 1.3, we have $\lim_{r\to\infty} u(r, t) = \zeta$ for all $t \in [0, T(u_0))$.

Radial steady states of (1.1) are solutions of the equation

$$\psi_{rr} + \frac{N-1}{r} \psi_r + f(\psi) = 0, \qquad r > 0, \tag{2.3}$$

and they also satisfy the condition $\psi_r(0) = 0$. We recall the following well-known properties.

Since (2.3) is a nonsingular ordinary differential equation on $(0,\infty)$, our standing assumption (F) implies that for any $(a,b) \in \mathbb{R}^2$ and $r_0 > 0$ the solution of (2.3) satisfying the initial conditions

$$\psi(r_0) = a, \quad \psi_r(r_0) = b, \tag{2.4}$$

is defined globally on $(0,\infty)$. Also, if $r_0 = 0$ and $b = 0$, the initial value problem (2.3), (2.4) is well posed: it has a unique solution $\psi(\cdot,a) \in C^1[0,\infty) \cap C^2(0,\infty)$, and for any $R > 0$ the $C^1[0,R]$-valued map $a \mapsto \psi(\cdot,a)$ is continuously differentiable.

For the proof, based on [9], of the following convergence property of solutions of (2.3) we refer the reader to [19, Sect. 2.1].

Lemma 2.2. *Assume* (ND), (ND2). *If ψ is a solution of* (2.3) *which is bounded in* $[1,\infty)$, *then*

$$\lim_{r\to\infty} (\psi(r), \psi'(r)) = (\eta, 0), \quad \text{where } \eta \in f^{-1}\{0\}. \tag{2.5}$$

2.2 Zero number and nodal curves

For two radial solutions u, $\tilde{u}$ of (1.1) on an open time interval J, the function $v := u - \tilde{u}$ solves the linear equation

$$v_t = \Delta v + c(x,t)v, \qquad x \in \mathbb{R}^N,\ t \in J, \tag{2.6}$$

where c is a continuous bounded radial function given by

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(u(x,t) - \tilde{u}(x,t))\, ds. \tag{2.7}$$

In the variables t and $r = |x|$, the equation for $v = v(r,t)$ takes the form

$$v_t = v_{rr} + \frac{N-1}{r} v_r + c(r,t)v, \qquad r > 0,\ t \in J, \tag{2.8}$$

and we also have $v_r(0,t) = 0$ for $t \in J$. The above in particular applies if $\tilde{u}$ is a radial steady state of (1.1), which will be the case in the next section.

In the remainder of this subsection, we assume that c is an arbitrary continuous bounded radial function on $\mathbb{R}^N \times J$. We recall some standard zero number properties of radial solutions of (2.6) and then add a result on global nodal curves of such solutions.

If $g : [0, \infty) \to \mathbb{R}$ is a continuous function, we denote by $z(g)$ the number (possibly infinite) of zeros of g.

The following lemma is proved in [5] (it is a radial version of results proved in [1, 4] for equations in one space dimension).

Lemma 2.3. *Let $v(r, t)$ be a nontrivial bounded radial solution of* (2.6)*. Then the following statements are valid:*

(i) For each $t \in J$, the zeros of $v(\cdot, t)$ in $[0, \infty)$ are isolated.

(ii) The function $t \mapsto z(v(\cdot, t))$ is monotone nonincreasing.

(iii) If for some $t_0 \in J$ the function $v(\cdot, t_0)$ has a multiple zero ρ_0 in $[0, \infty)$ (that is, $v(\rho_0, t_0) = v_\rho(\rho_0, t_0) = 0$) and $z(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in J$ with $t_1 < t_0 < t_2$, one has $z(v(\cdot, t_1)) > z(v(\cdot, t_2))$.

The existence of a global nodal curve, as stated in the following lemma, will play an important role in the proof of Theorem 1.3. We note that similar statements were proved in [1, 6] for equations on the real line.

Let v be as in Lemma 2.3. The *nodal set* of v refers to the set $v^{-1}(0)$; a *nodal curve* of v is a continuous curve contained in $v^{-1}(0)$. We will consider nodal curves of the form $\{(\xi(t), t) : \ t \in J_0\}$, where $J_0 \subset J$ is an interval and ξ is a continuous function on J_0. We say that a function $\phi(r)$ *changes sign* at $\bar{r}$ if there are sequences $r_n^- < \bar{r} < r_n^+$, both approaching $\bar{r}$, such that $\phi(r_n^-)\phi(r_n^+) < 0$.

