Spatial trajectories and convergence to traveling fronts for bistable reaction-diffusion equations

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Dedicated to Professor Djairo Guedes de Figueiredo
on the occasion of his 80th birthday

Abstract. We consider the semilinear parabolic equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, t > 0, \tag{A}$$

where f is a bistable nonlinearity. It is well-known that for a large class of initial data, the corresponding solutions converge to traveling fronts. We give a new proof of this classical result as well as some generalizations. Our proof uses a geometric method, which makes use of spatial trajectories $\{(u(x,t), u_x(x,t)) : x \in \mathbb{R}\}$ of solutions of (A).

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1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We assume that f is of a bistable type and $u_0(x)$ takes values between the two stable zeros of f and has a "front-like" shape (see below for precise hypotheses). Classical results then tell us that, under additional conditions on f and u_0, the solution of (1.1), (1.2) approaches the orbit of a traveling front. The main purpose of this note is to give a new proof of this result and relax its hypotheses somewhat.

To discuss the large-time behavior of solutions in more specific terms, we introduce two kinds of limit sets. Assuming that the solution of (1.1), (1.2), is bounded we set

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}, \tag{1.3}$$

$$\Omega(u) := \{\varphi : u(\cdot + x_n, t_n) \to \varphi \text{ for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}, \tag{1.4}$$

where the convergence is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence) in both cases. Since the solution u is determined uniquely by its initial value, we sometimes use the symbols $\omega(u_0)$, $\Omega(u_0)$ for $\omega(u)$, $\Omega(u)$.

By standard parabolic regularity estimates, the set $\{u(x+\cdot, t) : t \geq 1,\ x \in \mathbb{R}\}$ is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that both $\omega(u)$ and $\Omega(u)$ are nonempty, compact, and connected in $L^\infty_{loc}(\mathbb{R})$. Clearly, $\omega(u_0) \subset \Omega(u_0)$, but the opposite inclusion is not true in general. Both these limit sets give a useful information on the solution u: while $\Omega(u)$ gives a picture of the global shape of $u(\cdot, t)$ for large times and is also useful for investigating the behavior of $u(\cdot, t)$ in moving coordinate frames; $\omega(u)$ captures its large-time behavior in local regions.

To formulate our results, we first make precise our hypotheses. We assume the following conditions on f:

(Hf) $f \in C^1(\mathbb{R})$, $f(0) = f(1) = 0$ and there is $\alpha \in (0,1)$ such that

$$f < 0 \text{ in } (0, \alpha); \qquad f > 0 \text{ in } (\alpha, 1). \tag{1.5}$$

Since we only investigate solutions satisfying $0 \le u \le 1$, the values of $f(s)$ for $s \notin [0,1]$ are irrelevant. For convenience, we shall assume that

$$f > 0 \text{ in } (-\infty, 0); \qquad f < 0 \text{ in } (1, \infty); \quad f' \text{ is bounded.} \tag{1.6}$$

Thus 0, α, 1 are all the equilibria of the ordinary differential equation (ODE) $\dot{\xi} = f(\xi)$; 0, 1 are stable, whereas α is unstable, both from above and below (thus the name “bistable nonlinearity”). Obviously, the specific choice of the interval $[0,1]$ does not restrict generality; other bistable nonlinearities are brought to this form by a suitable scaling and translation. We often view 0, α, and 1 as constant functions and then they become steady states of (1.1).

Hypothesis (Hf) implies (see [2, 6, 16], for example) that there is a traveling front of (1.1) joining 0 and 1, that is, a solution U of the form

$$U(x,t) = \phi(x - \hat{c}t), \text{ where } \hat{c} \in \mathbb{R},\ \phi \in C^2(\mathbb{R}), \text{ and } \phi' > 0. \tag{1.7}$$

Moreover, both the increasing “profile” function ϕ and the “speed” $\hat{c}$ are uniquely determined, up to translations of ϕ, and $\operatorname{sign} \hat{c} = -\operatorname{sign} F(1)$, where

$$F(u) = \int_0^u f(s)\, ds. \tag{1.8}$$

For definiteness we shall assume that

$$F(1) = \int_0^1 f(s)\, ds \ge 0. \tag{1.9}$$

This means that the front “travels to the left” ($\hat{c} < 0$) or is a standing wave ($\hat{c} = 0$). Again, assumption (1.9) is at no cost to generality; the other case is completely analogous (or, one simply interchange the roles of the two stable equilibria). Note that $\tilde{U}(x,t) = U(-x,t)$ is also a traveling front; it has a decreasing profile function (namely, $\tilde{\phi} = \phi(-x)$) and the opposite speed.

With α as in (Hf), we assume the following conditions on u_0:

(Hu) $u_0 \in C(\mathbb{R})$, $0 \le u_0 \le 1$, and

$$\limsup_{x \to -\infty} u_0(x) < \alpha < \liminf_{x \to \infty} u_0(x). \tag{1.10}$$

In this sense, u_0 has a “front-like” shape.

Theorem 1.1. *Assume that* (Hf), (Hu) *hold and* $F(1) > 0$*. Let* u *be the solution of* (1.1), (1.2)*. Then*

$$\Omega(u) = \{0, 1\} \cup \{\phi(\cdot - \xi) : \xi \in \mathbb{R}\}, \tag{1.11}$$

where ϕ *is as in* (1.7)*.*

Below we also give a theorem for $F(1) = 0$ under an additional assumption on u_0.

Similar results on the approach of solutions to traveling fronts for bistable nonlinearities can be found in [6, 7], among many other publications. Let us discuss the relation of Theorem 1.1 to these classical results in more detail. It is not difficult to show (see Sect. 3) that (1.11) implies the following

Corollary 1.2. *Assume that the hypotheses of Theorem 1.1 hold. Then there is a* C^1*-function* $\gamma(t)$ *such that* $\gamma'(t) \to 0$ *as* $t \to \infty$*, and*

$$u(\cdot, t) - \phi(\cdot - \hat{c}t - \gamma(t)) \to 0 \text{ as } t \to \infty, \tag{1.12}$$

where the convergence is in $L^\infty(\mathbb{R})$*.*

This conclusion was proved in [7] under the extra assumption that u_0 is monotone. Note that Corollary 1.2 only says that the translation group orbit $\{\phi(\cdot - \xi) : \xi \in \mathbb{R}\}$ of ϕ attracts the solution; it does not say that the solution approaches a particular traveling front, or, in other words, that $\gamma(t)$ has a limit as $t \to \infty$. The latter was proved in [6] under the nondegeneracy condition

$$f'(0) < 0, \quad f'(1) < 0. \tag{1.13}$$

In this case, the monotonicity of u_0 is not assumed and one even gets the exponential rate of convergence in (1.12). There are many extensions of this convergence results, see for example, [2, 3, 9, 11, 12, 13, 14, 16] and references therein (for more bibliographical notes and a discussion of classical results for bistable and other types of nonlinearities see [16, Sect. 1.6]). Usually, the convergence is proved by first showing that the solution gets close to a particular traveling front at some time (this property follows from (1.12); in the nondegenerate case (1.13), Fife and McLeod [6] proved it by way of a Lyapunov functional) and then employing an asymptotic stability property of the front. Conditions (1.13), or similar nondegeneracy conditions, are typically needed to establish the linearized stability of the front. The convergence with the exponential rate is then obtained from the principle of linearized stability for parabolic equations [8, 15].

Here, we only treat the more general case with the weaker conclusion, as in Theorem 1.1, Corollary 1.2, without assuming the nondegeneracy conditions. Our objective is to give a relatively simple geometric proof of the result. The main technical tools of our method are intersection comparison (or zero number) arguments and analysis of spatial trajectories of solutions of (1.1). If u is a solution, then its *spatial trajectory at time* t is the set $\{(u(x,t), u_x(x,t)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Note that if u is a steady state, then its spatial trajectory is independent of t and it is a trajectory, in the usual sense, of the first-order system corresponding to the ODE $u_{xx} + f(u) = 0$. Likewise, if u is a traveling wave, then its spatial trajectory is independent of t and it a trajectory of the first order system corresponding to the equation $u_{xx} + cu_x + f(u) = 0$, where c is the speed of the wave. Our proof depends on a good understanding of how spatial trajectories of the solution of (1.1), (1.2) can intersect spatial trajectories of steady states and traveling waves.

