Locally uniform convergence to an equilibrium for nonlinear parabolic equations on $\mathbb{R}^N$

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Abstract

We consider bounded solutions of the Cauchy problem

$$\begin{cases} u_t - \Delta u = f(u), \ x \in \mathbb{R}^N, \ t > 0, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases}$$

where u_0 is a nonnegative function with compact support and f is a C^1 function on $\mathbb{R}$ with $f(0) = 0$. Assuming that f' is locally Hölder continuous and f satisfies a minor nondegeneracy condition, we prove that, as $t \to \infty$, the solution $u(\cdot, t)$ converges to an equilibrium φ locally uniformly in $\mathbb{R}^N$. Moreover, the limit function φ is either a constant equilibrium, or there is a point $x_0 \in \mathbb{R}^N$ such that φ is radially symmetric and radially decreasing about x_0, and it approaches a constant equilibrium as $|x - x_0| \to \infty$. The nondegeneracy condition

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only concerns a specific set of zeros of f and we make no assumption whatsoever on the nonconstant equilibria. The set of such equilibria can be very complicated and indeed a complete understanding of this set is usually beyond reach in dimension $N \geq 2$. Moreover, due to the symmetries of the equation, there are always continua of such equilibria. Our result shows that the assumption "u_0 has compact support" is powerful enough to guarantee that, first, the equilibria that can possibly be observed in the ω-limit set of u have a rather simple structure and, second, exactly one of them is selected. Our convergence result remains valid if Δu is replaced by a general elliptic operator of the form $\sum_{i,j} a_{ij} u_{x_i x_j}$ with constant coefficients a_{ij}.

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1 Introduction and statement of the main result

In this paper, we study the long-time behavior of bounded positive solutions of the Cauchy problem

$$u_t - \Delta u = f(u), \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$
$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where f is a C^1 function on $\mathbb{R}$ with $f(0) = 0$ and $u_0 \in L^\infty(\mathbb{R}^N)$ is a nonnegative function.

It is well-known that (1.1), (1.2) has a unique solution on the time interval $[0, \delta]$ for some $\delta > 0$. Here the solution refers to the mild solution, see for example [21]. This solution is of class $C^{2,1}$ on $\mathbb{R}^N \times (0, \delta]$, it satisfies (1.1) in the classical sense, and one has

$$\sup_{t\in(0,\delta]} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} < \infty, \quad u(\cdot,t) \to u_0, \text{ as } t \searrow 0, \text{ almost everywhere.}$$

The solution can be extended, as a classical solution, to a maximal time interval $(0, T)$. When referring to a solution, we have always in mind the maximally extended solution. If the solution of (1.1), (1.2) is bounded (meaning that $u \in L^\infty(\mathbb{R}^N \times (0, T))$, then it is global: $[0, T) = [0, \infty)$. We are concerned with the asymptotic behavior of bounded solutions as $t \to \infty$.

For the bounded-domain counterpart of (1.1), the asymptotic behavior of bounded solutions is relatively well understood. Consider, for example, the same equation on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ under Dirichlet boundary condition. Such an equation is gradient-like with respect to the energy functional

$$V(u) = \int_\Omega \left(\frac{|\nabla u(x)|^2}{2} - F(u(x)) \right) dx, \quad F(u) := \int_0^u f(\xi)\, d\xi. \tag{1.3}$$

This is to say that $V(u(\cdot,t))$ is finite and strictly decreasing along any solution with $u_t \not\equiv 0$. As a consequence, each bounded solution approaches a set of equilibria (steady states). In other words, the ω-limit set, $\omega(u)$, of any bounded solution u consists entirely of equilibria. Here

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some } t_n \to \infty\}, \tag{1.4}$$

with the convergence in $L^\infty(\Omega)$. By standard parabolic regularity estimates, the trajectory $\{u(\cdot,t) : t \geq 1\}$ of any bounded solution u is relatively compact in $L^\infty(\Omega)$. Therefore

$$\omega(u) \neq \emptyset \text{ and } \mathrm{dist}_{L^\infty(\Omega)}(u(\cdot,t), \omega(u)) \to 0 \text{ as } t \to \infty. \tag{1.5}$$

This gives a precise meaning to the statement that each bounded solution approaches a set of equilibria. If $N = 1$ or if f is analytic, then each bounded solution even converges to a single equilibrium, see [23, 33], [32], respectively. For a general $f = f(u)$, in dimension $N \geq 2$, the convergence issue remains unresolved; it is known, however, that nonconvergent bounded solutions do exist if f is allowed to depend on x: $f = f(x, u)$, see [24, 25].

For the problem (1.1), (1.2) on $\mathbb{R}^N$, the situation is far more complicated. To discuss the asymptotic behavior of bounded solutions, we shall again employ the ω-limit set. However, to retain the compactness of the trajectory and the properties (1.5), we use the topology of $L^\infty_{loc}(\mathbb{R}^N)$. Thus, in (1.4), we assume the convergence to be uniform on compact sets, but not necessarily uniform on $\mathbb{R}^N$.

With $\Omega = \mathbb{R}^N$, the energy functional (1.3) can still be used, although not for all solutions. To guarantee that the integral is convergent along a solution, one needs extra assumptions on u; at the minimum the decay condition

$$u(x,t) \to 0 \text{ as } |x| \to \infty \tag{1.6}$$

has to be imposed (usually combined with other assumptions). An approach to (1.1), (1.2), (1.6) based on energy estimates combined with a concentration compactness technique was taken up by Feireisl in [10]. Assuming $f'(0) < 0$, he proved that if for a sequence of times $t_n \to \infty$ one has $\sup_n \|u(\cdot,t_n)\|_{L^2(\mathbb{R}^N)} < \infty$, then along a subsequence of t_n the solution $u(\cdot,t)$ converges in $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ to an equilibrium. This result in particular implies that if $\sup_{t>1} \|u(\cdot,t)\|_{L^2(\mathbb{R}^N)} < \infty$, then $\omega(u)$ consists of equilibria.

More is known for solutions u which are *localized* in the sense that

$$u(x,t) \to 0 \text{ as } |x| \to \infty, \text{ uniformly in } t. \tag{1.7}$$

For the time being, let us continue to assume that $f'(0) < 0$. Using energy estimates, Busca et al. [2] proved the convergence to an equilibrium for an arbitrary bounded, positive solution of (1.1), (1.7), provided the decay in (1.7) is exponential. The last condition, which implies in particular that the energy functional is finite along $u(\cdot,t)$, is automatically satisfied if u_0 has compact support. See also [4, 11] for earlier convergence results for more specific nonlinearities. The result of [2] was improved in [14] by removing the requirement that the decay in (1.7) be exponential. The energy functional V still plays an important role in [14], however; while it may be infinite along the solution u itself, it is finite on $\omega(u)$. The latter applies equally well to localized solutions of asymptotically autonomous parabolic equations on $\mathbb{R}^N$ and it was used in [14] to prove the convergence result for such equations (see [3] for a different treatment of asymptotically autonomous equations based on the Lojasiewicz inequality).

For one-dimensional problems, there is an alternative approach to solutions of (1.1), (1.7) based on the intersection comparison techniques (also known as the zero number arguments). Using this approach, the convergence to an equilibrium of localized bounded solutions, not necessarily positive, was proved in [12]. This convergence result, being completely independent of the gradient-like structure, extends to time-periodic parabolic equations on $\mathbb{R}$.

Let us add a word of explanation about the uniform decay condition (1.7). Unlike (1.6), which is automatically satisfied if it is satisfied by $u(\cdot, 0) = u_0$, say if $u_0 \in C_0(\mathbb{R}^N)$, the validity of (1.7) cannot be easily ensured by an explicit condition on u_0. In fact, with $f'(0) < 0$, 0 is an asymptotically stable equilibrium of (1.1) and bounded solutions satisfying (1.7) will typically converge to this equilibrium as $t \to \infty$. Bounded solutions satisfying (1.7) which do not converge to 0 are commonly found as threshold solutions separating the solutions decaying to 0 and the solutions exhibiting a different kind of behavior, such as propagation or blowup in finite time (see [4, 7, 11, 27, 34] and references therein).

The convergence results mentioned above depend crucially on the condition $f'(0) < 0$. For nonlinearities not satisfying $f'(0) < 0$, even the behavior of localized bounded solutions is not well understood. It is known that for multidimensional problems such solutions, even the positive ones, do not necessarily converge to a single equilibrium. Two results to that effect were proved in [29, 30] for equations (1.1) with $N \geq 11$ and $f(u) = u^p$, where p is greater than a certain critical exponent. In [29], a positive, bounded, localized solution slowly oscillating up and down along a simply ordered family of equilibria was found (see also [31] for a related quasiconvergence result). In [30], a solution u of a "whack-a-mole" type was constructed: as time increases, $u(\cdot, t)$ repeatedly develops and diminishes humps at prescribed positions and at prescribed heights. Another result hinting at a possibility of more complicated asymptotic behavior of localized solutions was proved in [13]. In that paper, a spatially localized and radially symmetric, nonnegative homoclinic solution was found for $N \geq 3$ and $f(u) = u^p$, with $p_S < p < p_L$, where

$$p_S := \begin{cases} (N+2)/(N-2) & \text{if } N > 2 \\ \infty & \text{if } N \leq 2 \end{cases}$$

$$p_L := \begin{cases} (N-4)/(N-10) & \text{if } N > 10 \\ \infty & \text{if } N \leq 10 \end{cases}$$

By [28], no such homoclinic solution exists if $1 < p < p_S$, and in particular, for any $p > 1$, if $N = 1$ or $N = 2$. Also, it is not difficult to show, using intersection comparison arguments, that in dimension one, no spatially localized, nonnegative homoclinic solution can exist for any equation (1.1). It is not known whether the homoclinic solution, when it does exist, is contained in $\omega(u)$ for some solution u. In any case, its presence shows that even in studies of localized solutions, the role of the energy functional is much less prominent than for equations on bounded domains.

In this context, a result of Gallay and Sliepčevič [16] is very interesting. Given a bounded solution, not necessarily localized or even satisfying (1.6), they considered the energy integral (1.3) over B_R, the ball of radius R and center at the origin. For a single R, this integral is not decreasing along solutions and is not very useful in itself. However, they found valuable information on the dynamics of (1.1) by analyzing the behavior of this integral and some related quantities (energy flux into B_R and energy dissipation rate inside B_R), as $R \to \infty$. In particular they proved that in dimensions $N = 1$ and $N = 2$, the ω-limit set of any bounded solution u contains an equilibrium (see also [17] for recent improvements and generalizations of these results). Without any additional conditions on u, this result cannot be strengthened so as to say that all elements of $\omega(u)$ are equilibria. A construction of [8] yields a bounded solution $u(x_1, t)$ of an equation (1.1) on $\mathbb{R}$ whose ω-limit set contains nonequilibrium solutions. Of course, we can view u as a solution on $\mathbb{R}^N$, constant in $x_2, \dots, x_N$, thus obtaining an example in any dimension. It is an open problem whether in dimensions 3 and higher, the ω-limit set of any bounded solution u must contain an equilibrium.

In this paper, we focus our attention on positive, bounded solutions of (1.1) with *compact initial support*, that is, we assume that u_0 vanishes outside a compact set. It turns out that, for rather general f, we are able to give a very precise description of the asymptotic behavior of such solutions.