Lemma 2.4. *Let v be as in Lemma 2.3. The following statements are valid.*

(i) For any $(\rho_1, t_1) \in v^{-1}(0)$ and any $t_0 \in J$ with $t_0 < t_1$ there is a nodal curve $\{(\xi(t), t) : t \in [t_0, t_1]\}$ of v such that $\xi(t_1) = \rho_1$ and $\xi(t) > 0$ for all $t \in [t_0, t_1)$.

(ii) Let $\{(\xi_j(t), t) : t \in [t_1, t_2]\}$, $j = 1, 2$, be two nodal curves of v. If $\xi_1(t_2) < \xi_2(t_2)$, then $\xi_1(t) < \xi_2(t)$ for all $t \in [t_1, t_2]$.

(iii) Let $\{(\xi(t), t) : t \in [t_1, t_2]\}$ be a nodal curve of v with $\xi(t_2) > 0$. Then $\xi(t) > 0$ for all $t \in [t_1, t_2)$.

Proof. To prove statements (ii), (iii), we apply the maximum principle to (2.6). If (ii) is not valid, there is $\tau \in (t_1, t_2]$, such that $\xi_1(t) < \xi_2(t)$ for all

$t \in (\tau, t_2]$ and $\xi_1(\tau) = \xi_2(\tau)$. Then the solution v vanishes on the parabolic boundary of the domain

$$Q := \{(x,t) \in \mathbb{R}^N \times (\tau, t_2) : \xi_1(t) < |x| < \xi_2(t)\},$$

hence, by the maximum principle, it vanishes identically on Q. This is a contradiction to Lemma 2.3(i). If statement (iii) is not valid, then there is $\tau \in (t_1, t_2]$ such that $\xi(t) > 0$ for all $t \in (\tau, t_2]$ and $\xi(\tau) = 0$. Using the maximum principle on the domain

$$Q := \{(x,t) \in \mathbb{R}^N \times (\tau, t_2) : |x| < \xi(t)\},$$

we obtain $v \equiv 0$ on Q, which again is a contradiction to Lemma 2.3(i).

We now prove statement (i). Remember, that v, viewed as a function of r and t, satisfies equation (2.8) and the boundary condition

$$v_r(0,t) = 0, \quad t \in J. \tag{2.9}$$

Let $(\rho_1, t_1) \in v^{-1}(0)$ and $t_0 \in J$ with $t_0 < t_1$ be given. To start with, we claim that the following two statements are valid:

(s1) There exist $\delta > 0$ and a nodal curve $\Gamma = \{(\xi(t), t) : \ t \in [t_1 - \delta, t_1]\}$ of v such that $\xi(t_1) = \rho_1$, $\xi(t) > 0$ for all $t \in [t_1 - \delta, t_1)$, and $\xi(t)$ is a simple zero of $v(\cdot, t)$ for $t \in [t_1 - \delta, t_1)$.

(s2) If $\rho_1 > 0$, then there is $\delta > 0$ such that the nodal set of v in $[\rho_1 - \delta, \rho_1 + \delta] \times [t_1, t_1 + \delta]$ consists of the single point (ρ_1, t_1) or else it is equal to a nodal curve $\hat{\Gamma} = \{(\hat{\xi}(t), t) : \ t \in [t_1, t_1 + \delta]\}$ with $\xi(t_1) = \rho_1$.

If $v_r(\rho_1, t_1) \neq 0$ (which in particular means that $\rho_1 > 0$, cp. (2.9)), the validity of both statements, for a suitable $\delta > 0$, follows from the implicit function theorem. If ρ_1 is a multiple zero of $v(\cdot, t_1)$ and $\rho_1 > 0$, both statements follow from the local description of the nodal set of v given in [4]. Finally, if $\rho_1 = 0$ (so ρ_1 is automatically a multiple zero of $v(\cdot, t_1)$), the existence of a nodal curve as in (s1) follows from the local description of the nodal set given in [5].

Next we show that the nodal curve Γ given by statement (s1) can be continued globally, which yields a nodal curve as in statement (i).