We remark that spatial trajectories also appear, though not under this name, in [7]. In that paper, given a solution u with $u_x > 0$, the authors consider a function $p(u,t) = u_x(\zeta(u,t),t)$, where $\zeta(\cdot,t)$ is the inverse function to $u(\cdot,t)$. They show that p satisfies a degenerate parabolic equation and prove the attraction to traveling fronts by delicate comparison arguments for this equation. Observe that for any t, the graph of $p(\cdot,t)$ is the spatial trajectory $\tau(u(\cdot,t))$ of u. Obviously, for the spatial trajectory to be such a graph, the monotonicity in x is necessary. In contrast, we work with the spatial trajectories as curves in $\mathbb{R}^2$, thus we do not need the monotonicity assumption. Also, we do not use any transformed partial differential equations similar to the equation for p. All the essential information from which we can rather easily prove Theorem 1.1 is contained in phase diagrams of the ODEs $u_{xx} + cu_x + f(u) = 0$, for various c.

Theorem 1.1 can probably be proved in several different ways. For example, in [16] it is suggested that, under the additional assumption that

$$\lim_{x\to-\infty} u_0(x) = 0, \quad \lim_{x\to\infty} u_0(x) = 1, \tag{1.14}$$

the following approach should work. First one proves that there is a function $\epsilon(t) > 0$ such that $\epsilon(t) \to 0$ as $t \to \infty$ and u is increasing in x in the set $\{(x,t) : \epsilon(t) < u(x,t) < 1-\epsilon(t)\}$. Once this is established, one can modify the arguments in the monotone case, to get the conclusion in this more general situation. The method we use in the present paper is more direct and seems to be simpler than this suggested approach (and (1.14) is not needed).

Our method applies, with minor modifications, to more general situations where the existence and uniqueness of the traveling front can be established (see [6]), but for simplicity we just consider (1.5). The method is also useful in other problems; for example, in [10] we will use similar techniques in the proof of quasiconvergence of solutions with localized initial data. On the other hand, the scope of the method seems to be limited to the one-dimensional spatially homogeneous equations.

In the case $F(1) = 0$, we prove the same results as in Theorem 1.1 and Corollary 1.2, but we need a stronger assumption on u_0. For example, the following will do.

(Ha) Either $u_0 - \alpha$ has a unique zero, or the limits $u_0(\pm\infty)$ exist and one has
$$u_0(-\infty) \le u_0(x) \le u_0(\infty) \quad (x \in \mathbb{R}). \tag{1.15}$$

Note that (1.15) is trivially satisfied if (Hu) and (1.14) hold.

Theorem 1.3. *Assume that the hypotheses* (Hf), (Hu), (Ha) *hold and* $F(1) = 0$ *(so also* $\hat{c} = 0$*). Let* u *be the solution of* (1.1), (1.2). *Then the conclusions of Theorem 1.1 and Corollary 1.2 hold.*

Note that (1.11) in particular gives $\omega(u) \subset \{0,1\} \cup \{\phi(\cdot - \xi) : \xi \in \mathbb{R}\}$. One has $\{0,1\} \cap \omega(u) \ne \emptyset$ if and only if the function γ in (1.12) is unbounded. If $\gamma(t)$ has a finite limit ξ when $t \to \infty$, then $\omega(u)$ consists of the single equilibrium $\phi(\cdot - \xi)$ (and one can even take the uniform convergence in the definition of $\omega(u)$). This is the case if the stable zeros are nondegenerate: $f'(0), f'(1) < 0$ [6], but without this assumption the situation is not so clear. As far as we know, examples of solutions satisfying the present hypotheses for which $\omega(u)$ is not a singleton are not available.

The paper is organized as follows. In the next section, we recall several useful results concerning the zero number, Ω-limit sets, and solutions of the ODEs $u_{xx} + cu_x + f(u) = 0$, $c \le 0$. The proofs of the main results are given in Section 3.

2 Preliminaries

2.1 Phase space and traveling fronts

In this section we examine the solutions of the ODE

$$v_{xx} + cv_x + f(v) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

This is the equation satisfied by steady states of (1.1) (if $c = 0$) and by the profile functions of traveling fronts. Throughout the section we assume that the hypotheses (Hf), (1.6), (1.9) are satisfied.

The first-order system associated with (2.1) is

$$v_x = w, \quad w_x = -cw - f(v). \tag{2.2}$$

Its solutions are all global, by the Lipschitz continuity of f (see (1.6)). For $c = 0$, we obtain a Hamiltonian system,

$$v_x = w, \quad w_x = -f(v), \tag{2.3}$$

with the Hamiltonian energy

$$H(v,w) := w^2/2 + F(v).$$

In this case, the trajectories of (2.3) are contained in the level sets of H. Note in particular that the level sets are symmetric about the v axis. We now summarize a few basic properties of trajectories of system (2.3) (see Fig.1); they are all proved easily by an elementary phase plane analysis using the Hamiltonian and the standing hypotheses (Hf), (1.6), (1.9). System (2.3) has only four types of bounded orbits: equilibria (stationary solutions)–all of them on the v axis, nonconstant periodic orbits, homoclinic orbits–which exist only in the case $F(1) > 0$, and heteroclinic orbits–only in the case $F(1) = 0$. All bounded orbits, other than the equilibria $(0,0)$ and $(1,0)$ are contained in the open strip

$$S := \{(v,w) : 0 < v < 1\}.$$

This strip is covered by the level sets

$$L_\gamma := \{(v,w) : H(v,w) = \gamma\}, \;\; \gamma \in [F(\alpha), \infty).$$

For each $\gamma \in (F(\alpha), \infty)$, the level set L_γ intersects the vertical line $\{(\alpha, w) : w \in \mathbb{R}\}$, at exactly two points $(\alpha, \pm\sqrt{2(\gamma - F(\alpha))})$; for $\gamma = F(\alpha)$ there is just one intersection, the equilibrium $(\alpha, 0)$. For $\gamma > F(1)$, L_γ consists of two curves not intersecting the v-axis. The solutions v whose trajectories $\tau(v)$ are given by these curves are strictly monotone with infinite limits at $\pm\infty$. If $F(1) > 0$ and $\gamma \in (0, F(1))$, then the part of L_γ intersecting S coincides with a trajectory of a solution v with limits $v(\pm\infty) = -\infty$. If $\gamma = F(1) > 0$, then L_γ consists of $(1, 0)$ and the trajectories of solutions which converge to 1 as $x \to \infty$ or $x \to -\infty$. For $\gamma \in (F(\alpha), 0)$, the set $L_\gamma \cap S$ coincides with a nonstationary periodic orbit (or, closed orbit) of (2.3). For $\gamma = 0$, $L_\gamma \cap S$ is a homoclinic orbit to $(0, 0)$ (this is the case if $F(1) > 0$) or the union of two heteroclinic connections between the equilibria $(0, 0)$, $(1, 0)$ (if $F(1) > 0$). If (v, v_x) is a nonstationary periodic solution of (2.3), then $v - \alpha$ has infinitely many zeros. Of course, all these zeros are simple by the uniqueness for the Cauchy problem.