For one-dimensional problems (1.1), (1.2), several authors have addressed the asymptotic behavior of solutions with compact initial support, see [7, 9, 34] and references therein. The most general result can be found in [7]: with no additional assumption on u or f, the locally uniform convergence of u to an equilibrium is proved there (in fact, f does not even have to be of class C^1 for the result; Lipschitz continuity of f and $f(0) = 0$ are sufficient). Intersection comparison arguments and the fact that the equilibria are solutions of an ODE are key ingredients of the proof in [7]. None of these, of course, applies in higher dimensions and neither do energy arguments in general. It is therefore rather surprising that a similar convergence result also holds in any dimension, under just a minor nondegeneracy condition on f. The simplest sufficient condition for our convergence result to hold is the nondegeneracy of all zeros of f: $f'(\xi) \neq 0$ whenever $f(\xi) = 0$. We actually use a much weaker condition which only concerns a specific set of zeros of f, see condition (H) below. This condition does not entirely rule out degenerate zeros of f or even continua of such zeros.

We emphasize that there is no assumption whatsoever on the nonconstant equilibria. There is typically a vast variety of them in dimension $N \geq 2$, including families of ground states at various levels, functions which are periodic in some variables and decay in others [5], solutions which decay along all but finitely many rays emanating from the origin [22], as well as saddle shaped solutions and general multiple-end solutions [19]. Clearly, solutions in low dimensions can be viewed as solutions in higher dimensions by simply adding the missing variables along which the solution does not vary. A complete understanding of the set of nonconstant equilibria is in general still beyond reach in dimension $N \geq 2$, even for very simple nonlinearities f.

In this context, it is particularly interesting to note that the assumption "u_0 has compact support" is powerful enough to guarantee that only those equilibria with simple behavior (constant or radially symmetric and radially decreasing) have a chance to be selected in $\omega(u)$. Moreover, despite the fact that nonconstant equilibria always occur in continua due to the translation invariance of (1.1), exactly one of them is selected.

A simple nondegeneracy condition which is sufficient for our main convergence results is the following one.

(H) $f'(\gamma) \neq 0$ for each $\gamma \in \tilde{\Gamma}$, where

$$\tilde{\Gamma} := \Big\{\gamma \geq 0 : f(\gamma) = 0 \text{ and } \int_0^\gamma f(\eta)\,d\eta \geq \int_0^v f(\eta)\,d\eta \quad (0 \leq v \leq \gamma)\Big\}. \tag{1.8}$$

We use the following conventional, although a bit imprecise, terminology concerning (classical) solutions of

$$\Delta\varphi + f(\varphi) = 0, \quad x \in \mathbb{R}^N. \tag{1.9}$$

For a constant $\gamma \geq 0$, we say that φ is a *ground state of* (1.9) *based at* γ if $\varphi > \gamma$, and $\varphi(x) \to \gamma$ as $|x| \to \infty$. If $f(\gamma) = 0 > f'(\gamma)$, a well-known results of [18] implies that each ground state based at γ is radially symmetric and radially decreasing around some point $x_0 \in \mathbb{R}^N$.

Theorem 1.1. *Assume that $u_0 \in L^\infty(\mathbb{R}^N)$ is a nonnegative function with compact support. If $N \geq 2$, assume also that f' is locally Hölder continuous and* (H) *holds. If the solution u of* (1.1), (1.2) *is bounded, then one has*

$$\lim_{t\to\infty} u(\cdot,t) = \varphi \ \ \text{in } L^\infty_{loc}(\mathbb{R}^N), \tag{1.10}$$

where φ is a steady state of (1.1). *Moreover, there exists $\gamma \in \tilde{\Gamma}$ such that either $\varphi \equiv \gamma$ or φ is a ground state of* (1.9) *based at γ.*

As mentioned above, for $N = 1$, the convergence of u to an equilibrium φ was proved in [7]. It was also proved in [7] that if $\varphi \equiv \gamma$, then none of the following conditions can hold for any $\epsilon > 0$:

$$(a)\ f(s) \leq 0 \ \ (s \in (\gamma-\epsilon, \gamma+\epsilon)) \qquad (b)\ f(s) < 0 \ \ (s \in (\gamma-\epsilon, \gamma)).$$

The contribution of Theorem 1.1 in the one-dimensional case is that it gives a more precise information about the ground states and the constant steady states that can occur as the limit of u. Note, in particular, that there can be ground states at levels γ not contained in $\tilde{\Gamma}$, but these are not approached by any solution with compact initial support.

For $N \geq 2$, we can make the conclusion of Theorem 1.1 more precise. We shall also relax hypothesis (H) somewhat. Assume $f'(0) \neq 0$. We define a subset Γ of $\tilde{\Gamma}$ as follows. We include $\gamma = 0$ in Γ if and only if $f'(0) < 0$. If $\gamma \in \tilde{\Gamma} \setminus \{0\}$, we include γ in Γ if and only if

$$\int_0^\gamma f(\eta)\, d\eta > \int_0^v f(\eta)\, d\eta \quad (0 \leq v < \gamma). \tag{1.11}$$

Thus a zero $\gamma > 0$ of f belongs to Γ if and only if it is the strict maximizer of the function $F(v) = \int_0^v f(\eta)\, d\eta$ in $[0, \gamma]$, whereas it belongs to $\tilde{\Gamma}$ even if it is a nonstrict maximizer.

The following conditions can be used in place of (H).

(Hr1) $f'(0) \neq 0$ and $f'(\gamma) \neq 0$ for each $\gamma \in \Gamma \cap (0, \infty)$.

(Hr2) If $\gamma < \hat{\gamma}$ are two elements of Γ such that

$$(\gamma, \hat{\gamma}) \cap \Gamma = \emptyset \neq (\gamma, \hat{\gamma}) \cap \tilde{\Gamma}, \tag{1.12}$$

then $f'(\tilde{\gamma}) \neq 0$, where $\tilde{\gamma} := \max((\gamma, \hat{\gamma}) \cap \tilde{\Gamma})$.

Compared to (H), in (Hr1), (Hr2) we only require the nondegeneracy of the elements of $\Gamma \cup \{0\}$ and those elements of $\tilde{\Gamma} \setminus \Gamma$ which immediately precede some element of Γ. Note that the maximal element of $(\gamma, \hat{\gamma}) \cap \tilde{\Gamma}$ is well defined, if the set is nonempty, for $\tilde{\Gamma}$ is closed and $\hat{\gamma}$ is isolated in $\tilde{\Gamma}$ by (Hr1).

The hypothesis (Hr1)-(Hr2) is a little cumbersome, but it is weaker than (H) and allows for an even more complicated structure of the steady states of (1.1). An alternative to (Hr2), is the following condition:

(Hr2') If $\gamma < \hat{\gamma}$ are two elements of Γ such that (1.12) holds, then $\beta := \max\{\xi \in (\gamma, \hat{\gamma}) : f(\xi) = 0\}$ is not contained in $\tilde{\Gamma}$ (that is, β is not a maximizer of F in $[0, \beta)$).

Theorem 1.2. *Let $N \geq 2$. Assume that f' is locally Hölder continuous and that condition* (Hr1) *holds together with one of the conditions* (Hr2), (Hr2'). *Let $u_0 \in L^\infty(\mathbb{R}^N)$ be a nonnegative function with compact support. If the solution u of* (1.1), (1.2) *is bounded, then* (1.10) *holds and there exists $\gamma \in \Gamma$ such that either $\varphi \equiv \gamma$ or φ is a ground state of* (1.9) *based at γ.*

Remark 1.3. (i) If $N = 1$, the conclusion of Theorem 1.2 is not valid: it is not difficult to find solutions u for which φ is a ground state for some $\gamma \in \tilde{\Gamma} \setminus \Gamma$ (see Example 4.1).

(ii) Since $f'(\gamma) < 0$ for $\gamma \in \Gamma$, [18] implies that each ground state based at γ is radially symmetric around some point $x_0 \in \mathbb{R}^N$. Thus Theorem 1.2 entails an asymptotic spatial symmetry of bounded positive solutions with compact initial support (see [26] for a general background in asymptotic symmetry for parabolic equations).

(iii) In Theorem 1.2, a stronger conclusion than (1.10) is valid. As we will show in the proof of the theorem (see (3.40), (3.44)), there is a positive constant c such that

$$\sup_{|x| \leq ct} |u(x,t) - \varphi(x)| \to 0, \text{ as } t \to \infty.$$

(iv) If Δ is replaced by a general elliptic operator of the form

$$\mathcal{L}u := \sum_{i,j} a_{ij} u_{x_i x_j}$$

with constant coefficients a_{ij}, $i, j \in \{1, ..., N\}$, then the conclusions in Theorems 1.1 and 1.2 remain valid. Indeed, there is a linear change of variable $x = Ly$ such that $v(y,t) := u(Ly,t)$ satisfies

$$\begin{cases} v_t - \Delta v = f(v) \ \ (y \in \mathbb{R}^N, \ t > 0), \\ v(y,0) = v_0(y) := u_0(Ly) \ \ (y \in \mathbb{R}^N), \end{cases}$$

with v_0 having compact support. Therefore we can apply our results to v and the conclusions for u follow immediately.

(v) The assumption that f' is locally Hölder continuous is assumed for technical reasons only; it significantly simplifies some estimates in the proof of Theorem 1.2 (see the estimates following (3.45) at the end of Section 3.)

Let us give an outline of the proof of Theorem 1.2. A major step consists in proving that there exists $\gamma \in \Gamma$ such that $\liminf_{t\to\infty} u(x,t) \geq \gamma$ for each $x \in \mathbb{R}^N$ and the function u is localized at level γ, meaning that the nonnegative function $\max\{u, \gamma\} - \gamma$ is localized (at level 0). Having done this, we then construct a function $\tilde{u}$, which has the same ω-limit set as u, is also localized at γ, and, moreover, $\tilde{u} \geq \gamma$, a property which u itself does not have, unless $\gamma = 0$. We show that $\tilde{u}$ satisfies a nonautonomous equation $\tilde{u}_t = \Delta\tilde{u} + f(\tilde{u}) + h(x,t)$, with the perturbation term h decaying to zero exponentially in time. Applying to this asymptotically autonomous equation a convergence result of [14], we conclude that the ω-limit set of $\tilde{u}$, hence also of u, is either the constant equilibrium γ or a ground state based at γ.

The rest of the paper is organized as follows. Section 2 contains preliminary material. First we recall a reflection argument which implies a monotonicity property outside large balls for solutions with compact initial support. Then we examine the structure of solutions of $v_{rr} + f(v) = 0$. At several places below, solutions of this ODE and its perturbations are used in upper and lower estimates of solutions of (1.1). The last and most technical part of Section 2 is devoted to lower estimates of the solution of (1.1), (1.2). These estimates facilitate a construction of a suitable solution $\tilde{u}$ of the asymptotically autonomous equation in the above outline. In Section 3 we complete the proof of Theorem 1.2. The proof of Theorem 1.1 for $N = 1$ and an example justifying Remark 1.3(i) are given in Section 4.

Throughout the paper, the following are standing hypotheses:

(SH) $f \in C^1(\mathbb{R})$, $f(0) = 0$, $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$, and $\operatorname{spt} u_0$ is compact and nonempty.

Here $\operatorname{spt} u_0$ stands for the support of u_0 defined as the minimal closed set $A \subset \mathbb{R}^N$ such that $u_0 = 0$ a.e. in $\mathbb{R}^N \setminus A$. Modifying u_0 on a set of measure zero, if necessary, we will assume that

$$u_0 \equiv 0 \text{ on } \mathbb{R}^N \setminus \operatorname{spt} u_0.$$

The hypothesis of Theorem 1.2 will be assumed in Section 3 only.

2 Preliminaries

2.1 The monotonicity property

In this section, we assume that u is the solution of (1.1), (1.2), and that it is bounded. We derive some consequences of the compactness of $\operatorname{spt} u_0$.