Since $\xi(t)$ is a simple zero of $v(\cdot, t)$ for $t \in [t_1 - \delta, t_1)$, the function $v(\cdot, t)$ changes sign at $\xi(t)$. Therefore, there are points $0 < \xi_1^- < \xi(t_1 - \delta)$ and

$\xi_1^+ > \xi(t_1 - \delta)$ such that $v(\xi_1^-, t_1 - \delta)\, v(\xi_1^+, t_1 - \delta) < 0$. Let D^-, D^+ denote the connected components of the set $\{(r,t) \in [0,\infty) \times [t_0, t_1 - \delta] : v(r,t) \neq 0\}$ containing the points $(\xi_1^-, t_1 - \delta)$, $(\xi_1^+, t_1 - \delta))$, respectively.

Using the maximum principle (similarly as in [1, 5, 15]), we show that $D^\pm$ has to intersect the set $(0,\infty) \times \{t_0\}$. Indeed, otherwise $v(x,t)$, viewed as a function of $x \in \mathbb{R}^N$ and t, would vanish on the parabolic boundary of the set

$$\tilde{D}^\pm := \{(x,t) \in \mathbb{R}^N \times [t_0, t_1 - \delta] : (|x|, t) \in D^\pm\}.$$

The maximum principle would then imply that v vanishes on $D^\pm$, which is a contradiction. As a consequence, there is a (continuous) path $\mathcal{C}^\pm \subset D^\pm$ connecting $(\xi_1^\pm, t_1 - \delta)$ with a point $(\xi_0^\pm, t_0)$. Along each of the paths $\mathcal{C}^-$, $\mathcal{C}^+$, the function v is of one sign, and its signs on $\mathcal{C}^-$, $\mathcal{C}^+$ are opposite. In particular, $\mathcal{C}^-$, $\mathcal{C}^+$ are disjoint. It follows that if $\tau \in [t_0, t_1 - \delta]$ and $\tilde{\Gamma} := \{(\eta(t), t) :\ t \in [\tau, t_1 - \delta]\}$ is a nodal curve of v with $\eta(t_1 - \delta) = \xi(t_1 - \delta)$, then $\tilde{\Gamma}$ is contained in the region bounded by the paths $\mathcal{C}^-$, $\mathcal{C}^+$ and the segments $[\xi_1^-, \xi_1^+] \times \{t_1 - \delta\}$, $[\xi_0^-, \xi_0^+] \times \{t_0\}$. Moreover, $\tilde{\Gamma}$ is of positive distance to the paths $\mathcal{C}^-$, $\mathcal{C}^+$.

Let now τ_0 be the infimum of the values $\tau \in [t_0, t_1 - \delta]$ such that there is a nodal curve $\{(\xi(t), t) :\ t \in [\tau, t_1]\}$ containing Γ. Then there is a sequence $\Gamma_n = \{(\xi^n(t), t) :\ t \in [\tau_n, t_1]\}$ of such nodal curves with $\tau_n > \tau_0$ and $\tau_n \to \tau_0$. By the above remarks, the sequence $\{\xi^n(\tau_n)\}$ is bounded and of positive distance to 0. Passing to a subsequence, we may assume that $(\xi^n(\tau_n), \tau_n) \to (\rho_0, \tau_0)$ for some $\rho_0 \in (0, \infty)$.

Since $\rho_0 > 0$, the above statements (s1), (s2) both hold, for some $\delta > 0$, with (ρ_1, t_1) replaced by (ρ_0, τ_0). Moreover, since $(\xi^n(\tau_n), \tau_n) \to (\rho_0, \tau_0)$, the second alternative is statement (s2) must hold and if n is large enough the point $(\xi^n(\tau_n), \tau_n)$ lies on the nodal curve through the point (ρ_0, τ_0), as in statement (s2). Using this and statement (s1), we can now extend the function $\xi^n(t)$ to the interval $[\tau_0 - \delta, t_1]$ in such a way that $\tilde{\Gamma}_n := \{(\xi^n(t), t) : t \in [\tau_0 - \delta, t_1]\}$ is a nodal curve containing Γ. Then, by the definition of τ_0, $\tau_0 = t_0$, and we get a nodal curve as in statement (i) of Lemma 2.4 (even a slightly extended one). $\square$

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We assume that the hypotheses of the theorem are satisfied and the solution u of (1.1), (1.2) is bounded. Without loss of generality, we also assume that $\zeta = 0$. By Proposition 2.1, we have $u(\infty, t) := \lim_{r\to\infty} u(r,t) = 0$ for all $t \geq 0$.