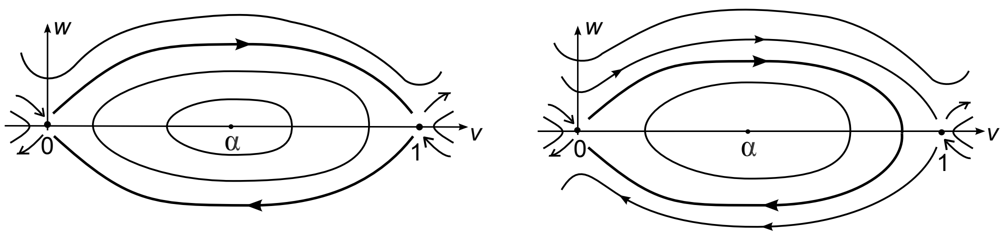

Figure 1: The phase diagram of system (2.3); the balanced case ($F(1) = 0$) is on the left, the unbalanced case ($F(1) > 0$) on the right

Let us now consider system (2.2) with $c < 0$. In this case, H is increasing along the solutions:

$$\frac{dH(v(x), w(x))}{dx} = -cw^2. \tag{2.4}$$

In particular, any bounded nonstationary solution of (2.2) with $c < 0$, is a heteroclinic solution between two different equilibria. For $c = \hat{c}$, and for this value only, (2.2) has a heteroclinic solution from $(0, 0)$ to $(1, 0)$, given by the profile function of the traveling front: $(v, w) \equiv (\phi, \phi_x)$ [2, 6, 16]. Obviously, for any solution (v, w) of (2.2), v is increasing (resp. decreasing) when $w > 0$ (resp. $w < 0$). One also shows easily that the sets

$$Q_1 := \{(v, w) : v \geq 1, w \geq 0\} \setminus \{(1, 0)\},$$
$$Q_3 := \{(v, w) : v \leq 0, w \leq 0\} \setminus \{(0, 0)\}$$

are positively invariant in the sense that if a solution satisfies $(v(0), w(0)) \in Q_i$, for $i = 1$ or $i = 3$, then for all $x > 0$ one has $(v(x), w(x)) \in \operatorname{Int} Q_i$ (the interior of Q_i). Similarly, the sets

$$Q_2 := \{(v, w) : v \le 0, w \ge 0\} \setminus \{(0,0)\},$$
$$Q_4 := \{(v, w) : v \ge 1, w \le 0\} \setminus \{(1,0)\}$$

are negatively invariant.

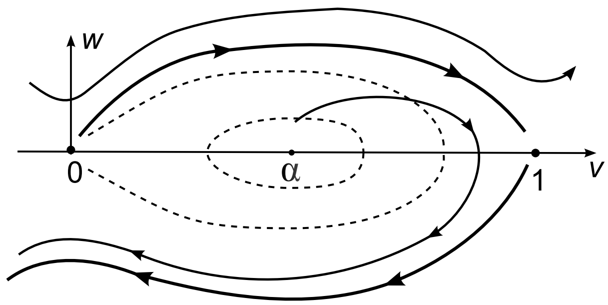

Figure 2: The phase diagram of system (2.2) with $c = \hat{c} < 0$; the dashed curves represent orbits of (2.3).

Let now $c = \hat{c}$, so that $\tau(\phi)$ is a heteroclinic orbit from $(0,0)$ to $(1,0)$. It is well known that, regardless of whether $f'(0)$, $f'(1)$ vanish or are negative, $\tau(\phi)$ contains all initial data $(\xi, \eta) \in S$ such that the solution ψ of (2.1) with

$$\psi(0) = \xi, \quad \psi_x(0) = \eta \tag{2.5}$$

satisfies either $(\psi(x), \psi'(x)) \to (1,0)$ as $x \to \infty$, or $(\psi(x), \psi'(x)) \to (0,0)$ as $x \to -\infty$ (see, for example, [2, Sect. 4]). Likewise, there is a solution $\tilde{\phi}$, such that $S \cap \tau(\tilde{\phi})$ is precisely the set of initial data $(\xi, \eta) \in S$ such that the solution ψ of (2.1), (2.5) satisfies $(\psi(x), \psi'(x)) \to (1,0)$ as $x \to -\infty$. If $\hat{c} = 0$, then $\tilde{\phi}$ is given simply by $\tilde{\phi}(x) = \phi(-x)$ and $\tau(\tilde{\phi})$ is a heteroclinic orbit from $(1,0)$ to $(0,0)$. If $\hat{c} < 0$, then $\tau(\tilde{\phi})$ intersects the halfline $\{(0, w) : w < 0\}$, hence by the positive invariance of Q_3 one has $(\tilde{\phi}(x), \tilde{\phi}_x(x)) \in Q_3$ for all large enough x (see Fig. 2). Since different trajectories of the autonomous system (2.2) cannot intersect, using the above properties of ϕ, $\tilde{\phi}$ and the invariance properties of the sets $Q_1 - Q_4$, we obtain the following characterization of the solutions of (2.1), (2.5) with

$$(\xi, \eta) \in S \setminus (\tau(\phi) \cup \tau(\tilde{\phi})). \tag{2.6}$$

Lemma 2.1. *Let $c = \hat{c}$ and let ψ be the solution* (2.1), (2.5), *where* (ξ, η) *is as in* (2.6). *Consider the statements (ai)–(aiii) below. If $\hat{c} = 0$, then one of the statements (ai), (aii) holds; if $\hat{c} < 0$, then either $\psi \equiv \alpha$ or one of the statements (aii), (aiii) holds.*

(ai) *ψ is a periodic solution with $0 < \psi < 1$ (that is, either $\psi \equiv \alpha$ or it is a nonconstant periodic solution).*

(aii) *There are numbers $x_1 < x_2$ such that*

$$\begin{aligned} &\psi(x) \in (0,1) \quad (x \in (x_1, x_2)), \\ &\psi(x) \notin (0,1) \quad (x \in \mathbb{R} \setminus (x_1, x_2)), \\ &\psi(x_1) \neq \psi(x_2). \end{aligned} \tag{2.7}$$

(aiii) $(\psi(x), \psi'(x)) \to (\alpha, 0)$ *as $x \to -\infty$, and there is $x_0 \in \mathbb{R}$ such that*

$$0 < \psi(x) < 1 \ \ (x \in (-\infty, x_0)); \quad \psi(x) < 0 \ \ (x \in (x_0, \infty)). \tag{2.8}$$

Note that in (aii) and (aiii), (ψ, ψ') is not an equilibrium, hence $\psi'(x) \neq 0$ whenever $\psi(x) = 0$ or $\psi(x) = 1$. This and (2.7) imply that in (aii) we have

$$\psi(x_1), \psi(x_2) \in \{0, 1\}; \quad \psi'(x_1) \neq 0, \ \psi'(x_2) \neq 0, \tag{2.9}$$

and in (aiii)

$$\psi(x_0) = 0, \quad \psi'(x_0) < 0. \tag{2.10}$$

Corollary 2.2. *Assume that $\hat{c} < 0$ and fix $(\xi, \eta) \in S \setminus \tau(\phi)$. If $c \in (\hat{c}, 0)$ is sufficiently close to $\hat{c}$ and ψ is the solution* (2.1), (2.5), *then either $\psi \equiv \alpha$ or one of the statements (aii), (aiii) holds.*

Proof. Denote the solution of (2.1), (2.5) by ψ^c; ψ being the solution for $c = \hat{c}$ as in Lemma 2.1. If $(\xi, \eta) = (\alpha, 0)$, then of course $\psi^c \equiv \alpha$.

Assume that $(\xi, \eta) \neq (\alpha, 0)$. For now assume also that (ξ, η) is as in (2.6), so that (aii) or (aiii) hold for $c = \hat{c}$. We claim that these are robust properties, so, due to the continuous dependence of the solution ψ^c on c, they remain valid–possibly with slightly perturbed x_1, x_2, or x_0–if ψ is replaced with ψ^c and $c > \hat{c}$, $c \approx \hat{c}$. This is obviously the case with (aii) because of (2.9) and the invariance properties of the sets $Q_1 - Q_4$. If (aiii) holds for $c = \hat{c}$, there is $x_3 < x_0$ such that $(\psi(x_3), \psi_x(x_3))$ is contained inside a periodic orbit of the Hamiltonian system (2.3). The same is then true if ψ is replaced with ψ^c and $c \approx \hat{c}$. Then $(\psi^c(x), \psi_x(x))$ is "trapped" inside this periodic orbit, by the monotonicity of H, and $(\psi^c(x), \psi^c_x(x))$ has to converge to the equilibrium $(\alpha, 0)$ as $x \to -\infty$. The rest of the properties in (aiii) are clearly robust, due to (2.10) and the positive invariance of Q_3.