Lemma 2.1. (i) *Assume that y is a point in $\mathbb{R}^N$, which is separated from $\operatorname{spt} u_0$ by a hyperplane H. Let P denote the reflection about H. Then*

$$u(Py, t) > u(y, t) \quad (t > 0). \tag{2.1}$$

(ii) *Let B_0 be the minimal ball centered at the origin such that $\operatorname{spt} u_0 \subset B_0$. Then*

$$u_r(x, t) := \nabla u(x, t) \cdot x/|x| < 0 \quad (x \in \mathbb{R}^N \setminus B_0,\ t > 0).$$

These properties are established by a well-known reflection argument. The function $u(Px, t) - u(x, t)$ solves a linear parabolic equation, it vanishes on H, and its initial value is a nonnegative and nonzero function in the connected component of $\mathbb{R}^N \setminus H$ not containing $\operatorname{spt} u_0$. Statement (i) follows directly from this and the maximum principle. Statement (ii) follows from the Hopf boundary principle applied to the same linear problem with any hyperplane H not intersecting B_0 (for more details see [2, Appendix], for example).

Lemma 2.2. *Given any $\theta > 0$, assume that there is a sequence $(y_j, t_j) \in \mathbb{R}^N \times (0, \infty)$ such that $|y_j| \to \infty$ and $u(y_j, t_j) \geq \theta$, $j = 1, 2, \ldots$ Then for each ball $B \subset \mathbb{R}^N$, there is $\ell = \ell(B)$ such that*

$$u(x, t_j) > \theta \quad (x \in \bar{B},\ j \geq \ell). \tag{2.2}$$

Proof. Clearly, it is sufficient to prove the conclusion for all sufficiently large balls. Thus let $B \subset \mathbb{R}^N$ be any ball which contains $\operatorname{spt} u_0$. Suppose that the conclusion is false. Then there are sequences $x_n \in \bar{B}$ and $j_n \in \mathbb{N}$ such that $j_n \to \infty$ and $u(x_n, t_{j_n}) \leq \theta$, $n = 1, 2, \ldots$ Let H_n be the hyperplane which contains the point $(y_{j_n} + x_n)/2$ and is perpendicular to the vector $y_{j_n} - x_n$. Since $|y_{j_n}| \to \infty$, for all large n the hyperplane H_n separates y_{j_n} from $\operatorname{spt} u_0$. Denoting by P_n the reflection about H_n, we obtain from Lemma 2.1 that for all large n

$$u(x_n, t) = u(P_n(y_{j_n}), t) > u(y_{j_n}, t) \quad (t > 0).$$

In particular, taking $t = t_{j_n}$, we get a contradiction: $\theta \geq u(x_n, t_{j_n}) > u(y_{j_n}, t_{j_n}) \geq \theta$. □

2.2 A phase plane analysis

In this section, we are concerned with solutions of the ODE

$$v_{rr} + f(v) = 0. \tag{2.3}$$

The results below follow from a standard and elementary phase analysis. Therefore we just give brief proofs.

Consider the first-order system

$$v_r = w, \quad w_r = -f(v) \tag{2.4}$$

associated with (2.3). It is a Hamiltonian system with respect to the energy

$$H(v, w) := w^2/2 + F(v),$$

where F is as in (1.3): $F(v) = \int_0^v f(\xi)\, d\xi$. Thus the trajectories of (2.4) are contained in the level sets of H. Note that these level sets are symmetric about the v axis.

Lemma 2.3. *Let $\hat{\gamma} > \gamma$ be two nonnegative zeros of f.*

(i) *If $\beta^* \in (\gamma, \hat{\gamma})$ is such that*

$$F(\beta^*) \geq F(v) \quad (v \in (\gamma, \beta^*)), \text{ and} \tag{2.5}$$

$$f(v) > 0 \quad (v \in (\beta^*, \hat{\gamma})), \tag{2.6}$$

then for each $\theta \in (\beta^*, \hat{\gamma})$ there is r_θ such that the solution of (2.3) with $v(0) = \theta$, $v_r(0) = 0$ satisfies $v(r_\theta) = \gamma$ and $v_r < 0$ on $(0, r_\theta]$.

(ii) *If* $\beta^* \in (\gamma, \hat{\gamma})$ *is such that* (2.5) *holds with the strict inequality,* $F(\beta^*) = F(\gamma)$*, and* $f(\beta^*) \neq 0$*, then the solution of* (2.3) *with* $v(0) = \beta^*$*,* $v_r(0) = 0$ *is defined for all* $r \in \mathbb{R}$*,* $v_r < 0$ *on* $(0, \infty)$*, and* $v(r) \to \gamma$*, as* $r \to \infty$*. If* $f'(\gamma) \neq 0$*, then the convergence* $v(r) \to \gamma$ *is exponential.*

(iii) *If* $F(\hat{\gamma}) = F(\gamma)$ *and*

$$F(v) < F(\gamma) \quad (v \in (\gamma, \hat{\gamma})), \tag{2.7}$$

then there is a solution v *of* (2.3) *such that* $v(r) \to \gamma$*, as* $r \to \infty$*, and* $v(r) \to \hat{\gamma}$*, as* $r \to -\infty$*. If* $f'(\gamma) \neq 0$*, then the convergence* $v(r) \to \gamma$ *is exponential.*

Figure 1 illustrates the assumptions in (i)-(iii) and the trajectories of system (2.4).

Proof of Lemma 2.3. Under the conditions in (i), the connected component of the level set

$$\{(v, w) : v \geq \gamma,\ H(v, w) = F(\theta)\}$$

containing the point $(\theta, 0)$ intersects the vertical line $v = \gamma$ at two points $(\gamma, \pm w_\theta)$ with $w_\theta \neq 0$. Moreover, it intersects the v-axis at $(\theta, 0)$ only, so it does not contain any equilibrium of (2.4). This level set contains the trajectory (v, v_r) of the solution with $v(0) = \theta$, $v_r(0) = 0$, from which statement (i) follows.

Under the conditions in (ii), there is a homoclinic orbit to the equilibrium $(\gamma, 0)$ which coincides with a connected component of the level set $\{(v, w) : H(v, w) = F(\gamma)\}$ in $\{(v, w) : v > \gamma\}$. The point $(\beta^*, 0)$ is where the homoclinic orbit intersects the v-axis. Choosing a time parameterization such that the intersection occurs at $r = 0$, we obtain the solution v as in the conclusion of the lemma. If $f'(\gamma) \neq 0$, then the equilibrium $(\gamma, 0)$ is hyperbolic, which implies that the convergence $v(r) \to \gamma$ is exponential.

Now assume that the condition in (iii) hold. This time, a connected component of the level set $\{(v, w) : H(v, w) = F(\gamma)\}$ in $\{(v, w) : v > \gamma\}$ intersects the v-axis at $(\hat{\gamma}, 0)$ and it is composed of $(\hat{\gamma}, 0)$ and two heteroclinic orbits between the equilibria $(\gamma, 0)$ and $(\hat{\gamma}, 0)$. The heteroclinic orbit from $(\hat{\gamma}, 0)$ to $(\gamma, 0)$ corresponds to a solution in the conclusion of (iii) (which is unique, up to translations). Again, if $f'(\gamma) \neq 0$, then the hyperbolicity of $(\gamma, 0)$ implies that the convergence $v(r) \to \gamma$ is exponential. □

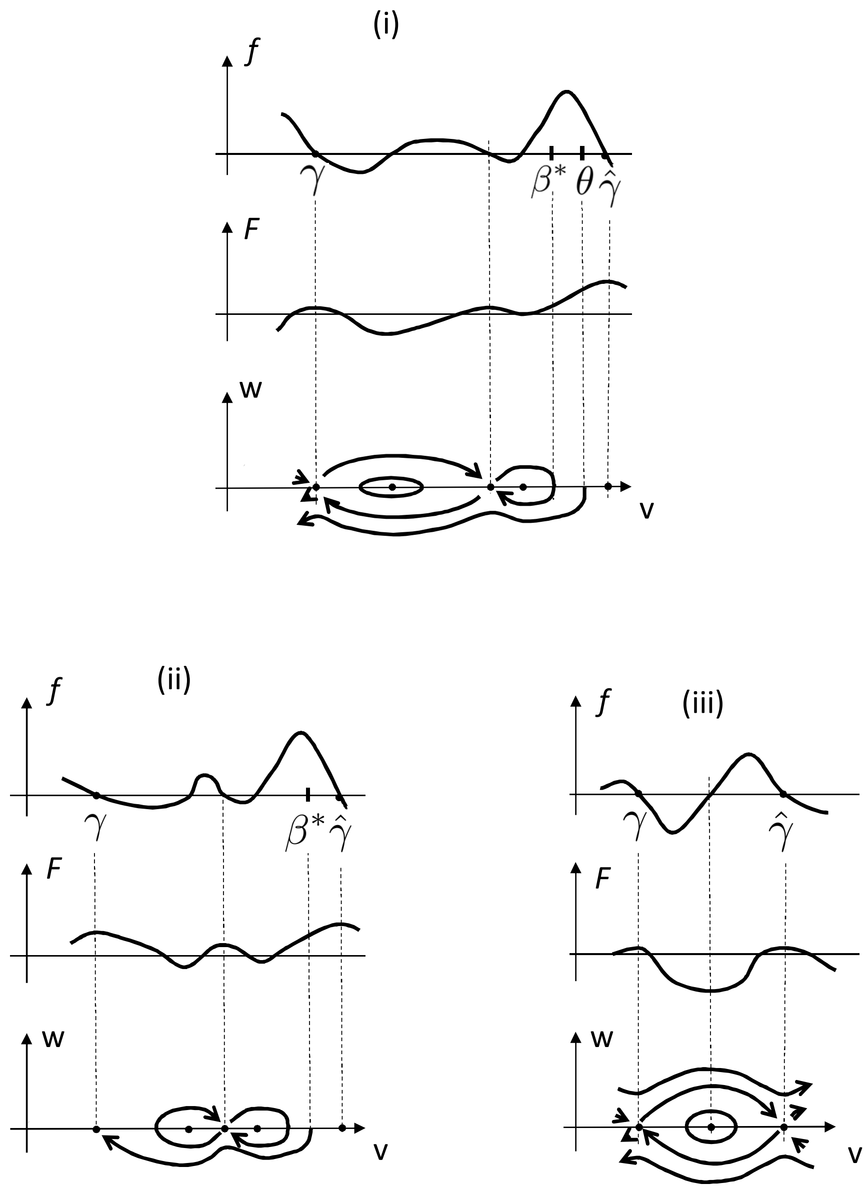

Figure 1: The graphs of the functions f and F as in statements (i)–(iii) of Lemma 2.3. The bottom figures indicate trajectories of the corresponding systems (2.4).

2.3 Lower estimates

In this section we prove useful lower estimates on the solution of (1.1), (1.2). The following lemma will be used in comparison arguments (F is as in (1.3)).