We already know by Theorem 1.2 that u is quasiconvergent: $\omega(u)$ consists of radial steady states. In other words,

$$\omega(u) = \{\psi(\cdot, a) : a \in \mathcal{T}\}, \tag{3.1}$$

where $\psi(r, a)$ denotes the solution of (2.3), (2.4) with $r_0 = 0$ and $b = 0$, and

$$\mathcal{T} := \{\psi(0) : \psi \in \omega(u)\}.$$

Since $\omega(u)$ is connected and compact in $L^\infty_{loc}(\mathbb{R}^N)$, $\mathcal{T}$ is a compact interval or a set consisting of a single point $a \in \mathbb{R}$. The latter gives the desired convergence conclusion: $\omega(u) = \{\psi(\cdot, a)\}$; so our goal is to rule out the former. This will be achieved by means of a contradiction argument, which we facilitate by the following three lemmas scrutinizing the possibility that $\mathcal{T}$ is an interval.

Lemma 3.1. *If $a \in \operatorname{int} \mathcal{T}$, then the following relations are valid*

$$z(u(\cdot, t) - \psi(\cdot, a)) = \infty \quad (t > 0), \tag{3.2}$$

$$\psi(\infty, a) := \lim_{r\to\infty} \psi(r, a) = 0. \tag{3.3}$$

For the proof of this lemma we refer the reader to [19, Proposition 3.5].

Lemma 3.2. *If $a \in \operatorname{int} \mathcal{T}$, then there is a sequence $t_n \to \infty$, such that $u(\cdot, t_n) \to \psi(\cdot, a)$ in $L^\infty[0, \infty)$.*

The point of this statement is that the convergence is in $L^\infty[0, \infty)$, not just in $L^\infty_{loc}(\mathbb{R}^N)$ as in the definition of $\omega(u)$. We emphasize that we only prove this for interior points $\mathcal{T}$, the existence of which we eventually want to rule out. It is worth mentioning that in the proof of this lemma, we do not employ the strict inequality $f'(0) < 0$; we only use the fact that there is $\delta > 0$ such that

$$f'(s) \leq 0 \quad (s \in [-\delta, \delta]). \tag{3.4}$$

Proof of Lemma 3.2. Fix any $a \in \operatorname{int} \mathcal{T}$. To simplify the notation, we set $\psi := \psi(\cdot, a)$, and further let $v := u - \psi$. Clearly, v is a nontrivial radial solution of a linear equation (2.6) with c given by (2.7).

By the definition of $\mathcal{T}$, from $a \in \operatorname{int} \mathcal{T}$ it follows that there is an increasing sequence $\bar{t}_n$ in $(0, \infty)$ such that $\bar{t}_n \to \infty$ and $u(0, \bar{t}_n) = a$, that is, $v(0, \bar{t}_n) = 0$, for $n = 1, 2, \ldots$. Pick any $t_0 \in (0, \bar{t}_1)$. By Lemma 2.4(i), there are nodal curves

$$\{(\xi_n(t), t) : \ t \in [t_0, \bar{t}_n]\}, \quad n = 1, 2, \ldots,$$

of v such that $\xi_n(\bar{t}_n) = 0$ and $\xi_n(t) > 0$ for all $t \in [t_0, t_n)$. In particular, for each n, $\xi_n(\bar{t}_n) = 0 < \xi_{n+1}(\bar{t}_n)$, and therefore, by Lemma 2.4(ii),

$$\xi_n(t) < \xi_{n+1}(t) \quad (t \in [t_0, \bar{t}_n]). \tag{3.5}$$

These relations and the fact that the zeros of $v(\cdot, t_0)$ are isolated (see Lemma 2.3) imply that

$$\xi_n(t_0) \nearrow \infty \quad n \to \infty. \tag{3.6}$$

By Lemma 3.1, $\psi(\infty) = 0 = u(\infty, t)$. Let now $\delta > 0$ be as in (3.4) and pick $\rho > 0$ such that

$$|\psi(r)| < \delta \quad (r \geq \rho). \tag{3.7}$$

By (3.6), if n_0 large enough, then for all $n \geq n_0$ we have $\xi_n(t_0) > \rho$, which implies that there is $t_n \in (t_0, \bar{t}_n)$ such that

$$\xi_n(t) > \rho \ \ (t \in [t_0, t_n)) \quad \text{and} \quad \xi_n(t_n) = \rho. \tag{3.8}$$