It remains to consider the case $(\xi, \eta) \in \tau(\tilde{\phi})$. Shifting $\tilde{\phi}$, we may assume that $(\xi, \eta) = (\tilde{\phi}(0), \tilde{\phi}_x(0))$, so that

$$(\tilde{\phi}(0), \tilde{\phi}_x(0)) = (\psi^c(0), \psi^c_x(0)). \tag{2.11}$$

It is clear that if $c \approx \hat{c}$, then, going forward (that is, increasing x), the trajectory of (ψ^c, ψ^c_x) leaves S and then stays in Q_3, just as the trajectory of $(\tilde{\phi}, \tilde{\phi}_x)$ does. Going backward, one can use comparison of solutions of (2.2) with different values of c (as carried out in [2, Sect. 4], for example), to conclude from (2.11) that if $c \in (\hat{c}, 0)$ then the trajectory of (ψ^c, ψ^c_x) leaves S through the halfline $\{(1, w) : w < 0\}$. Then it stays in Q_4 by the negative invariance. Hence (aii) holds with ψ replaced by ψ^c if $c \in (\hat{c}, 0)$ and $c \approx \hat{c}$. $\square$

The following lemma will be used in a comparison argument below.

Lemma 2.3. *If $\hat{c} < 0$, then there exists numbers $c_n \in (\hat{c}, 0)$, $n = 1, 2, \dots$, and functions $\psi_n \in C(\mathbb{R})$, $n = 1, 2, \dots$, such that for each n, ψ_n is a solution of (2.1) with $c = c_n$, $\psi_n < 1$, $\limsup_{|x| \to \infty} \psi_n(x) < 0$, and, as $n \to \infty$, one has $c_n \to \hat{c}$ and $\max_{x \in \mathbb{R}} \psi_n \to 1$.*

Proof. Take any $(\xi, \eta) \in \tau(\phi)$ with $\xi > \alpha$ and let ψ^c be the solution of (2.1) with $\psi^c(0) = \xi$, $\psi^c_x(0) = \eta$ (at this point, $c \in (\hat{c}, 0)$ is arbitrary). A comparison of solutions of (2.2) with different values of c [2, Sect. 4] shows that there are $x_1 < 0 < x_2$ (depending on c) such that

$$\psi^c(x_1) = 0,\ \psi^c_x(x_1) > 0, \quad \psi^c(x_2) \in (\alpha, 1),\ \psi^c_x(x_2) = 0. \tag{2.12}$$

By the negative invariance of Q_2, $(\psi^c(x), \psi^c_x(x)) \in Q_2$ for $x < x_1$. Now, if c is close to $\hat{c}$, then, by the continuity with respect to initial data, x_2 is large and $\psi^c(x_2)$ is close to 1. Using this and the structure of the level set of H for $F(1) = -\hat{c} > 0$ (cf. Fig. 1), one shows easily that for $x > x_2$, $(\psi^c(x), \psi^c_x(x))$ stays in the lower half plane and eventually enters the positively invariant quadrant Q_3. Then $\psi^c(x_2) \approx 1$ is the maximum of ψ^c, hence $\psi^c < 1$. Taking a sequence $\{c_n\}$ in $(\hat{c}, 0)$ with $c_n \to \hat{c}$ and setting $\psi_n := \psi^{c_n}$, we obtain sequences with the stated properties. $\square$

2.2 Properties of $\Omega(u)$

In this section we consider bounded solutions of the problem

$$u_t = u_{xx} + cu_x + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{2.13}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{2.14}$$

where $f \in C^1(\mathbb{R})$, $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $c \in \mathbb{R}$ (other assumptions are not needed in this section). We define the Ω-limit set of a bounded solution u as in (1.4). We shall denote this set by $\Omega(u)$ or $\Omega(u_0)$, but it is useful to clarify the following. If u is a bounded solution of (1.1), then the function $\tilde{u}(x,t) := u(x+ct,t)$ is a bounded solution of (2.13). Obviously, $\Omega(u) = \Omega(\tilde{u})$ and $\tilde{u}(\cdot,0) \equiv u(\cdot,0)$. In other words, if u_0 is given, then $\Omega(u_0)$ is independent of the choice of c in the problem (2.13), (2.14).

Assume that the solution u of (2.13), (2.14) is bounded. Then, the usual parabolic regularity estimates imply that the derivatives u_t, u_x, u_{xx} are bounded on $\mathbb{R} \times [1,\infty)$ and they are globally α-Hölder on this set for each $\alpha \in (0,1)$. The following results are standard consequences of this regularity property: $\Omega(u_0)$ is a nonempty, compact, connected subset of $L^\infty_{loc}(\mathbb{R})$. Moreover, in (1.4) one can take the convergence in $C^1_{loc}(\mathbb{R})$, and $\Omega(u_0)$ is compact and connected in that space as well. The latter implies that the set

$$\{(\varphi(x), \varphi_x(x)) : \varphi \in \Omega(u_0),\ x \in \mathbb{R}\}$$

is compact and connected in $\mathbb{R}^2$.

We now recall the invariance property of $\Omega(u_0)$. Let $\varphi \in \Omega(u)$, so that $u(x_n + \cdot, t_n) \to \varphi$ for some sequence $\{(x_n, t_n)\}$ with $t_n \to \infty$. Then, passing to a subsequence if necessary, one can show that the sequence $u(x_n + \cdot, t_n + \cdot)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to a function U which is an entire solution of (2.13) (that is, a solution of (2.13) on $\mathbb{R}^2$). Obviously, $U(\cdot,0) = \varphi$.

Finally, we note that $\Omega(u_0)$ is also translation-invariant: with each $\varphi \in \Omega(u_0)$, $\Omega(u_0)$ contains the whole translation group orbit of φ, $\{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}$. This follows directly from the definition of $\Omega(u_0)$.

2.3 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + cv_x + a(x,t)v, \quad x \in \mathbb{R},\ t \in (s,T), \tag{2.15}$$

where $-\infty < s < T \leq \infty$, a is a bounded continuous function on $\mathbb{R} \times [s, T)$, and c is a constant. In the next section we use the following fact, often without notice. If u, $\tilde{u}$ are bounded solutions of the nonlinear equation (2.13), then their difference $v = u - \tilde{u}$ satisfies a linear equation (2.15).

We denote by $z(v(\cdot, t))$ the number, possibly infinite, of the zero points $x \in \mathbb{R}$ of the function $x \to v(x, t)$.

The following intersection-comparison principle holds (see [1, 4]).

Lemma 2.4. *Let $v \in C(\mathbb{R} \times [s, T))$ be a nontrivial solution of* (2.15) (2.15) *on $\mathbb{R} \times (s, T)$. Then the following statements hold true:*

(i) *For each $t \in (s, T)$, all zeros of $v(\cdot, t)$ are isolated.*

(ii) *$t \mapsto z(v(\cdot, t))$ is a monotone nonincreasing function on $[s, T)$ with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s, T)$, the function $v(\cdot, t_0)$ has a multiple zero and $z(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in (s, T)$ with $t_1 < t_0 < t_2$ one has*

$$z(v(\cdot, t_1)) > z(v(\cdot, t_2)). \tag{2.16}$$

If (2.16) holds, we say that $z(v(\cdot, t))$ drops in the interval (t_1, t_2).

Remark 2.5. It is clear that if $z(v(\cdot, s_0)) < \infty$ for some $s_0 \in (s, T)$, then $z(v(\cdot, t))$ can drop at most finitely many times in (s_0, T) and if it is constant on (s_0, T), then $v(\cdot, t)$ has only simple zeros for each $t \in (s_0, T)$.