Lemma 2.4. *Assume that $\hat{\gamma} > \beta \geq 0$ are zeros of f such that*

$$f(v) > 0 \quad (v \in (\beta, \hat{\gamma})), \tag{2.8}$$

$$F(v) < F(\hat{\gamma}) \quad (v \in [0, \hat{\gamma})). \tag{2.9}$$

Given any $\theta \in (\beta, \hat{\gamma})$, there exists a constant $R = R(\theta) > 0$ such that the solution of (1.1) *with*

$$u(x,0) = \begin{cases} \theta \text{ for } |x| \leq R, \\ 0 \text{ for } |x| > R, \end{cases} \tag{2.10}$$

has the following property. For each $\epsilon > 0$ there exist positive constants $\tilde{T}$ and $\tilde{c}$ such that

$$u(x,t) \geq \hat{\gamma} - \epsilon \quad (t \geq \tilde{T},\ |x| \leq \tilde{c}\,t). \tag{2.11}$$

In particular,

$$\lim_{t\to\infty} u(\cdot,t) \to \hat{\gamma} \ \text{ in } L^\infty_{loc}(\mathbb{R}^N). \tag{2.12}$$

Proof. We use similar arguments as in [27, Proof of Lemma 3.5 and Addendum]. By (2.9), (2.8), there is a unique $\beta^* \in [\beta, \hat{\gamma})$ with $F(\beta^*) = \max\{F(v) : v \in [0,\beta]\}$. By (2.8), for this β^* we also have $F(v) \leq F(\beta^*)$ for each $v \in [0, \beta^*]$. Hence the hypotheses of Lemma 2.3(i) are satisfied with $\gamma = 0$.

We now claim that for each $\xi \in (\beta^*, \hat{\gamma})$, there is $c > 0$ such that the solution of

$$v'' + cv' + f(v) = 0, \quad v(0) = \xi,\ v'(0) = 0 \tag{2.13}$$

satisfies, for some $r_0 > 0$, the relations $v' < 0$ in $(0, r_0]$ and $v(r_0) = 0$. Indeed, for $c = 0$ this is guaranteed by Lemma 2.3(i) and a continuity argument then gives the result for each sufficiently small $c > 0$. Choose such a c, and denote the solution of (2.13) by q^ξ. Next pick $\rho > (N-1)/c$ and $c_1 \in (0, c - (N-1)/\rho)$, and define a function U_0^ξ by

$$U_0^\xi(r) := \begin{cases} \xi & \text{for } r \leq \rho, \\ q^\xi(r-\rho) & \text{for } \rho < r \leq \rho + r_0, \\ 0 & \text{for } r > \rho + r_0, \end{cases}$$

Let U^ξ be the solution of (1.1) with $U^\xi(x,0) = U_0^\xi(|x|)$. As shown in [1, Lemma 5.1, p. 64], U^ξ has the following two properties:

$$\lim_{t\to\infty} U^\xi(\cdot,t) \to \hat{\gamma} \ \text{ in } L^\infty_{loc}(\mathbb{R}^N), \tag{2.14}$$

$$U^\xi(x,t) > \xi \quad (t > 0,\ |x| \leq \rho + c_1 t). \tag{2.15}$$

Suppose for a moment that we can prove that the solution of (1.1), (2.10) satisfies (2.12). We show that (2.11) is then true as well. Fix any $\epsilon > 0$ and choose $\xi \in (\beta^*, \hat{\gamma})$ with $\xi > \hat{\gamma} - \epsilon$. From (2.12) it follows that there is $\hat{t} > 0$ such that $u(\cdot, \hat{t}) \geq U_0^\xi$. Then the comparison principle and (2.15) give

$$u(x, \hat{t} + t) \geq U^\xi(x, t) > \xi > \hat{\gamma} - \epsilon \quad (t > 0,\ |x| \leq \rho + c_1 t).$$

Consequently,

$$u(x, t) > \hat{\gamma} - \epsilon \quad (t \geq \hat{t},\ |x| \leq \rho + c_1(t - \hat{t})).$$

Therefore, for each $\tilde{c} \in (0, c_1)$, we can find $\tilde{T} > \hat{t}$ such that (2.11) holds.

It remains to prove that given any $\theta \in (\beta, \hat{\gamma})$, the solution of (1.1), (2.10) satisfies (2.12), provided R is sufficiently large. For that aim, we pick $\xi \in (\beta^*, \hat{\gamma})$, choose c, r_0, and ρ as above, and define the corresponding solution U^ξ. Set $R_0 := \rho + r_0$. It is sufficient to prove that for some t_0 one has

$$u(x, t_0) > \xi \quad (|x| \leq R_0). \tag{2.16}$$

Indeed, we then have $u(\cdot, t_0) \geq U_0^\xi$ and (2.12) follows from (2.14) via a comparison argument.

To prove (2.16), let us first consider the solution $\bar{u}$ of (1.1) with initial condition identical to θ. Obviously, $\bar{u}$ coincides with the solution of the ODE $\bar{u}_t = f(\bar{u})$ with $\bar{u}(0) = \theta$. Since $f > 0$ in $(\beta, \hat{\gamma})$, there exists $t_0 > 0$ such that $\bar{u}(t_0) > \xi$. We now claim that the solution u^R of (1.1), (2.10) satisfies $u^R(\cdot, t_0) \to \bar{u}(t_0)$, as $R \to \infty$, uniformly on each ball; in particular, (2.16) holds for all large enough R. The claim here is a consequence of the continuity of the solutions of (1.1) with respect to their initial data. More precisely, consider the space $\mathcal{B}$ of all continuous functions on $\mathbb{R}^N$ taking values in $[0, \hat{\gamma}]$. We equip $\mathcal{B}$ with the metric given by the weighted sup-norm

$$\|v\|_\rho \equiv \sup_{x \in \mathbb{R}} \rho(x)|v(x)|, \tag{2.17}$$

where $\rho(x) := 1/(1 + |x|^2)$. Then, given any $T > 0$ and any two solutions u, $\tilde{u}$ of (1.1) with $u(\cdot, 0), \tilde{u}(\cdot, 0) \in \mathcal{B}$, one has

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_\rho \leq L(T)\|u(\cdot, 0) - \tilde{u}(\cdot, 0)\|_\rho \quad (t \in [0, T]), \tag{2.18}$$

where $L(T)$ is a constant depending on T but not on the solutions. This continuity result, which clearly implies the claim, is proved easily by considering the linear parabolic equation satisfied by $w(x,t) := \rho(x)(u(x,t)-\tilde{u}(x,t))$. As one verifies by a simple computation, the linear equation has bounded coefficients, hence (2.18) follows by standard parabolic estimates. For $N=1$, the details can be found in [12, Proof of Lemma 6.2] and a similar computation applies in any dimension. □

Lemma 2.4 and a comparison argument immediately give the following result.

Corollary 2.5. *Under the hypotheses of Lemma 2.4, given any $\theta \in (\beta, \hat{\gamma})$, there exists a constant $R = R(\theta) > 0$ with the following property. If the solution of* (1.1), (1.2) *is global and for some $t_1 \geq 0$ one has*

$$u(x,t_1) \geq \theta \quad (|x| \leq R(\theta)), \tag{2.19}$$

then for each $\epsilon > 0$ there exist positive constants $\tilde{T}$ and $\tilde{c}$ such that

$$u(x,t) \geq \hat{\gamma} - \epsilon \quad (t \geq \tilde{T},\ |x| \leq \tilde{c}\, t); \tag{2.20}$$

in particular, $\liminf_{t\to\infty} u(x,t) \geq \hat{\gamma}$, uniformly for x in compact sets.

If $f'(\hat{\gamma}) < 0$, we can say more:

Lemma 2.6. *Assume that the hypotheses of Lemma 2.4 are satisfied and moreover $f'(\hat{\gamma}) < 0$. Given $\theta \in (\beta, \hat{\gamma})$, let $R(\theta) > 0$ be as in Corollary 2.5. Assume that the solution u of* (1.1), (1.2) *is global and* (2.19) *is satisfied for some $t_1 \geq 0$. Then there exist positive constants M, c_0, σ, and t_0 such that*

$$u(x,t) \geq \hat{\gamma} - Me^{-\sigma t} \quad (t \geq t_0,\ |x| \leq c_0 t). \tag{2.21}$$

Proof. Since $f'(\hat{\gamma}) < 0$, we can find positive constants ϵ and δ such that

$$f(u) \geq \delta(\hat{\gamma} - u) \quad (u \in [\hat{\gamma} - \epsilon, \hat{\gamma}]).$$

With $R > 0$ to be chosen later, we now consider the auxiliary problem

$$\begin{array}{ll} \psi_t - \Delta\psi = \delta(\hat{\gamma} - \psi), & t > 0,\ |x| < R, \\ \psi = \hat{\gamma} - \epsilon, & t > 0,\ |x| = R, \\ \psi = \hat{\gamma} - \epsilon, & t = 0,\ |x| < R. \end{array}$$

By the comparison principle, the unique solution of this problem, denoted by ψ_R, satisfies,

$$\hat{\gamma} - \epsilon \leq \psi_R(x,t) \leq \hat{\gamma} \quad (t > 0,\ |x| < R).$$

Therefore $\delta(\hat{\gamma} - \psi_R) \leq f(\psi_R)$ and we can use (2.20) and the comparison principle to conclude that

$$u(x, t + \frac{R}{\tilde{c}}) \geq \psi_R(x,t) \quad (|x| < R,\ t > 0), \tag{2.22}$$

provided that

$$R \geq \tilde{c}\tilde{T}.$$

We next estimate ψ_R. To this end, we introduce the function

$$\Psi = \Psi_R = e^{\delta t}(\psi_R - \hat{\gamma} + \epsilon). \tag{2.23}$$

A simple calculation shows that Ψ satisfies

$$\begin{cases} \Psi_t - \Delta\Psi = \epsilon\delta e^{\delta t}, & t > 0,\ |x| < R, \\ \Psi = 0, & t > 0,\ |x| = R, \\ \Psi = 0, & t = 0,\ |x| < R. \end{cases} \tag{2.24}$$

For any $T \geq \tilde{T}$, we take $R = \tilde{c}T$ in (2.24) and compare its unique solution $\Psi_{\tilde{c}T}$ with $\tilde{\Psi}_{cT}$ which stands for the unique solution of the problem

$$\begin{cases} \tilde{\Psi}_t - \Delta\tilde{\Psi} = \epsilon\delta e^{\delta t}, & (x,t) \in Q_{cT} \times (0,\infty), \\ \tilde{\Psi} = 0, & (x,t) \in \partial Q_{cT} \times (0,\infty), \\ \tilde{\Psi} = 0, & (x,t) \in Q_{cT} \times \{0\}, \end{cases} \tag{2.25}$$

where

$$c = \tilde{c}/\sqrt{N}, \quad Q_{cT} := \{x \in \mathbb{R}^N : |x_i| < cT \text{ for } i = 1, ..., N\}.$$

Clearly $Q_{cT} \subset B_{\tilde{c}T}(0)$. Therefore, by the comparison principle,

$$\Psi_{\tilde{c}T}(x,t) \geq \tilde{\Psi}_{cT}(x,t) \quad ((x,t) \in Q_{cT} \times [0,\infty)).$$

The Green function associated with (2.25) (see page 84 of [15]) is given by

$$G(x,t;\xi,\tau) = \prod_{i=1}^{N} G_1(x_i, t; \xi_i, \tau),$$

where G_1 is the Green function of (2.25) in one space dimension:

$$G_1(x_i, t; \xi_i, \tau) = \sum_{n\in\mathbb{Z}} (-1)^n \frac{1}{\sqrt{4\pi(t-\tau)}} \exp\left[-\frac{(x_i - \xi_i - 2ncT)^2}{4(t-\tau)}\right].$$

Thus we can write

$$\tilde{\Psi}_{cT}(x,t) = \int_0^t \epsilon\delta e^{\delta(t-\tau)} \left(\prod_{i=1}^{N} \int_{-cT}^{cT} G_1(x_i,t;\xi_i,\tau)d\xi_i \right) d\tau.$$

We choose $\eta \in (0,1)$, as specified below, and consider points $(x,t) \in \mathbb{R}^{N+1}$ satisfying

$$|x_i| < (1-\eta)cT \ \ (i=1,...,N), \quad 0 \le t \le \frac{\eta^2 c^2}{4} T. \tag{2.26}$$

We claim that with $T_1 := \max\{1, \tilde{T}\}$, for $T \ge T_1$, $\tau \in (0,t)$, and (x,t) satisfying (2.26), one has

$$\int_{-cT}^{cT} G_1(x_i,t;\xi_i,\tau)d\xi_i \ge 1 - \frac{4}{\sqrt{\pi}} e^{-T/2} > 0. \tag{2.27}$$

This estimate has been shown in the proof of Lemma 6.5 in [6]. For completeness, we repeat the argument here.