Making n_0 larger if necessary, we also have $|u(r, t_0)| < \delta$ for all $r \geq \xi_n(t_0)$ with $n \geq n_0$.

We claim that for all $n \geq n_0$,

$$|u(r, t)| \leq \delta \quad (r \geq \xi_n(t), \ t \in [t_0, t_n]). \tag{3.9}$$

To prove this, first observe that the relations $f(0) = 0$ and $f' \leq 0$ in $[-\delta, \delta]$ yield $f(\delta) \leq 0 \leq f(-\delta)$. Thus, the constants δ, $-\delta$ are, respectively, a supersolution and a subsolution of equation (1.1). We now apply the comparison principle on the domain

$$Q_n := \{(r, t) \in \mathbb{R}^2 : r > \xi_n(t), \ t \in (t_0, t_n))\}.$$

By the choice of n_0, we have $|u(r, t_0)| < \delta$ for all $r \geq \xi_n(t_0)$. The "side" part of parabolic boundary of Q is a nodal curve of $v = u - \psi$ on which

$$|u(\xi_n(t), t)| = |\psi(\xi_n(t))| < \delta \quad (t \in (t_0, t_n)),$$

due to (3.7), (3.8). Therefore, the comparison principle does apply and (3.9) follows.

Consider now the function $v = u - \psi$. Relations (3.7), (3.9) imply that the coefficient $c(r,t)$ in the linear equation (2.6) satisfied by v is nonpositive on the domain Q_n. As already mentioned above, the "side" part of the parabolic boundary of Q_n is a nodal line of v, and we also have $v(\infty, t) = 0$. Therefore, applying the maximum and minimum principles on the domain Q_n, we obtain

$$\max_{r \geq \rho} |v(r, t_n)| \leq \max_{r \geq \xi_n(t_0)} |v(r, t_0)|. \tag{3.10}$$

Since $v(\infty, t_0) = 0$, (3.10) and (3.6) give

$$\max_{r \geq \rho} |u(r, t_n) - \psi(r)| \leq \max_{r \geq \xi_n(t_0)} |v(r, t_0)| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

This in particular implies that $t_n \to \infty$, as $u(\cdot, t) - \psi$ cannot vanish on an interval for any $t > 0$. Replacing the sequence $\{t_n\}$ by a subsequence if necessary, we have $u(\cdot, t_n) \to \tilde{\psi}$ in $L^\infty_{loc}[0, \infty)$ for some $\tilde{\psi} \in \omega(u)$. By Theorem 1.2, $\tilde{\psi}$ is a radial steady state, and, by (3.11), necessarily $\tilde{\psi} = \psi$. Thus, in addition to (3.11), we also have

$$\max_{r \leq \rho} |u(r, t_n) - \psi(r)| \to 0,$$

which yields the desired conclusion that $u(\cdot, t_n) \to \psi$ in $L^\infty[0, \infty)$. □

The next lemma, when applied to $\mathcal{J} = \operatorname{int} \mathcal{T}$, shows that if $\operatorname{int} \mathcal{T}$ is nonempty, then for some subinterval $\mathcal{I} \subset \operatorname{int} \mathcal{T}$ the steady states $\psi(\cdot, a)$, $a \in \mathcal{I}$, form a Lipschitz manifold in $L^\infty(0, \infty)$.

Lemma 3.3. *Assume $\mathcal{J} \subset \mathbb{R}$ is an open interval such that $\psi(\infty, a) = 0$ for all $a \in \mathcal{J}$. Then there is an open subinterval $\mathcal{I} \subset \mathcal{J}$ such that the map*

$$a \mapsto \psi(\cdot, a) : \mathcal{I} \to L^\infty(0, \infty) \tag{3.12}$$

is Lipschitz continuous.