Corollary 2.6. *Assume that v is a solution of* (2.15) *such that for some $s_0 \in (s, T)$ one has*

$$\liminf_{|x| \to \infty} |v(x, s_0)| > 0. \tag{2.17}$$

Then there is $t_0 > 0$ such that for $t \geq t_0$ the function $v(\cdot, t)$ has only finitely many zeros and all of them are simple.

Proof. Since the zeros of $v(\cdot, s_0)$ are isolated, (2.17) implies that there is only a finite number of them. The conclusion now follows directly from Lemma 2.4 and Remark 2.5. □

The next lemma shows that the property for a solution to have multiple zeros is robust.

Lemma 2.7. *Assume that v is a nontrivial solution of* (2.15) *such that for some $s_0 \in (s, T)$ the function $v(\cdot, s_0)$ has a multiple zero at some x_0: $v(x_0, s_0) = v_x(x_0, s_0) = 0$. Assume further that for some $\delta, \epsilon > 0$, v_n is a sequence in $C^1([x_0 - \delta, x_0 + \delta] \times [s_0 - \epsilon, s_0 + \epsilon])$ which converges in this space to the function v. Then for all sufficiently large n the function $v_n(\cdot, t)$ has a multiple zero in $(x_0 - \delta, x_0 + \delta)$ for some $t \in (s_0 - \epsilon, s_0 + \epsilon)$.*

This can be proved using a version of Lemma 2.4 on a small interval around x_0 and the implicit function theorem, see [5, Lemma 2.6] for details. Note that the v_n are not required to be solutions of any equation.

3 Proofs of Theorems 1.1, 1.3, and Corollary 1.2

Throughout this section we assume hypotheses (Hf), (Hu), (1.6), and (1.9) to be satisfied, and let u be the solution of (1.1), (1.2). Recall that $\hat{c} \leq 0$ is the speed of the traveling front and ϕ is its profile function. Here, we choose the specific translation of the profile function such that $\phi(0) = \alpha$.

If ψ is a nonconstant periodic steady state of (1.1), we denote by $\mathrm{Int}(\tau(\psi))$ the interior of $\tau(\psi)$ (viewing $\tau(\psi)$ as a Jordan curve).

We start with the following estimates.

Lemma 3.1. *One has*

$$\lim_{t\to\infty} (\liminf_{x\to\infty} u(x,t)) = 1, \qquad \lim_{t\to\infty} (\limsup_{x\to\infty} |u_x(x,t)|) = 0. \tag{3.1}$$

$$\lim_{t\to\infty} (\limsup_{x\to-\infty} u(x,t)) = 0, \qquad \lim_{t\to\infty} (\limsup_{x\to-\infty} |u_x(x,t)|) = 0, \tag{3.2}$$

Moreover, if $\hat{c} < 0$ and $c \in (\hat{c}, 0]$ then for any $x_0 \in \mathbb{R}$ one has

$$\inf_{x \geq x_0} u(x + ct, t) \to 1 \text{ as } t \to \infty. \tag{3.3}$$

Proof. We prove (3.1) and omit the proof of (3.2), which is completely analogous. It is sufficient to prove the first relation in (3.1), the second one then follows by standard parabolic regularity estimates for the function $1 - u$ (which solves a linear equation (2.15)). We can always replace u_0 by a nondecreasing function $\tilde{u}_0$, which still satisfies the assumptions of Theorem 1.1 and is such that $\tilde{u}_0 \leq u_0$. By the comparison principle, if we prove the first relation in (3.1) for $\tilde{u}_0$, then it also holds for the original function u_0. We thus proceed assuming that u_0 itself is nondecreasing. Then $u(x,t)$ is nondecreasing in x for each $t \geq 0$. Therefore the limit $\rho(t) := \lim_{y\to\infty} u(y,t)$ exists for each $t \geq 0$. The function ρ is continuous on $[0,\infty)$ and it solves the ODE $\dot{\rho} = f(\rho)$ on $(0,\infty)$ (see, for example, [16, Theorem 5.5.2]). Since $\rho(0) \in (\alpha, 1]$ by assumption, we have $\rho(t) \to 1$, as $t \to \infty$. This completes the proof of (3.1).

We now prove (3.3). Again, without loss of generality, we may assume that u_0 is nondecreasing. Let c_n and ψ_n be as in Lemma 2.3. Given any $\epsilon > 0$, we can choose n such that $c_n \in (\hat{c}, c)$ and $\max \psi_n \in (1-\epsilon, 1)$. Shifting ψ_n, we may assume that $\psi_n(0) = \max \psi_n$. By (3.1) and Lemma 2.3, we can further choose positive constants t_0 and y_0 such that

$$u(x + c_n t_0, t_0) > \psi_n(x - y_0) \quad (x \in \mathbb{R}).$$

Since the functions $u(x + c_n t, t)$ and $\psi_n(x - y_0)$ satisfy the same equation, equation (2.13) with $c = c_n$, the comparison principle gives

$$u(x + c_n t, t) > \psi_n(x - y_0) \quad (x \in \mathbb{R},\ t \geq t_0).$$

Using the monotonicity of $u(\cdot, t)$, we in particular obtain

$$u(x + ct, t) \geq u(y_0 + c_n t) > \psi_n(0) > 1 - \epsilon \quad (x \geq y_0 + (c_n - c)t,\ t \geq t_0).$$

Since $c_n < c$ and ϵ can be taken arbitrarily small, it is clear that (3.3) holds for any x_0. $\square$

Relations (3.1), (3.2), and the definition of $\Omega(u)$ immediately give the following.

Corollary 3.2. *The constant steady states* 0 *and* 1 *are elements of* $\Omega(u)$.

The next lemma comprises the crux of the proof of Theorem 1.1.

Lemma 3.3. *Let $c \in [\hat{c}, 0]$ and let ψ be a solution of* (2.1). *Assume that either one of the statements (ai), (aii) in Lemma 2.1 holds, or $\hat{c} < c < 0$ and statement (aiii) holds. Then there is T such that*

$$\tau(u(\cdot,t)) \cap \tau(\psi) = \emptyset \quad (t \geq T). \tag{3.4}$$

Proof. The proof is by contradiction. We assume that

$$\tau(u(\cdot, t_n)) \cap \tau(\psi) \neq \emptyset \text{ for some sequence } t_n \to \infty. \tag{3.5}$$

First we show that this leads to a contradiction if (ai) holds. If $\psi \equiv \alpha$, then $\tau(\psi) = (\alpha, 0)$ and (3.5) means that $u(\cdot, t_n) - \alpha$ has a multiple zero for $n = 1, 2, \dots$. We know that this is not possible due to Corollary 2.6 and Lemma 3.1. Thus, we can proceed assuming that ψ is a nonconstant periodic solution (and $c = 0$). Let $\rho > 0$ be the minimal period of ψ. According to (3.5), for each n there is $y_n \in [0, \rho)$ such that the function $u(\cdot, t_n) - \psi(\cdot - y_n)$ has a multiple zero, say z_n. Consequently, $x = 0$ is a multiple zero of the function $u(\cdot + z_n, t_n) - \psi(\cdot + z_n - y_n)$. Write $z_n = k_n\rho + \zeta_n$, where $k_n \in \mathbb{Z}$ and $\zeta_n \in [0, \rho)$. We may assume, passing to a subsequence if necessary, that $\zeta_n \to \zeta_0 \in [0, \rho]$ and $y_n \to y_0 \in [0, \rho]$, hence

$$\psi(\cdot + z_n - y_n) = \psi(\cdot + \zeta_n - y_n) \to \psi(\cdot + \zeta_0 - y_0) \text{ in } C^1_b(\mathbb{R}).$$