Set

$$G_0(r,t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t}}.$$

Then

$$\begin{aligned} G_1(x_i,t;\xi_i,\tau) &\ge G_0(x_i-\xi_i,t-\tau) \\ &\quad - G_0(x_i-\xi_i-2cT,t-\tau) - G_0(x_i-\xi_i+2cT,t-\tau). \end{aligned}$$

For $T \ge T_1$, $\tau \in (0,t)$, and (x,t) satisfying (2.26), we have

$$\frac{cT \pm x_i}{2\sqrt{t-\tau}} \ge \frac{\eta cT}{2\sqrt{t-\tau}} \ge \frac{\eta cT}{2\sqrt{t}} = \sqrt{T} \cdot \frac{\eta c\sqrt{T}}{2\sqrt{t}} \ge \sqrt{T} \ge 1. \tag{2.28}$$

Also,

$$\begin{aligned} &\int_{-cT}^{cT} G_0(x_i-\xi_i,t-\tau)d\xi_i \\ &\qquad = \left(\int_{-\infty}^{\infty} - \int_{-\infty}^{-cT} - \int_{cT}^{\infty} \right) G_0(x_i-\xi_i,t-\tau)d\xi_i = 1 - I_1 - I_2 \end{aligned}$$

where

$$I_1 := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-\frac{cT+x_i}{2\sqrt{t-\tau}}} e^{-r^2} dr, \ I_2 := \frac{1}{\sqrt{\pi}} \int_{\frac{cT-x_i}{2\sqrt{t-\tau}}}^{\infty} e^{-r^2} dr.$$

Using the elementary inequality

$$\int_y^\infty e^{-r^2} dr \le \int_y^\infty r e^{-r^2/2} dr = e^{-y^2/2} \text{ for all } y \ge 1,$$

$(cT \pm x_i) \ge \eta cT$, and (2.28), we deduce

$$I_1, \ I_2 \le \frac{1}{\sqrt{\pi}} e^{-\frac{(\eta cT)^2}{8(t-\tau)}}.$$

Relations (2.28) also give

$$\frac{(\eta cT)^2}{8(t-\tau)} \ge T/2.$$

Thus

$$I_1, \ I_2 \le \frac{1}{\sqrt{\pi}} e^{-T/2},$$

and

$$\int_{-cT}^{cT} G_0(x_i - \xi_i, t-\tau) d\xi_i \ge 1 - \frac{2}{\sqrt{\pi}} e^{-T/2}.$$

Similarly,

$$\begin{aligned}
\int_{-cT}^{cT} G_0(x_i - \xi_i - 2cT, t-\tau) d\xi_i &= \frac{1}{\sqrt{\pi}} \int_{-cT}^{cT} \frac{1}{2\sqrt{t-\tau}} e^{-\frac{(x_i-\xi_i-2cT)^2}{4(t-\tau)}} d\xi_i \\
&\le \frac{1}{\sqrt{\pi}} \int_{-cT}^{\infty} \frac{1}{2\sqrt{t-\tau}} e^{-\frac{(x_i-\xi_i-2cT)^2}{4(t-\tau)}} d\xi_i \\
&= \frac{1}{\sqrt{\pi}} \int_{\frac{cT-x_i}{2\sqrt{t-\tau}}}^{\infty} e^{-r^2} dr \le \frac{1}{\sqrt{\pi}} e^{-T/2},
\end{aligned}$$

and

$$\int_{-cT}^{cT} G_0(x_i - \xi_i + 2cT, t-\tau) d\xi_i \le \frac{1}{\sqrt{\pi}} e^{-T/2}.$$

Consequently, for $T \ge T_1$, $\tau \in (0,t)$, and (x,t) satisfying (2.26), we have

$$\int_{-cT}^{cT} G_1(x_i, t; \xi_i, \tau) d\xi_i \ge 1 - \frac{4}{\sqrt{\pi}} e^{-T/2},$$

as claimed.

It follows that, for all large T, say $T \geq T_2$ ($\geq T_1$), there exists $M_0 > 0$ such that

$$\prod_{i=1}^{N} \int_{-cT}^{cT} G_1(x_i, t; \xi_i, \tau) d\xi_i \geq (1 - \frac{4}{\sqrt{\pi}} e^{-T/2})^N \geq 1 - M_0 e^{-T/2}$$

provided that (2.26) holds and $0 \leq \tau < t$. We thus obtain

$$\tilde{\Psi}_{cT}(x,t) \geq \epsilon\delta \int_0^t e^{\delta(t-\tau)}(1 - M_0 e^{-T/2}) d\tau = \epsilon(1 - M_0 e^{-T/2})(e^{\delta t} - 1)$$

for $T \geq T_2$ and (x,t) satisfying (2.26). Therefore, for such T and (x,t),

$$\begin{aligned} \psi_{\tilde{c}T}(x,t) - \hat{\gamma} \quad &= e^{-\delta t} \Psi_{\tilde{c}T}(x,t) - \epsilon \\ &\geq e^{-\delta t} \tilde{\Psi}_{cT}(x,t) - \epsilon \\ &\geq \epsilon(1 - M_0 e^{-T/2})(1 - e^{-\delta t}) - \epsilon \\ &\geq \epsilon[-M_0 e^{-T/2} - e^{-\delta t}]. \end{aligned}$$

It follows, recalling (2.22), (2.23), that

$$u(x, t+T) \geq \psi_{\tilde{c}T}(x,t) \geq \hat{\gamma} - \epsilon(M_0 e^{-T/2} + e^{-\delta t})$$

for $|x| \leq (1-\eta)cT$ and $0 \leq t \leq \frac{\eta^2 c^2}{4} T$, with $c = \tilde{c}/\sqrt{N}$. Taking $t = \frac{\eta^2 c^2}{4} T$, we obtain

$$u(x, (1 + \frac{\eta^2 c^2}{4})T) \geq \hat{\gamma} - \epsilon(M_0 + 1) e^{-\delta \frac{\eta^2 c^2}{4} T}$$

for $|x| \leq (1-\eta)cT$, $T \geq T_2$, provided that $\eta \in (0,1)$ is chosen such that $\delta \frac{\eta^2 c^2}{4} \leq \frac{1}{2}$.

We now fix η with the above property and write $t = (1 + \frac{\eta^2 c^2}{4})T$, $M = \epsilon(M_0 + 1)$, $\sigma = \delta \frac{\eta^2 c^2}{4} (1 + \frac{\eta^2 c^2}{4})^{-1}$ and $c_0 = (1-\eta)c\,(1 + \frac{\eta^2 c^2}{4})^{-1}$. Then

$$u(x,t) \geq \hat{\gamma} - M e^{-\sigma t} \text{ for } |x| \leq c_0 t \text{ and all large } t.$$

$\square$

3 Proof of Theorem 1.2

Throughout this section, u stands for the solution of (1.1), (1.2) with $N \geq 2$. We assume, in addition to the standing hypotheses (SH), that the solution u is bounded, and that (Hr1) holds together with one of the conditions (Hr2), (Hr2').

Let b_0 be a bound on u, thus

$$0 < u(x,t) \leq b_0 \quad (x \in \mathbb{R}^N,\ t > 0). \tag{3.1}$$

Without affecting the validity of Theorem 1.2, we can modify f outside $[0, b_0]$ so that it still satisfies the above assumptions;, in addition, it is globally Lipschitz and there is $\gamma > b_0$ such that $f(\gamma) = 0 > f'(\gamma)$, $f < 0$ on $[\gamma, \infty)$, and $F(\gamma) > F(v)$ for each $v \in [0, \gamma)$. This means in particular that the set Γ, as defined in Section 1, has a maximal point, which is greater than b_0.

As one checks easily, the set Γ is closed, hence, by the above modification of f, it is compact and nonempty. The nondegeneracy condition (Hr1) then implies that Γ is finite, so for some integer $m \geq 1$ one has

$$\Gamma \cap (0, \infty) = \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \text{ with } \gamma_1 < \gamma_2 < \cdots < \gamma_m \text{ and } \gamma_m > b_0. \tag{3.2}$$

If $0 \in \Gamma$ (that is, if $f'(0) < 0$), we define $\gamma_0 := 0$.

By definition,

$$\begin{gathered} F(v) < F(\gamma_k) \ \ (v \in [0, \gamma_k)), \text{ and hence} \\ F(\gamma_1) < F(\gamma_2) < \cdots < F(\gamma_m). \end{gathered} \tag{3.3}$$

This and (Hr1) imply

$$f'(\gamma_j) < 0 \quad (j = 1, \ldots, k). \tag{3.4}$$

For a constant θ, we say that u is *localized at level* θ if

$$\lim_{|x| \to \infty} \max\{u(x,t), \theta\} = \theta \quad \text{uniformly in } t > 0.$$

The next lemma is a key step in the proof of Theorem 1.2.

Lemma 3.1. *For each $k \in \{1, 2, \ldots, m\}$ one of the following statements is valid:*

(si) $\liminf_{t \to \infty} u(x,t) \geq \gamma_k$, *uniformly for x in compact sets;*

(sii) *u is localized at level γ_{k-1}.*

Moreover, if (si) *holds, then there are positive constants M, c_0, σ, and t_0 such that*

$$u(x,t) \geq \gamma_k - Me^{-\sigma t} \quad (t \geq t_0,\ |x| \leq c_0 t); \tag{3.5}$$

and if (sii) *holds, then there are positive constants M_1, σ_1 such that*

$$u(x,t) \leq \gamma_{k-1} + M_1 e^{-\sigma_1 |x|} \quad (x \in \mathbb{R}^N,\ t > 0). \tag{3.6}$$

Implicitly contained in this statement is the fact that γ_{k-1} is always defined, even for $k = 1$, if (si) does not hold. In other words, if $k = 1$ and (si) does not hold, then necessarily $f'(0) < 0$.

Remark 3.2. It is mainly in this lemma that the cases $N = 1$ and $N \geq 2$ differ. If $N = 1$, for the above alternative to be valid, (sii) has to be replaced with the following weaker statement (see the remarks following (3.17)).

(sii') *u is localized at the level $\tilde{\gamma}_{k-1}$, where $\tilde{\gamma}_{k-1}$ is the maximal element of the set $\tilde{\Gamma} \cap [\gamma_{k-1}, \gamma_k)$.*

The proof of Lemma 3.1 involves several steps carried out in Lemmas 3.3-3.7 below. The conclusion of Lemma 3.1 follows directly from Lemmas 3.3 and 3.5. We remark that the assumption $N \neq 1$ is only needed in Lemma 3.7 and the second part of the proof of Lemma 3.5.

We shall use the following notation. Given $k \geq 1$, let

$$\beta_k := \max\{\beta \in [0, \gamma_k) : f(\beta) = 0\}. \tag{3.7}$$

Since $f'(\gamma_k) < 0$, β_k is well defined and one has

$$f(v) > 0 \quad (v \in (\beta_k, \gamma_k)). \tag{3.8}$$

For any $\theta > 0$, let

$$D_\theta := \{y \in \mathbb{R}^N : u(y,t) \geq \theta \text{ for some } t > 0\}. \tag{3.9}$$

We distinguish the following two cases.

(a) For each $\theta \in (\beta_k, \gamma_k)$ the set D_θ is bounded.