Proof. We introduce the notation

$$\beta^+ := \inf\{\eta > 0 : f(\eta) = 0\}, \quad \beta^- := \sup\{\eta < 0 : f(\eta) = 0\}, \tag{3.13}$$

with the convention that $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$. Note that the assumption $f'(0) < 0$ implies that $\beta^- < 0 < \beta^+$ and

$$f(\eta) < 0 \quad (\eta \in (0, \beta^+)), \qquad f(\eta) > 0 \quad (\eta \in (\beta^-, 0)). \tag{3.14}$$

Moreover, if $\beta = \beta^-$ or $\beta = \beta^+$ and β finite, then β is a zero of f, hence a solution of (2.3). Therefore, for any $a \in \mathcal{J}$, the assumption $\psi(\infty, a) = 0$ implies that $\psi(\cdot, a) - \beta$ has only finitely many zeros, all of them simple. If β is infinite, then, trivially, $z(\psi(\cdot, a) - \beta)) = 0$. Thus,

$$\mathcal{J} = \bigcup_{k=0}^{\infty} M_k, \qquad M_k := \{a \in \mathcal{J} : z(\psi(\cdot, a) - \beta^+) \leq k\}.$$

Recall that for each (finite) $R > 0$ the map

$$a \mapsto \psi(\cdot, a) : \mathcal{J} \to C^1[0, R] \tag{3.15}$$

is continuously differentiable. Therefore, the simple zeros of $\psi(\cdot, a) - \beta^+$, if there are any, persist and change continuously as a is perturbed slightly. This implies that for each k the set $\mathcal{J} \setminus M_k$ is open, hence M_k is closed. Using the Baire category theorem, we obtain that for some k the set M_k has nonempty interior. Therefore, replacing $\mathcal{J}$ by an open subinterval if necessary, we may assume that $z(\psi(\cdot, a) - \beta^+) \leq k$ for all $a \in \mathcal{J}$. If $a_0 \in \mathcal{J}$ is such that $z(\psi(\cdot, a) - \beta^+)$ is maximal possible; then due to the maximality and persistence of simple zeros, $z(\psi(\cdot, a) - \beta^+) = z(\psi(\cdot, a_0) - \beta^+)$ for all $a \approx a_0$. Thus, we can again make the interval $\mathcal{J}$ smaller so that $z(\psi(\cdot, a) - \beta^+)$ is independent of $a \in \mathcal{J}$. Arguing similarly, we replace $\mathcal{J}$ by a yet smaller subinterval so that both $z(\psi(\cdot, a) - \beta^+)$ and $z(\psi(\cdot, a) - \beta^-)$ are independent of $a \in \mathcal{J}$.

Pick now any $a_0 \in \mathcal{J}$, and let $\delta > 0$ be such that $f' < 0$ in $[-\delta, \delta]$. Clearly, $\beta^- < -\delta$, $\delta < \beta^+$. Fix a large enough $\rho > 0$ so that $|\psi(r, a_0)| < \delta$ for all $r \geq \rho$. In particular, all zeros of $\psi(\cdot, a_0) - \beta^\pm$ are contained in $(0, \rho)$. We now show that if $\mathcal{I} \subset \mathcal{J}$ is a small enough open interval containing a_0, then the following two conditions are satisfied:

(a) $|\psi(\rho, a)| < \delta \quad (a \in \mathcal{I})$,

(b) $\psi(r, a) \in (\beta^-, \beta^+) \quad (r \geq \rho,\ a \in \mathcal{I})$.

The fact that (a) holds for $a \approx a_0$ follows immediately from the continuity of the map (3.15) with $R = \rho$. For $a \approx a_0$, we also have $z(\psi(\cdot, a) - \beta^\pm) = z(\psi(\cdot, a_0) - \beta^\pm)$ and the zeros of $\psi(\cdot, a) - \beta^\pm$ are small perturbations of the zeros of $\psi(\cdot, a_0) - \beta^\pm$. In particular, all zeros of $\psi(\cdot, a) - \beta^\pm$ are contained in $(0, \rho)$, which, in conjunction with (a), gives (b).