We may also assume that $u(\cdot + z_n, t_n) \to \varphi$ for some $\varphi \in \Omega(u)$, and $u(\cdot + z_n, \cdot + t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$, where U is an entire solution of (1.1) with $U(\cdot, 0) = \varphi$ (see Sect. 2.2). Clearly, the function $U(\cdot, 0) - \psi(\cdot + \zeta_0 - y_0) = \varphi - \psi(\cdot + \zeta_0 - y_0)$ has a multiple zero at $x = 0$ and $u(\cdot + z_n, \cdot + t_n) - \psi(\cdot + z_n - y_0) \to U - \psi(\cdot - \zeta_0 - y_0)$ in $C^1_{loc}(\mathbb{R}^2)$. Now, $V := U - \psi(\cdot - \zeta_0 - y_0)$ is an entire solution of a linear equation (2.15) (with $c = 0$) and we verify in a moment that $V(\cdot, 0) = \varphi - \psi(\cdot - \zeta_0 - y_0) \not\equiv 0$. Therefore, Lemma 2.7 implies that for each sufficiently large n, the function $u(\cdot + z_n, s + t_n) - \psi(\cdot + z_n - y_0)$ has a multiple zero (near $x = 0$) for some small s. However, by Corollary 2.6 and Lemma 3.1, $u(\cdot, t) - \psi(\cdot - y_0)$ has only simple zeros for all sufficiently large t. Since $t_n + s \to \infty$, we have a desired contradiction.

To verify that $\varphi - \psi(\cdot - \zeta_0 - y_0) \not\equiv 0$, we note that for $t > 0$, the function $u(\cdot, t) - \alpha$ has a finite number of zeros and this number is independent of t if t is large enough (see Corollary (2.6)). On the other hand, as $\psi(\cdot - \zeta_0 - y_0) - \alpha$ has infinitely many simple zeros (see Sect. 2.1), the relations $\psi(\cdot - \zeta_0 - y_0) \equiv \varphi = \lim u(\cdot, t_n)$ would give a contradictory conclusion that $\lim z(u(\cdot, t_n) - \alpha) \to \infty$. This shows that $\psi(\cdot - \zeta_0 - y_0) \equiv \varphi$ cannot hold. The proof under condition (ai) is now complete.

Now assume that (aii) holds. Let $\tilde{u}(x, t) = u(x + ct, t)$, so that $\tilde{u}$ and ψ satisfy the same equation (2.13). Obviously, $\tau(\tilde{u}(\cdot, t)) = \tau(u(\cdot, t))$ for any t, thus (3.5) means that there is $y_n \in \mathbb{R}$ such that

$$\tilde{u}(\cdot, t_n) - \psi(\cdot - y_n) \text{ has a multiple zero } z_n. \tag{3.6}$$

In particular, $\psi(z_n - y_n) = \tilde{u}(z_n, t_n) \in (0,1)$, which implies that $z_n - y_n \in (x_1, x_2)$ (cp. (aii)). We distinguish the following two possibilities regarding the sequence $\{y_n\}$:

(a) $\{y_n\}$ is bounded (b) $\{y_n\}$ is not bounded.

If (a) holds, then $\{z_n\}$ is bounded as well. We now use similar arguments as above for (ai). Passing to subsequences we may assume that for some $y_0, z_0 \in \mathbb{R}$ and $\varphi \in \Omega(u)$, one has $y_n \to y_0$, $z_n \to z_0$, $\tilde{u}(\cdot, t_n) \to \varphi$ in $C^1_{loc}(\mathbb{R})$. Also, we may assume that $\tilde{u}(\cdot, \cdot + t_n) \to \tilde{U}$ in $C^1_{loc}(\mathbb{R}^2)$, where $\tilde{U}$ is an entire solution of (2.13) with $\tilde{U}(\cdot, 0) = \varphi$ (see Sect. 2.2). Clearly, z_0 is a multiple zero of $\tilde{U}(\cdot,0) - \psi(\cdot - y_0)$ and one has $\tilde{u}(\cdot, \cdot + t_n) - \psi(\cdot + y_0) \to U - \psi(\cdot - y_0)$. The function $V := \tilde{U} - \psi(\cdot - \zeta_0 - y_0)$ is a entire solution of a linear equation (2.15) and $V \not\equiv 0$ by (aii) and the fact that $0 \leq \tilde{U} \leq 1$. Lemma 2.7 implies that for each sufficiently large n, the function $\tilde{u}(\cdot, s + t_n) - \psi(\cdot + y_0)$ has a multiple zero for some $s \approx 0$. However, by Corollary (2.6) and (aii), $\tilde{u}(\cdot, t) - \psi(\cdot + y_0)$ has only simple zeros for all sufficiently large t, and we have a contradiction.

Next we consider the possibility (b). For definiteness we assume that, after passing to a subsequence, one has $y_n \to -\infty$; the case $y_n \to \infty$ can be treated in an analogous way. By (2.7), (2.9), there is $\epsilon > 0$ such that $|\psi'(x)| > \epsilon$, whenever $x \in [x_1, x_2]$ and $\psi(x) < \epsilon$. By Lemma 3.1, there are positive constants r and t_0 such that $u(x, t_0) + |u_x(x, t_0)| < \epsilon$ if $x < r$. For $\tilde{u}$ this means that $\tilde{u}(x, t_0) + |\tilde{u}_x(x, t_0)| < \epsilon$ if $x < \tilde{r} := r - ct_0$. Consequently, if n is so large that $x_2 + y_n < \tilde{r}$, then $\tilde{u}(\cdot, t_0) - \psi(\cdot - y_n)$ has a unique zero in the interval $[x_1 + y_n, x_2 + y_n]$. Of course, by (2.7), $\tilde{u}(\cdot, t_0) - \psi(\cdot - y_n)$ has no zero outside this interval, hence $z(\tilde{u}(\cdot, t_0) - \psi(\cdot - y_n)) = 1$. Clearly, by (2.7), $z(\tilde{u}(\cdot, t) - \psi(\cdot - y_n)) \geq 1$ for all t, hence the equality must hold here by the monotonicity of the zero number (see Lemma 2.4). The unique zero of $\tilde{u}(\cdot, t) - \psi(\cdot - y_n)$ has to be simple for all $t > t_0$ (see Remark 2.5). This holds for all sufficiently large n, in particular, we can choose n so that also $t_n > t_0$. We thus have a contradiction to (3.6).

Finally, we assume that $\hat{c} < c < 0$ and (aiii) holds. As above, (3.6) holds with $\tilde{u}(x, t) := u(x + ct, t)$. The possibilities that $\{y_n\}$ is bounded, or $\{y_n\}$ has a subsequence converging to $-\infty$, can be treated similarly as in the case (aii); the only possibility that requires a different consideration is that y_n (replaced by a subsequence) converges to ∞. Assuming that $y_n \to \infty$, choose $\epsilon > 0$ such that $1 - \epsilon > \psi$ everywhere. By (3.3), there is t_0 such that

$$\tilde{u}(x, t) > 1 - \epsilon \quad (x \geq 0,\ t \geq t_0).$$

This implies that if n is large enough, then all zeros of $\tilde{u}(\cdot, t_n) - \psi(\cdot - y_n)$ are located in $(-\infty, 0]$; in particular, $z_n \leq 0$, where z_n is the multiple zero in (3.6). Hence, by (aiii) and the assumption that $y_n \to \infty$,

$$(\tilde{u}(z_n, t_n), \tilde{u}_x(z_n, t_n)) = (\psi(z_n - y_n), \psi_x(z_n - y_n)) \to (\alpha, 0). \tag{3.7}$$

We now take a periodic steady state $\tilde{\psi}$ of (1.1) such that $0 < \tilde{\psi} < 1$ and $(\alpha, 0) \in \mathrm{Int}(\tau(\tilde{\psi}))$ (see Sect. 2.1 and cp. Fig. 1). Then (3.7) implies that for large n the spatial trajectory $\tau(\tilde{u}(\cdot, t_n)) = \tau(u(\cdot, t_n))$ has to intersect $\tau(\tilde{\psi})$ (it cannot be contained entirely in $\mathrm{Int}(\tau(\tilde{\psi}))$ because of Lemma 3.1). Thus we have a contradiction to the result proved above in the case (ai). $\square$