(b) There exists $\theta \in (\beta_k, \gamma_k)$ such that the set D_θ is unbounded.

We will prove that (a) implies (sii) and (b) implies (si). We start with the simpler case (b):

Lemma 3.3. *If $k \in \{1, 2, \ldots, m\}$ is such that* (b) *holds, then statement* (si) *is valid and there are positive constants M, c_0, σ, and t_0 for which* (3.5) *holds.*

Proof. The first relation in (3.3) and (3.8) show that all hypotheses of Lemma 2.4 and Corollary 2.5 are satisfied with $\beta = \beta_k$, $\hat{\gamma} = \gamma_k$. Let $R(\theta)$ be as in Corollary 2.5. Condition (b) implies that there is a sequence $(y_j, t_j) \in \mathbb{R}^N \times (0, \infty)$ such that $|y_j| \to \infty$ and $u(y_j, t_j) \geq \theta$, $j = 1, 2, \ldots$. Using Lemma 2.2, we find an integer j such that

$$u(x, t_j) > \theta \quad (|x| \leq R(\theta)). \tag{3.10}$$

Corollary 2.5 now implies that (si) holds and Lemma 2.6 yields constants M, c_0, σ, and t_0 for which (3.5) holds. $\square$

The following lemma shows that if $k = 1$ and $f'(0) > 0$, then (si) and (b) must hold.

Lemma 3.4. *If $f'(0) > 0$, then $\liminf_{t \to \infty} u(x, t) \geq \gamma_1$, uniformly for x in compact sets.*

Proof. Under the condition $f'(0) > 0$, γ_1 is the first positive zero of f (and $\beta_1 = 0$). The result now follows from well known results on the "monostable" nonlinearity $f\big|_{[0,\gamma_1]}$. In fact, the solution $\bar{u}$ of (1.1) with the initial condition $\bar{u}(x, 0) := \min\{u_0(x), \gamma_1\}$ converges to γ_1 in $L^\infty_{loc}(\mathbb{R}^N)$, see for example [1, Corollary 1, p. 66]. The conclusion of Lemma 3.4 follows from this via a comparison argument. $\square$

Lemma 3.5. *If $k \in \{1, 2, \ldots, m\}$ is such that* (a) *holds, then statement* (sii) *is valid and there are positive constants M_1, σ_1 for which* (3.6) *holds.*

We split the proof of this lemma into two parts. In the first part, we derive an estimate on u, which allows us to complete the proof of Lemma 3.5 under the extra condition that

$$\tilde{\Gamma} \cap (\gamma_{k-1}, \gamma_k) = \emptyset. \tag{3.11}$$

If this condition is not satisfied, additional estimates are needed and these will be given in the second part.

Proof of Lemma 3.5, Part 1. In view of Lemma 3.4, condition (a) implies that either $k \geq 2$ or $k = 1$ and $f'(0) < 0$. Hence γ_{k-1} is always defined.

From (3.3), (3.8) we infer that there is a unique $\beta^* \in [\beta_k, \gamma_k)$ with $F(\beta^*) = \max\{F(v) : v \in [\gamma_{k-1}, \beta_k]\}$ and

$$F(v) \leq F(\beta^*) \quad (v \in [\gamma_{k-1}, \beta^*]). \tag{3.12}$$

We claim that $\beta^* > \beta_k$. Indeed, since $f(\beta_k) = 0$, $\beta^* = \beta_k$ would mean that $\beta_k \in \tilde{\Gamma}$ and it is the maximal zero of f in (γ_{k-1}, γ_k), as well as the maximal element of $\tilde{\Gamma} \cap (\gamma_{k-1}, \gamma_k)$. We immediately obtain a contradiction if (Hr2') is assumed since it implies $\beta_k \notin \tilde{\Gamma}$. If (Hr2) holds, we obtain $f'(\beta_k) \neq 0$ and (3.12) then forces $f'(\beta_k) < 0$, in contradiction to (3.8).

Thus $\beta^* > \beta_k$ and in particular $f(\beta^*) > 0$. We now pick

$$\theta \in (\beta_k, \beta^*)$$

and fix it for the rest of the proof of Lemma 3.5 (including the second part).

Since we are assuming condition (a), there is $\rho > 0$ such that the bounded set $D_\theta \cup \operatorname{spt} u_0$ is contained in B_ρ, the open ball of radius ρ centered at the origin. This means that

$$u(x,t) < \theta \quad (|x| \geq \rho, t > 0), \ \operatorname{spt} u_0 \subset B_\rho. \tag{3.13}$$

We now complete the proof of Lemma 3.5 in the case that (3.11) holds. In view of (3.8) and $\beta^* > \beta_k$, (3.11) implies that the inequality in (3.12) is strict:

$$F(v) < F(\beta^*) \quad (v \in (\gamma_{k-1}, \beta^*)), \tag{3.14}$$

and $F(\beta^*) = F(\gamma_{k-1})$. Hence the assumptions of Lemma 2.3(ii) are satisfied with $\gamma = \gamma_{k-1}$ and $\hat{\gamma} = \gamma_k$. Consequently, the solution of

$$v_{rr} + f(v) = 0, \quad v(0) = \beta^*, \ v_r(0) = 0 \tag{3.15}$$

is defined for all $r \in \mathbb{R}$, $v_r < 0$ on $(0, \infty)$, and $v(r) \to \gamma_{k-1}$, as $r \to \infty$, exponentially. Define

$$\Psi(x) = v(|x| - \rho) \quad (x \in \mathbb{R}^N, \ |x| \geq \rho).$$

This radially symmetric function satisfies

$$\begin{aligned}\Delta\Psi(x) + f(\Psi(x)) &= v_{rr}(r-\rho) + \frac{N-1}{r} v_r(r-\rho) + f(v(r-\rho)) \\ &= \frac{N-1}{r} v_r(r-\rho) < 0 \quad (r = |x| > \rho),\end{aligned}$$

since $v_r < 0$. Hence Ψ is a supersolution of (1.1) in the exterior of B_ρ. Moreover, by (3.13), $\Psi > 0 \equiv u_0$ in $\mathbb{R}^N \setminus B_\rho$ and

$$\Psi(x) = v(0) = \beta^* > \theta > u(x,t) \quad (|x| = \rho, t > 0).$$

Therefore, by comparison,

$$u(x,t) \leq \Psi(x) = v(|x| - \rho) \quad (|x| \geq \rho, t > 0). \tag{3.16}$$

This proves that (sii) holds. Moreover, since the convergence of $v(r) \to \gamma_{k-1}$ is exponential, and u is bounded, there are constants M_1, σ_1 such that (3.6) holds. □

We now need to deal with the case

$$\tilde{\Gamma}_k := \tilde{\Gamma} \cap (\gamma_{k-1}, \gamma_k) \neq \emptyset. \tag{3.17}$$

We can still use the supersolution Ψ constructed above from the solution v of (3.15). However, this time, $v(r)$ converges to $\tilde{\gamma}_{k-1}$, the maximal element of $\tilde{\Gamma}_k$, and not γ_{k-1} (see Figure 2). This only gives that u is localized at level $\tilde{\gamma}_{k-1}$, hence statement (sii') in Remark 3.2 holds (for any $N \geq 1$). If $N = 1$, this is the best one can do.

To prove that u is localized at level γ_{k-1} if $N \geq 2$, we construct a different supersolution by gluing together solutions of two different ODEs, see (3.21) and (3.30) below. The details will be given in the second part of the proof, after we have examined suitable solutions of these ODEs.

We keep the notation from the first part of the proof of Lemma 3.5. Recall in particular that we have fixed $\theta \in (\beta_k, \beta^*)$ and chosen ρ such that conditions (3.13) hold. Also, we are assuming that (3.17) is satisfied.

Let $\hat{\gamma}_{k-1}$ be the minimal point of $\tilde{\Gamma}_k$. It is well defined due to the condition $f'(\gamma_{k-1}) \neq 0$. Clearly,

$$F(\xi) < F(\gamma_{k-1}) = F(\hat{\gamma}_{k-1}) \quad (\xi \in (\gamma_{k-1}, \hat{\gamma}_{k-1})), \ \text{ and } \hat{\gamma}_{k-1} \leq \beta_k < \theta.$$

Hence the assumptions of Lemma 2.3(iii) are satisfied with $\gamma = \gamma_{k-1}$ and $\hat{\gamma} = \hat{\gamma}_{k-1}$. Consequently, there is a solution of

$$\zeta_{rr} + f(\zeta) = 0 \tag{3.18}$$

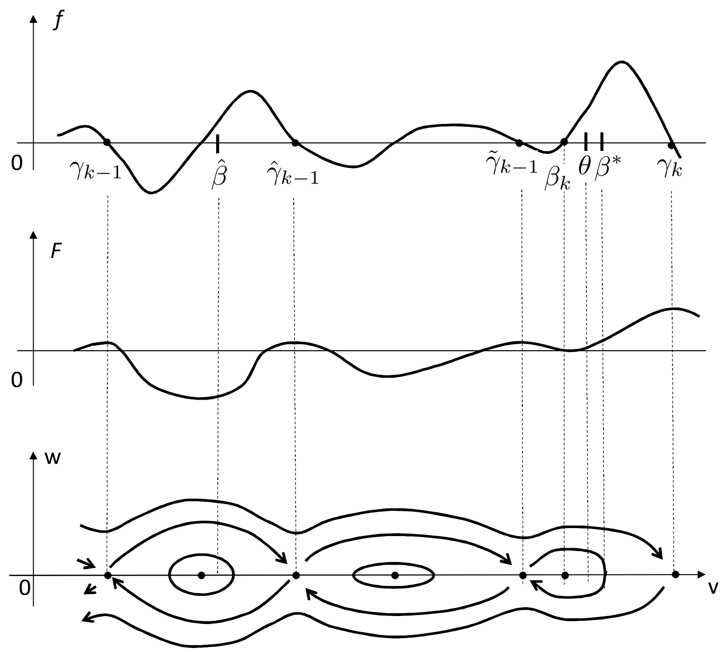

Figure 2: Sketch of the graphs of the functions f and F, and trajectories of the corresponding system (2.4). The trajectory of (ζ, ζ_r) is the heteroclinic orbit from $(\hat\gamma_{k-1}, 0)$ to $(\gamma_{k-1}, 0)$.

such that

$$\begin{aligned} &\zeta_r(r) < 0 \quad (r \in \mathbb{R}), \\ &\zeta(r) \to \hat\gamma_{k-1}, \text{ as } r \to -\infty, \\ &\zeta(r) \to \gamma_{k-1}, \text{ as } r \to \infty, \end{aligned}$$

where the last convergence is exponential.

Pick some $\hat\beta \in (\gamma_{k-1}, \hat\gamma_{k-1})$. Replacing ζ with its translation, we can assume that $\zeta(0) = \hat\beta$. Set

$$q_0 := \zeta_r(0) < 0, \quad \lambda_0 := F(\gamma_{k-1}) = F(\hat\gamma_{k-1}). \tag{3.19}$$

Note that λ_0 is the value of the Hamiltonian $H(v, v_r) = v_r^2/2 + F(v)$ along the trajectory of (ζ, ζ_r) (cp. Fig. 2). In particular,

$$q_0 = -\sqrt{2(\lambda_0 - F(\hat{\beta}))}. \tag{3.20}$$

Let now $z(r, \epsilon)$ be the solution of

$$z_{rr} + \epsilon z_r + f(z) = 0, \tag{3.21}$$

$$z(0) = \hat{\beta}, \quad z_r(0) = q_0. \tag{3.22}$$

Since we have modified f to be globally Lipschitz, z is defined for all $(r, \epsilon) \in \mathbb{R}^2$, it is of class C^1 in (r, ϵ) and C^2 in r. Obviously, $z(\cdot, 0) = \zeta$. We need the following result.