We next prove that conditions (a) and (b) imply that if $a \in \mathcal{I}$ then

$$|\psi(r,a)| < \delta \quad (r \geq \rho). \tag{3.16}$$

Indeed, if not, then, due to the relations (a) and $\psi(\infty, a) = 0$, there is $r > \rho$ such that $\psi(r,a)$ is a positive local maximum or a negative local minimum of $\psi(\cdot, a)$. In either case, we get a contradiction using equation (2.3) and the relations in (b) and (3.14).

Take now any $a, \tilde{a} \in \mathcal{I}$. The difference $v := \psi(\cdot, a) - \psi(\cdot, \tilde{a})$ satisfies a linear equation

$$v_{rr} + \frac{N-1}{r} v_r + c(r)v = 0, \quad r > 0,$$

where

$$c(r) = \int_0^1 f'(s\psi(r,a) + (1-s)\psi(r,\tilde{a}))\, ds.$$

Due to (3.16), we have $c(r) \leq 0$ for all $r > \rho$. Since $v(\infty) = \psi(\infty, a) - \psi(\infty, \tilde{a}) = 0$, the maximum principle gives

$$|\psi(r,a) - \psi(r,\tilde{a})| \leq |\psi(\rho, a) - \psi(\rho, \tilde{a})| \quad (r \geq \rho).$$

Combining this with the differentiability property of the map (3.15), we obtain that the map (3.12) is indeed Lipschitz continuous on some open subinterval of $\mathcal{J}$. □

We can now complete the proof of Theorem 1.3. We go by contradiction. Assume that $\operatorname{int} \mathcal{T} \neq \emptyset$. Applying Lemma 3.3 to $\mathcal{J} = \mathcal{T}$, we find an open interval $\mathcal{I} \subset \mathcal{T}$ such that the map (3.12) is Lipschitz continuous.

Pick some $a_0 \in \operatorname{int} \mathcal{I}$, and set

$$\psi := \psi(\cdot, a_0). \tag{3.17}$$

We intend to apply an abstract convergence theorem of [2], similarly as in [3, 7], to conclude that $u(\cdot, t) \to \psi$ in $L^\infty(0,\infty)$ as $t \to \infty$. This obviously is a contradiction to (3.1).

The setting in [2] is as follows

(H1) X is a Banach and Π is a continuous map on X with a fixed point ψ.

(H2) Π is of class C^1 on a neighborhood of ψ and $\Pi'(\psi)$ admits a spectral decomposition:

$$\sigma(\Pi'(\psi)) = \sigma^u \cup \sigma^c \cup \sigma^s,$$

where σ^u, σ^c, σ^s are closed subsets of $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$, $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, respectively.

Condition (H2) implies that we can write X as the direct sum

$$X = X^u \oplus X^c \oplus X^s,$$

where X^i is the image of the spectral projection of $\Pi'(\psi)$ associated with the spectral set σ^i, $i = u, c, s$ (see [10]).

We denote by $\omega_\Pi(u_0)$ the ω-limit set of a point $u_0 \in X$ relative to the iterates of the map Π (the convergence is in X):

$$\omega_\Pi(u_0) = \{\phi \in X : \Pi^{n_k}(u_0) \to \phi \text{ for some sequence } n_k \to \infty\}.$$

By $\mathrm{Fix}(\Pi)$ we denote the set of all fixed points of Π.

The following result is contained in [2, Theorem B].

Theorem 3.4. *Assume* (H1), (H2) *and let* u_0 *be a point in* X *such that* $\psi \in \omega_\Pi(u_0) \subset \mathrm{Fix}(\Pi)$. *Assume that either* X^u *is finite-dimensional or the orbit* $\{\Pi^n(u_0) : n = 0, 1, \dots\}$ *is relatively compact in* X. *Further assume that the following hypothesis is satisfied*

(M) $m := \dim X^c < \infty$ *and there is a submanifold* $M \subset X$ *with* $\dim M = m$ *such that* $\psi \in M \subset \mathrm{Fix}(F)$.

Then $\omega_\Pi(u_0) = \{\psi\}$.