Corollary 3.4. *Let c and ψ be as in Lemma 3.3. Then for any $\varphi \in \Omega(u)$ one has $\tau(\varphi) \cap \tau(\psi) = \emptyset$.*

Proof. Assume this is not true. Then for some y_0 the function $\varphi - \psi(\cdot - y_0)$ has a multiple zero. There is an entire solution U of (2.13) (with the same c as in the statement of the lemma) such that $U(\cdot, 0) = \varphi$ and $u(\cdot + x_n + ct_n, \cdot + t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$ for some sequences $x_n \in \mathbb{R}$, $t_n \to \infty$ (see Sect. 2.2). Then $V := U - \psi(\cdot - y_0)$ is a solution of a linear equation (2.15) and $V \not\equiv 0$, as noted in the proof of Lemma 3.3 (see case (ai) in the proof; if (aii) or (aiii) holds, then $V \not\equiv 0$ is trivial). Thus, using Lemma 2.7 as in the previous proof, we find a sequences $\tilde{t}_n \approx t_n$, $\tilde{x}_n \in \mathbb{R}$, $n = 1, 2, \ldots$, such that $\tilde{t}_n \to \infty$ and $u(\cdot + \tilde{x}_n, \tilde{t}_n) - \psi(\cdot + y_0)$ has a multiple zero for $n = 1, 2, \ldots$. This contradicts (3.4). $\square$

We next consider the set

$$K_\Omega := \cup_{\varphi \in \Omega(u)} \tau(\varphi) = \{(\varphi(x), \varphi_x(x)) : \varphi \in \Omega(u),\, x \in \mathbb{R}\}. \tag{3.8}$$

This is a compact, connected subset of $\mathbb{R}^2$ (cp. Sect. 2.2).

Lemma 3.5. *One has $K_\Omega \subset \Sigma$, where*

$$\Sigma := \begin{cases} \{(0,0), (1,0)\} \cup \tau(\phi) & \textit{if } \hat{c} < 0, \\ \{(0,0), (1,0)\} \cup \tau(\phi) \cup \tau(\tilde{\phi}) & \textit{if } \hat{c} = 0, \end{cases} \tag{3.9}$$

and $\tilde{\phi}$ is defined by $\tilde{\phi}(x) = \phi(-x)$ (as in Sect. 2.1).

Proof. Assume that $K_\Omega \not\subset \Sigma$. Then there are $(\xi, \eta) \in \mathbb{R}^2 \setminus \Sigma$ and $\varphi \in \Omega(u)$ such that $(\varphi(x_0), \varphi_x(x_0)) = (\xi, \eta)$ for some x_0. Obviously, $0 \le \varphi \le 1$ and the existence of an entire solution through φ (see Sect. 2.2) and the comparison principle show that either $\varphi \equiv 0$, or $\varphi \equiv 1$, or else $0 < \varphi < 1$. Since $(\xi, \eta) \notin \{(0,0),(1,0)\}$, the relations $0 < \varphi < 1$ must hold and, in particular, $0 < \xi < 1$. By Lemma 2.1 and Corollary 2.2, there are $c \in [\hat{c}, 0]$ and a solution ψ, such that $(\psi(0), \psi'(0)) = (\xi, \eta)$ and the assumptions of Lemma 3.3 are satisfied. For this ψ, we have $\tau(\varphi) \cap \tau(\psi) \neq \emptyset$, in contradiction to Corollary 3.4. $\square$

Completion of the proof of Theorem 1.1. Let $\hat{c} < 0$. Corollary 3.2 implies that K_Ω contains the points $(0,0)$, $(0,1)$. Therefore, Lemma 3.5 and the connectedness of K_Ω imply that

$$K_\Omega = \{(0,0),(1,0)\} \cup \tau(\phi). \tag{3.10}$$

Take now any $\varphi \in \Omega(u)$. As noted in the proof of Lemma 3.5, if φ is not one of the constant steady states 0, 1, then $0 < \varphi < 1$. In this case, (3.10) implies that $\tau(\varphi) \subset \tau(\phi)$. Since $\phi' > 0$, this means that for each $x \in \mathbb{R}$ there is a unique $\zeta(x)$, such that

$$\varphi(x) = \phi(\zeta(x)), \quad \varphi'(x) = \phi'(\zeta(x)). \tag{3.11}$$

Moreover, $\zeta \in C^1$ by the implicit function theorem. Differentiating the first identity and comparing to the second one, we obtain that $\zeta' \equiv 1$. Thus there is $\xi \in \mathbb{R}$ such that $\varphi \equiv \psi(\cdot - \xi)$. This proves (1.11). $\square$

Proof of Theorem 1.3, Part 1. Assume that $\hat{c} = F(1) = 0$. Also assume the additional hypothesis on u_0, (Ha), to be satisfied. In this part of the proof we show that (1.11) holds.

The arguments from the previous proof apply here, the only difference is that the specific statement of Lemma 3.5 for $\hat{c} = 0$ has to be used. Thus, in place of (3.10), we can only say that one of the following possibilities occurs:

(oi) $\Omega(u) = \{0,1\} \cup \{\phi(\cdot - \xi) : \xi \in \mathbb{R}\}$ (as stated in Theorem 1.3),

(oii) $\tilde{\phi} \in \Omega(u)$.

We just need to rule out (oii); (1.11) then follows from (oi), as in the proof of Theorem 1.1. Assume that $\tilde{\phi} \in \Omega(u)$: there are $x_n \in \mathbb{R}$, $t_n > 0$, $n = 1, 2, \ldots$

such that $t_n \to \infty$ and $u(\cdot + x_n, t_n) \to \tilde{\phi}$. From this and Lemma 3.1 it follows that for all large enough n, the function $u(\cdot, t_n) - \alpha$ has at least three zeros whose mutual distances go to infinity as $n \to \infty$. To contradict this conclusion, we employ hypothesis (Ha).

First we note that the monotonicity of the zero number (see Lemma 2.4) implies that $z(u(\cdot,t)-\alpha) \geq 3$. Thus, if $z(u_0-\alpha) = 1$, we have a contradiction already and we are done. We proceed assuming that the other condition of (Ha) holds. We claim that this condition is preserved at positive times: For each $t > 0$ the limits $u(\pm\infty, t)$ exist and one has

$$u(-\infty, t) < u(x,t) < u(\infty, t) \quad (x \in \mathbb{R},\ t > 0). \tag{3.12}$$

Indeed, the existence of the limits $u_0(\pm)$ implies that the limits $\rho^{\pm}(t) := u(\pm\infty, t)$ exist for all $t \geq 0$ and they satisfy the ODE $\dot{\rho} = f(\rho)$ with the initial conditions $\rho^{\pm}(0) := u_0(\pm\infty)$ (see [16, Theorem 5.5.2]). Relations (1.15) give $\rho^{-}(0) \leq u_0 \leq \rho^{-}(0)$. Of course, none of these inequalities is an identity by (Ha). Relations (3.12) now follow from the strong comparison principle.

By Corollary 2.6, we can choose $t_0 > 0$ such that for $t \geq t_0$, the zeros of $u(\cdot, t) - \alpha$ are all simple, and their number, say k, is finite and independent of t. Let $\zeta_1(t) < \cdots < \zeta_k(t)$ denote the zeros of $u(\cdot,t) - \alpha$ for $t \geq t_0$. Since they are simple, the functions $\zeta_1, \dots \zeta_k$ are C^1 on $[t_0, \infty)$.