Lemma 3.6. *Under the above assumptions and notation, there is a function $\tau : (0, \infty) \to (0, \infty)$ with the following properties:*

$$z_r(r, \epsilon) < 0 \quad (r \in [-\tau(\epsilon), 0], \epsilon > 0), \tag{3.23}$$

$$z(-\tau(\epsilon), \epsilon) = \theta \quad (\epsilon > 0), \tag{3.24}$$

$$\tau(\epsilon) \to \infty \textit{ and } \epsilon\tau(\epsilon) \to 0, \textit{ as } \epsilon \to 0. \tag{3.25}$$

Proof. **S**ince $q_0 < 0$, we have $z_r(\cdot, \epsilon) < 0$ on an interval $(-\varsigma(\epsilon), 0]$, where we choose $\varsigma(\epsilon) \in (0, \infty]$ maximal possible. We claim that if $\epsilon > 0$, then the interval $(-\varsigma(\epsilon), 0)$ contains a point $-\tau(\epsilon)$ such that (3.24) holds. Assume it does not. Then

$$\gamma_{k-1} < \hat{\beta} \leq z(r, \epsilon) < \theta \quad (r \in (-\varsigma(\epsilon), 0]). \tag{3.26}$$

From (3.21) we get

$$\frac{\partial}{\partial r}\left(\frac{z_r^2}{2} + F(z)\right) = -\epsilon z_r^2 < 0. \tag{3.27}$$

This implies, that there is $\delta(\epsilon) > 0$ such that for $r \in (-\varsigma(\epsilon), -\min\{\varsigma(\epsilon)/2, 1\})$

$$\frac{z_r^2(r, \epsilon)}{2} + F(z(r, \epsilon)) - \delta(\epsilon) > \frac{z_r^2(0, \epsilon)}{2} + F(z(0, \epsilon)) = \lambda_0 = F(\gamma_{k-1}),$$

where we have used (3.22), (3.20), and (3.19). Since $F \leq F(\gamma_{k-1})$ on $(\gamma_{k-1}, \beta^*) \supset (\hat{\beta}, \theta)$, we have

$$\frac{z_r^2(r, \epsilon)}{2} > \delta(\epsilon) + F(\gamma_{k-1}) - F(z(r, \epsilon) \geq \delta(\epsilon) \tag{3.28}$$

for $r \in (-\varsigma(\epsilon), -\min\{\varsigma(\epsilon)/2, 1\})$. This immediately gives a contradiction if $\varsigma(\epsilon)$ is finite, for then $z_r(\varsigma(\epsilon), \epsilon) = 0$ due to the maximality of $\varsigma(\epsilon)$. If $\varsigma(\epsilon) = \infty$, then due to (3.28), $z(\cdot, \epsilon)$ cannot be bounded, which is a contradiction to (3.26).

Thus there indeed exists $-\tau(\epsilon) \in (-\varsigma(\epsilon), 0)$ such that (3.24) holds. Since $z_r(\cdot, \epsilon) < 0$ on $(-\varsigma(\epsilon), 0)$, (3.23) holds and $\tau(\epsilon)$ is uniquely determined. It remains to prove (3.25).

Since $z(r, 0) = \zeta(r) < \hat{\gamma}_{k-1} < \theta$ for each r, we have $\tau(\epsilon) \to \infty$ as $\epsilon \to 0$. Fix $\delta > 0$ and let $\epsilon_0 > 0$ be small enough so that $\tau(\epsilon) > \delta$ for $\epsilon \in (0, \epsilon_0)$.

Set

$$\lambda(r, \epsilon) := \frac{z_r^2(r, \epsilon)}{2} + F(z(r, \epsilon)).$$

Note that $\lambda(0, \epsilon) = \lambda_0 = F(\gamma_{k-1})$. By (3.27), we have for $r \in [-\infty, -\delta)$

$$\begin{aligned}\lambda_0 - \lambda(r, \epsilon) &= -\epsilon \left(\int_r^{-\delta} z_r^2(s, \epsilon)\, ds + \int_{-\delta}^{0} z_r^2(s, \epsilon)\, ds \right) \\ &\leq -\epsilon \int_{-\delta}^{0} z_r^2(s, \epsilon)\, ds := -\epsilon \kappa(\epsilon),\end{aligned}$$

with

$$\kappa(\epsilon) = \int_{-\delta}^{0} z_r^2(s, \epsilon)\, ds$$

a C^1 function of ϵ satisfying

$$\kappa(0) = \int_{-\delta}^{0} \zeta_r^2(s)\, ds > 0.$$

From $\lambda(r, \epsilon) \geq \lambda_0 + \epsilon \kappa(\epsilon)$, we obtain

$$\frac{z_r^2(r, \epsilon)}{2} \geq \lambda_0 - F(z(r, \epsilon)) + \epsilon \kappa(\epsilon) \quad (r \in (-\tau(\epsilon), -\delta)). \tag{3.29}$$

Since $z(r, \epsilon) \in (\hat{\beta}, \theta) \subset (\gamma_{k-1}, \beta^*)$ for $r \in (-\tau(\epsilon), -\delta)$, we have

$$\lambda_0 - F(z(r, \epsilon)) = F(\gamma_{k-1}) - F(z(r, \epsilon)) \geq 0.$$

Therefore, (3.29) and the fact that $z_r(\cdot, \epsilon) < 0$ on $[-\tau(\epsilon), -\delta]$ imply

$$z_r(r, \epsilon) \leq -\sqrt{\epsilon \kappa(\epsilon)}.$$

Integrating this for r from $-\tau(\epsilon)$ to $-\delta$ and using (3.24), we obtain

$$z(-\delta,\epsilon)-\theta \le (\delta-\tau(\epsilon))\sqrt{\epsilon\kappa(\epsilon)},$$

or, equivalently,

$$\tau(\epsilon) \le \delta + (\epsilon\kappa(\epsilon))^{-\frac{1}{2}}(\theta - z(-\delta,\epsilon)).$$

Since $\kappa(0)>0$ and $z(-\delta,0)=\zeta(0)\in\mathbb{R}$, this gives $\epsilon\tau(\epsilon)\to 0$, as claimed. $\square$

For a suitable $a>0$, we next consider the following problem

$$\eta_{rr}+\frac{N-1}{r}\eta_r+f(\eta)=0, \tag{3.30}$$

$$\eta(a)=\hat{\beta}, \quad \eta_r(a)=q_0. \tag{3.31}$$

Here $\hat{\beta}$ and q_0 are the same as in (3.22), (3.19), so η and $z(\cdot-a,\epsilon)$ share the initial conditions at $r=a$. By (3.19),

$$\frac{\eta_r^2(a)}{2}+F(\eta(a))=\lambda_0. \tag{3.32}$$

Lemma 3.7. *Under the above assumptions, the solution η of* (3.30), (3.31) *satisfies*

$$\gamma_{k-1}<\eta(r)<\hat{\gamma}_{k-1} \quad (r>a). \tag{3.33}$$

Proof. Since f is Lipschitz, η is defined for all $r\ge a$. Similarly as in (3.27), we have

$$\frac{\partial}{\partial r}\left(\frac{\eta_r^2}{2}+F(\eta)\right)=-\frac{N-1}{r}\eta_r^2.$$

Hence, by (3.32),

$$\frac{\eta_r^2(r)}{2}+F(\eta(r))<\lambda_0 \quad (r>a).$$

This implies that the trajectory $\{(\eta(r),\eta_r(r)):r>a\}$ has to stay inside the planar region whose boundary is formed by the curves

$$\{(\zeta(r),\zeta_r(r)):r\in\mathbb{R}\}, \quad \{(\zeta(r),-\zeta_r(r)):r\in\mathbb{R}\}$$

and their limit points $(\gamma_{k-1},0)$, $(\hat{\gamma}_{k-1},0)$ (see Fig. 3). These are four trajectories of the system associated with (3.18) forming a closed level curve of the Hamiltonian $\zeta_r^2/2+F(\zeta)$. Since $\zeta_r<0$, this planar region is contained in the strip $\{(v,w):\gamma_{k-1}\le v\le\hat{\gamma}_{k-1}\}$, which gives (3.33). $\square$

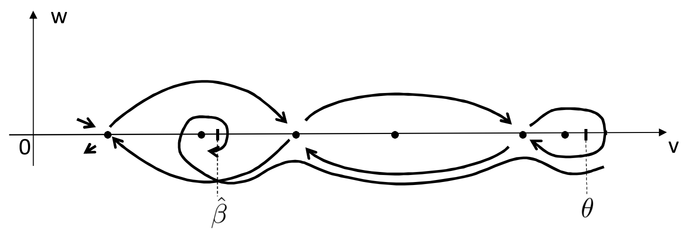

Figure 3: The trajectory of (η, η_r) is “trapped” inside a heteroclinic loop of system (2.4). It is matched with the trajectory of (z, z_r) which intersects the line $\{v = \theta\}$.

Proof of Lemma 3.5, Part 2. We now prove, in two steps, that the solution u is localized at level γ_{k-1}. First we show that for some $a > 0$ one has

$$u(x,t) < \hat\beta \quad (|x| \geq a,\ t > 0) \tag{3.34}$$

with $\hat\beta$ as in (3.22), (3.31). For that we construct a supersolution gluing together the solution z of (3.21) and the solution η of (3.30).

With ρ as in (3.13) and with $\tau(\epsilon)$ as in Lemma 3.6, we choose $\epsilon > 0$ so small that

$$\frac{N-1}{\epsilon} - \tau(\epsilon) > \rho. \tag{3.35}$$

This choice is possible by (3.25). Set

$$a := \frac{N-1}{\epsilon}, \quad \tau := \tau(\epsilon),$$

and define a function Φ on $[a-\tau, \infty) \subset (\rho, \infty)$ by

$$\Phi(r) = \begin{cases} z(r-a, \epsilon) & \text{if } r \in [a-\tau, a] \\ \eta(r) & \text{if } r \geq a, \end{cases}$$

with z and η as in (3.21), (3.22), and (3.30), (3.31), respectively. By Lemma 3.7, Φ is a bounded positive function. Also it is of class C^2 on $[a-\tau, a]$ and $[a, \infty)$, and since η and $z(\cdot - a, \epsilon)$ share the initial conditions at $r = a$, Φ is of class C^1 on $[a-\tau,\infty)$. Defining $\mu(r)$ by

$$\mu(r) = \begin{cases} \epsilon = \dfrac{N-1}{a} & \text{if } r \in [a-\tau, a) \\ \dfrac{N-1}{r} & \text{if } r \geq a, \end{cases}$$

we see that Φ is a solution of the ODE

$$\Phi_{rr} + \mu(r)\Phi_r + f(\Phi) = 0, \quad r \geq a-\tau. \tag{3.36}$$

On the interval $[a-\tau, a)$, we have $\Phi_r(r) = z_r(r-a) < 0$ and $\mu(r) = (N-1)/a \leq (N-1)/r$. Therefore

$$\Phi_{rr} + \frac{N-1}{r}\Phi_r + f(\Phi) = \begin{cases} \left(\dfrac{N-1}{r} - \mu(r)\right)\Phi_r \leq 0 & \text{if } r \in [a-\tau, a), \\ 0 \quad \text{if } r \geq a. \end{cases}$$

We conclude that the function $\Phi(|x|)$ is a supersolution of (1.1) on the set $\{x \in \mathbb{R}^N : |x| \geq a-\tau\}$. Since $\operatorname{spt} u_0 \subset B_\rho$ and $a-\tau > \rho$ (see (3.35)), we have

$$\Phi(|x|) > 0 = u_0(x) \quad (|x| \geq a-\tau).$$

By (3.24), (3.13),

$$\Phi(a-\tau) = z(-\tau, \epsilon) = \theta > u(x,t) \quad (|x| = a-\tau,\ t > 0).$$

Thus, by the comparison principle,

$$u(x,t) < \Phi(|x|) \quad (|x| > a-\tau,\ t > 0).$$

In particular,

$$u(x,t) < \Phi(a) = z(0,\epsilon) = \hat{\beta} \quad (|x| = a > \rho),$$

which implies (3.34), as u is radially decreasing outside B_ρ (see Lemma 2.1(ii)).