We apply this theorem to the time-1 map of (1.1). This can be done in the $C_0(\mathbb{R}^N)$-setting or $L^\infty(\mathbb{R}^N)$-setting; equation (1.1) is well posed on either of these Banach spaces due to $f(0) = 0$. We choose the $C_0(\mathbb{R}^N)$-setting, as in [3]. Note that for the solution u considered in this section we have $u(\cdot, t) \in C_0(\mathbb{R}^N)$, although we do not necessarily have the compactness of the orbit $\{u(\cdot, t) : t \geq 1\}$ in $C_0(\mathbb{R}^N)$ (or in $L^\infty(\mathbb{R}^N)$, for that matter). This is a minor difference from [3, 7], and is not a complication as the compactness is not needed for the application of Theorem 3.4.

Let $Y := C_0(\mathbb{R}^N)$. As in the introduction, $u(x, t, u_0)$ denotes the solution of (1.1), (1.2), where we now take $u_0 \in Y$. The solution is defined globally due to the assumption that f' is bounded.

The subspace $X \subset Y$ consisting of all radial functions in Y is closed in Y and invariant under the semiflow of (1.1). We equip X with the induced norm (the supremum norm). Consider the time-1 map of (1.1) on X:

$$\Pi : u_0 \mapsto u(\cdot, 1; u_0).$$

By the differentiable dependence of solutions on the initial data (see [8, 12]), Π is a C^1-map on X. Obviously, $\Pi^n(u_0) = u(\cdot, n, u_0)$, $n = 0, 1, \ldots$, and the steady states of (1.1) contained in X are fixed points of Π. In particular, due to (3.3), the steady states $\psi(\cdot, a)$, $a \in \mathcal{T}$, are fixed points of Π.

Consider the linearization $\Pi'(\psi)$ at the fixed point ψ (cp. (3.17)). It satisfies hypothesis (H2) with $\dim X^u < \infty$ and $\dim X^c \leq 1$. This can be verified using spectral properties of the Schrödinger operator $\Delta + f'(\psi(r))$—note that the radial potential in this operator satisfies $\lim_{r\to\infty} f'(\psi(r)) = f'(0) < 0$—and the spectral mapping theorem which relates the spectrum of $\Pi'(\psi)$ to the spectrum of the Schrödinger operator. The details can be found in [3, Section 2.2]. In our case, the arguments in [3, Section 2.2] can be simplified a little, as we only consider the operator in the radial space X and thus do not need to worry about the symmetry of eigenfunctions and simplicity of the corresponding eigenvalues as in [3, Section 2.2].

Recall that we have chosen $a_0 \in \mathcal{I}$, where $\mathcal{I} \subset \mathcal{T}$ is an open interval on which the map (3.12) is Lipschitz continuous. In view of (3.3), this means that the set

$$M := \{\psi(\cdot, a) : a \in \mathcal{I}\} \tag{3.18}$$

is a Lipschitz curve of fixed points of Π containing ψ. This implies, by the implicit function theorem, that the kernel of Π' is nontrivial. Consequently, $\dim X^c = 1$ and we have also verified hypothesis (M) of Theorem 3.4.

We now show that if u is as in Theorem 1.3 and $u_0 = u(\cdot, 0)$, then $\psi \in \omega_\Pi(u_0)$. By Lemma 3.2, there is a sequence $t_n \to \infty$ such that $u(\cdot, t_n) \to \psi$ in X. Since ψ is a steady state, we consequently have

$$u(\cdot, t_n + t) \to \psi \tag{3.19}$$

uniformly for $t \in [-m, m]$, $m = 1, 2, \ldots$. Therefore, there is a sequence of positive integers $n_k \to \infty$ such that

$$\Pi^{n_k}(u_0) = u(\cdot, n_k) \to \psi,$$

as required.

Also, any element of $\omega_\Pi(u_0)$ is clearly an element of $\omega(u) \cap X$, which implies, due to Theorem 1.2, that $\omega_\Pi(u_0) \subset \mathrm{Fix}(\Pi)$.

Theorem 3.4 now implies that $\Pi^n(u_0) \to \psi$ in X. Using (3.19) with $t_n := n$, we obtain that $u(\cdot, t) \to \psi$ in X as $t \to \infty$. This is a contradiction to (3.1).

By this contradiction we have ruled out the possibility that $\mathcal{T}$ has nonempty interior. Thus $\mathcal{T}$ is a singleton, proving the convergence of u (in $L^\infty_{loc}(\mathbb{R}^N)$).

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