Using (3.12), one shows easily that there is a smooth increasing function $\tilde{u}_0$ such that

$$\begin{aligned} u(-\infty, t_0) < \tilde{u}_0(-\infty) < \min_{\zeta_1(t_0) \leq x \leq \zeta_k(t_0)} u(x, t_0), \\ u(\infty, t_0) > \tilde{u}_0(\infty) > \max_{\zeta_1(t_0) \leq x \leq \zeta_k(t_0)} u(x, t_0). \end{aligned} \tag{3.13}$$

Clearly, for such $\tilde{u}_0$, if η is large enough, then

$$u(x, t_0) < \tilde{u}_0(x + \eta) \quad (x \leq \zeta_k(t_0)), \tag{3.14}$$

$$u(x, t_0) > \tilde{u}_0(x - \eta) \quad (x \geq \zeta_1(t_0)). \tag{3.15}$$

Let $\tilde{u}$ be the solution of (1.1) on (t_0, ∞) with the initial condition $\tilde{u}(\cdot, t_0) = \tilde{u}_0$. Then $\tilde{u}(x,t)$ is continuous on $\mathbb{R} \times [t_0, \infty)$ and increasing in x. By (3.13), the relations $\tilde{u}(-\infty, t) < \alpha < \tilde{u}(-\infty, t)$ hold for $t = t_0$, hence they continue to hold for all for all $t \geq t_0$ (see Lemma 3.1). Therefore, for each $t \geq t_0$ the function $\tilde{u}(x,t) - \alpha$ has a unique zero $\xi(t)$ and $t \mapsto \xi(t)$ is continuous on $[t_0, \infty)$.

Consider now the relations

$$\xi(t) - \eta < \zeta_1(t), \quad \zeta_k(t) < \xi(t) + \eta. \tag{3.16}$$

They are both satisfied for $t = t_0$ (use the monotonicity of $u(\cdot, t)$ and the relations (3.14), (3.15), with $x = \zeta_1(t_0)$, $x = \zeta_k(t_0)$, respectively). By continuity, they are also satisfied it $t > t_0$ is sufficiently close to t_0. On the other hand, (3.16) cannot be satisfied for all $t > t_0$ by the properties of the sequence $\{t_n\}$ stated above: $\zeta_k(t_n) - \zeta_1(t_n) \to \infty$. Thus there is $t_1 > t_0$ such that relations (3.16) hold for all $t \in [t_0, t_1)$ and either $\xi(t_1) - \eta = \zeta_1(t_1)$ or $\zeta_k(t_1) = \xi(t_1) + \eta$. Assume that the former holds (the latter can be dealt with in an analogous way). Then

$$\tilde{u}(\zeta_1(t_1) + \eta, t_1) = \tilde{u}(\xi(t_1), t_1) = \alpha = u(\zeta_1(t_1), t_1). \tag{3.17}$$

Since $\xi(t) - \eta$ is the unique zero of the function $\tilde{u}(\cdot + \eta, t) - \alpha$ and $\zeta_k(t) > \zeta_1(t)$, the first relation in (3.16) yields

$$\tilde{u}(\zeta_k(t) + \eta, t) > \alpha = u(\zeta_k(t), t) \quad (t_0 \leq t \leq t_1). \tag{3.18}$$

Using this, (3.14), and the strong comparison principle, we obtain

$$\tilde{u}(x + \eta, t) > u(x, t) \quad (x < \zeta_k(t),\, t_0 \leq t \leq t_1), \tag{3.19}$$

contradicting (3.17).

With this contradiction, the proof of (1.11) is complete. $\square$

Proof of Corollary 1.2 and Proof of Theorem 1.3, Part 2. We ssume that $\operatorname{sign} \hat{c} = -\operatorname{sign} F(1) < 0$ or that $\hat{c} = F(1) = 0$ and the additional assumption (Ha) is satisfied. Under these assumptions, we have already proved that (1.11) holds. Since $\phi' > 0$, this implies in particular that if t is large enough, then $u_x(x, t) > 0$ whenever $u(x, t) = \alpha$. Consequently, for large t there is a unique $\gamma(t)$ such that $u(\gamma(t) + \hat{c}t, t) = \alpha$. Moreover, $\gamma \in C^1$, by the implicit function theorem. Denote $\tilde{u}(x, t) := u(x + \hat{c}t, x)$, so $\tilde{u}$ and ϕ solve the same equation (2.13), with $c = \hat{c}$. Any sequence $t_n \to \infty$ can be replaced by a subsequence such that $\tilde{u}(\cdot + \gamma(t_n), t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$ for some $\varphi \in \Omega(u)$. Necessarily, $\varphi(0) = \alpha$. Therefore, by (1.11) and our choice $\phi(0) = \alpha$, we have $\varphi = \phi$. Since this limit is always the same, we have

$$\tilde{u}(\cdot + \gamma(t), t) \to \phi \text{ as } t \to \infty, \tag{3.20}$$

with the convergence in $L^\infty_{loc}(\mathbb{R})$.

To complete the proof, we need to prove that the convergence takes place in $L^\infty(\mathbb{R})$ and $\gamma'(t) \to 0$ as $t \to \infty$. We start with the latter. Recall, that any sequence $t_n \to \infty$ can be replaced by a subsequence such that $\tilde{u}(\cdot + \gamma(t_n), \cdot + t_n)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to an entire solution U of equation (2.13) with $U(\cdot, 0) = \phi$. Since ϕ is a steady state of (2.13), we have $U \equiv \phi$, by uniqueness and backward uniqueness for (2.13). Thus the convergence in $C^1_{loc}(\mathbb{R}^2)$ yields

$$(\tilde{u}(\cdot + \gamma(t_n), \cdot + t_n), \tilde{u}_x(\cdot + \gamma(t_n), \cdot + t_n), \tilde{u}_t(\cdot + \gamma(t_n), \cdot + t_n) \to (\phi, \phi_x, 0).$$

Since this is true for any sequence $t_n \to \infty$, we have, in particular,

$$(\tilde{u}(\gamma(t), t), \tilde{u}_x(\gamma(t), t), \tilde{u}_t(\gamma(t), t) \to (\alpha, \phi_x(0), 0), \tag{3.21}$$

as $t \to \infty$. Now, differentiating the relation $\tilde{u}(\gamma(t), t) = \alpha$, we obtain $\tilde{u}_x(\gamma(t), t)\gamma'(t) + u_t(\gamma(t), t) = 0$. Since $\phi_x(0) \neq 0$, from (3.21) we conclude that $\gamma'(t) \to 0$ as $t \to \infty$.

It remains to prove that the convergence in (3.20) is uniform on $\mathbb{R}$. Assume it is not. Then there exist $\delta > 0$ and sequences $\{x_n\}$, $\{t_n\}$ such that $|x_n| \to \infty$, $t_n \to \infty$, and

$$|\tilde{u}(x_n + \gamma(t_n), t_n) - \phi(x_n)| > 2\delta. \tag{3.22}$$

Assume for definiteness that $\{x_n\}$ can be replaced by a subsequence so that $x_n \to -\infty$ (the case when $x_n \to \infty$ can be treated similarly). Since $\phi(-\infty) = 0$, (3.22) in particular implies that for all large enough n one has $\tilde{u}(x_n + \gamma(t_n), t_n) > \delta$. On the other hand, using $\phi(-\infty) = 0$ and (3.20), we find x_0 such that $\tilde{u}(x_0 + \gamma(t), t) < \delta$ for all sufficiently large t. These relations imply that if n is sufficiently large, then there is y_n between x_n and x_0, such that

$$\tilde{u}(y_n + \gamma(t_n), t_n) = \delta, \quad \tilde{u}_x(y_n + \gamma(t_n), t_n) \leq 0. \tag{3.23}$$

Take now a subsequence of $\tilde{u}(\cdot + y_n + \gamma(t_n), t_n)$, which converges in $C^1_{loc}(\mathbb{R})$ to some $\varphi \in \Omega(u)$. By (3.23), $\varphi(0) = \delta$, $\varphi'(0) \leq 0$. However, by (1.11), $\varphi = \phi(\cdot - \xi)$ for some ξ, hence $\varphi' > 0$. This contradiction completes the proof. $\square$

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