With (3.34) at hand, we can further estimate u in much the same way as in (3.16). Just replace (3.13) with (3.34), v with ζ (see (3.18)), and β^* with $\hat{\beta}$, and use similar arguments as those leading to (3.16). This gives

$$u(x,t) \leq \zeta(|x|-a) \quad (|x| > a,\ t > 0).$$

Hence statement (sii) holds and, since $\zeta(r) \to \gamma_{k-1}$, as $r \to \infty$, exponentially, and u is bounded, (3.6) holds for some M_1, σ_1. The proof of Lemma 3.1 is complete. $\square$

We remark that in the above proof, one could alternatively define $\Phi(r) = \zeta(r-a)$ for $r \geq a$, where ζ is as in (3.18). This way one could bypass Lemma 3.7 at the expense of working with weak, rather than classical, supersolutions.

Before we proceed to the proof of Theorem 1.2, we recall a result of [14] concerning asymptotically autonomous equations of the form

$$\tilde{u}_t = \Delta \tilde{u} + f(\tilde{u}) + h(x,t), \quad x \in \mathbb{R}^N, \ t > t_0. \tag{3.37}$$

Assume that for some $t_0 > 0$ and $\alpha \in (0,1)$, h is a continuous function on $\mathbb{R}^N \times (t_0, \infty)$, which is α-Hölder continuous in x, and there is $\varepsilon > 0$ such that

$$\sup_{t>t_0} \|h(\cdot,t)\|_{C^\alpha(\mathbb{R}^N)} e^{\varepsilon t} < \infty. \tag{3.38}$$

Here $\|\cdot\|_{C^\alpha(\mathbb{R}^N)}$ stands for a standard norm on $C^\alpha(\mathbb{R}^N)$, the space of bounded, α-Hölder continuous functions on $\mathbb{R}$.

Theorem 3.8. *Under the above hypotheses, let $\tilde{u}$ be a bounded solution of* (3.37)*. Assume that there is $\gamma \geq 0$ such that $f(\gamma) = 0 > f'(\gamma)$,*

$$\tilde{u}(x,t) \geq \gamma \ \ (x \in \mathbb{R}^N, \ t > t_0), \quad \textit{and } u \textit{ is localized at level } \gamma. \tag{3.39}$$

Then, as $t \to \infty$,

$$\tilde{u}(\cdot,t) \to \varphi \textit{ in } L^\infty(\mathbb{R}^N), \tag{3.40}$$

where $\varphi \equiv \gamma$ or φ is a ground state of (1.9) *based at γ.*

In [14] this result was proved for nonnegative solutions under the assumptions $f(0) = 0 > f'(0)$. Theorem 3.8 follows from this result applied to $\tilde{u} - \gamma$.

Proof of Theorem 1.2. Let k be the maximal nonnegative integer for which statement (si) of Lemma 3.1 holds. In view of (3.1), (3.2), we have $k \leq m-1$. Since (si) does not hold with k replaced with $k+1$, Lemma 3.1 implies that u is localized at level γ_k. From Lemma 3.1 we further obtain that there are positive constants M, c_0, σ, t_0, M_1, σ_1 such that

$$u(x,t) \geq \gamma_k - Me^{-\sigma t} \quad (t \geq t_0, \ |x| \leq c_0 t), \tag{3.41}$$

$$u(x,t) \leq \gamma_k + M_1 e^{-\sigma_1 |x|} \quad (x \in \mathbb{R}^N, \ t > 0). \tag{3.42}$$

Our goal is to find a bounded function $\tilde{u}$ on $\mathbb{R}^N \times (t_0, \infty)$ with the following properties:

(pi) $\tilde{u}(x,t) \geq \gamma_k$ $(x \in \mathbb{R}^N,\, t > t_0)$ and $\tilde{u}$ is localized at level γ_k,

(pii) $\tilde{u}$ is a solution of equation (3.37), where h is a continuous function satisfying (3.38) for some positive constants $\varepsilon > 0$ and $\alpha \in (0,1)$,

(piii) $\tilde{u}(\cdot,t) - u(\cdot,t) \to 0$, as $t \to \infty$, in $L^\infty_{loc}(\mathbb{R}^N)$.

Clearly, once such a function $\tilde{u}$ is found, a reference to Theorem 3.8 completes the proof of Theorem 1.2.

To define $\tilde{u}$, choose a function $\varrho_1 \in C^\infty(\mathbb{R})$ such that $0 \leq \varrho_1 \leq 1$ everywhere and

$$\varrho_1(r) = \begin{cases} 1 & (r \leq 0), \\ 0 & (r \geq 1). \end{cases}$$

Set $c := c_0/4$ and

$$\varrho(x,t) := \varrho_1\left(\frac{|x| - ct}{ct}\right) = \varrho_1\left(\frac{|x|}{ct} - 1\right).$$

Then $\varrho \in C^\infty(\mathbb{R}^N)$, $0 \leq \varrho \leq 1$, and

$$\varrho(x,t) = \begin{cases} 1 & (t > 0,\ |x| \leq ct), \\ 0 & (t > 0,\ |x| \geq 2ct). \end{cases}$$

It is easy to verify that all derivatives of ϱ are bounded on $\mathbb{R}^N \times [\delta, \infty)$, for each $\delta > 0$.

Now define, for $(x,t) \in \mathbb{R}^N \times (0,\infty)$,

$$\begin{aligned} \tilde{u}(x,t) :&= (u(x,t) + Me^{-\sigma t}))\varrho(x,t) + \gamma_k(1 - \varrho(x,t)) \\ &= u(x,t)\varrho(x,t) + \gamma_k(1 - \varrho(x,t)) + Me^{-\sigma t}\varrho(x,t). \end{aligned}$$

We show that (pi)-(piii) hold true.

Since $2c < c_0$, we have $u(x,t) + Me^{-\sigma t} \geq \gamma_k$ in the set $\{(x,t) : t > t_0, |x| \leq 2ct\}$ (see (3.41)). Outside this set $\varrho \equiv 0$, hence $\tilde{u}(x,t) \geq \gamma_k$ for any $t \geq t_0$, $x \in \mathbb{R}^N$. Clearly, $\tilde{u}$ is bounded. Now, since $0 \leq \varrho \leq 1$,

$$\tilde{u}(x,t) \leq u(x,t) + \gamma_k + Me^{-\sigma t}\varrho(x,t). \tag{3.43}$$

We know that u is localized at level γ_k (see (3.42); we show that so is $\gamma_k + Me^{-\sigma t}\varrho(x,t)$. If $|x| \leq 2ct$, then

$$Me^{-\sigma t}\varrho(x,t) \leq Me^{-\sigma t} \leq Me^{-\frac{\sigma}{2c}|x|}.$$

The same is trivially true if $|x| \geq 2ct$, for then $\rho(x,t) = 0$. This proves that $\tilde{u}$ is localized at level γ_k. Statement (pi) is thus proved.

Since $\varrho(x,t) = 1$ for $|x| \leq ct$, we have

$$\tilde{u}(x,t) - u(x,t) = Me^{-\sigma t} \quad (t > t_0,\, |x| \leq ct). \tag{3.44}$$

This implies (piii).

It remains to verify (pii). The function $\tilde{u}$ is a solution of (3.37) with

$$h := \tilde{u}_t - \Delta\tilde{u} - f(\tilde{u}).$$

We show that for all sufficiently small $\varepsilon > 0$ the functions

$$e^{\varepsilon t}h(x,t), \quad e^{\varepsilon t}\nabla h(x,t) \tag{3.45}$$

are bounded on $\mathbb{R}^N \times (t_0, \infty)$ (here and below $\nabla = \nabla_x$). This is easily seen to be true in the regions $\{(x,t) : t > t_0,\, |x| \geq 2ct\}$ and $\{(x,t) : t > t_0,\, |x| \leq ct\}$, where $\tilde{u}$ coincides with γ_k and $u + Me^{-\sigma t}$, respectively, thus

$$\begin{aligned}
&h \equiv -f(\gamma_k) = 0 \ \text{ in } \{(x,t) : t > 0,\, |x| \geq 2ct\},\\
&h \equiv -\sigma Me^{-\sigma t} + f(u) - f(u + Me^{-\sigma t}) \ \text{ in } \{(x,t) : t > 0,\, |x| \leq ct\}.
\end{aligned}$$

One just uses the boundedness of u and ∇u, and the assumption that f' is locally Hölder continuous (this is the only place in the paper where the Hölder continuity of f' is needed).

We next estimate the functions in (3.45) in the set

$$\{(x,t) : t > t_0,\, ct < |x| < 2ct\}.$$

We have, omitting the argument (x,t) in u, ϱ, etc.,

$$\begin{aligned}
h &= (u_t - \Delta u)\varrho - f(u\varrho + \gamma_k(1-\varrho) + Me^{-\sigma t}\varrho)\\
&\qquad + (\varrho_t - \Delta\varrho)(u - \gamma_k + Me^{-\sigma t}) - 2\nabla u \cdot \nabla\varrho - \sigma Me^{-\sigma t}\varrho\\
&= f(u)\varrho - f(u\varrho + \gamma_k(1-\varrho) + Me^{-\sigma t}\varrho)\\
&\qquad + (\varrho_t - \Delta\varrho)(u - \gamma_k) - 2\nabla u \cdot \nabla\varrho + Me^{-\sigma t}(\varrho_t - \Delta\varrho - \sigma\varrho).
\end{aligned}$$

As all derivatives of ϱ are bounded, we only need to estimate the functions

$$(u - \gamma_k),\ \nabla u,\ f(u),\ f(u\varrho + \gamma_k(1-\varrho) + Me^{-\sigma t}\varrho), \tag{3.46}$$

and their x-derivatives in $\{(x,t) : t > t_0,\ ct < |x| < 2ct\}$.

From (3.41), (3.42), we have

$$|u(x,t) - \gamma_k| \leq M_2 \max\{e^{-\sigma t},\ e^{-\sigma_1 |x|}\} \quad (t > t_0,\ |x| \leq c_0 t), \tag{3.47}$$

with $M_2 = \max\{M,\ M_1\}$.

We now intend to use the following parabolic estimates:

$$|\nabla u(x,t)|,\ |D^2 u(x,t)| \leq C \sup_{(y,s)\in B_1(x)\times(t-1,t)} |u(y,s) - \gamma_k| \tag{3.48}$$

for $t \geq 2,\ x \in \mathbb{R}^N$, where $B_1(x)$ stands for the ball of radius 1 centered at x and C is a constant. Let us first prove estimate (3.48) for $\nabla u(x,t)$. The function $w = u - \gamma_k$ satisfies a linear parabolic equation $w_t = \Delta w + c(x,t)w$ with a bounded coefficient c. By standard interior L^p-estimates, for each $p > N+2$, there is a constant C_p such that

$$\|u - \gamma_k\|_{W_p^{2,1}(B_{\frac{1}{2}}(x)\times(t-\frac{1}{2},t)} \leq C_p \|u - \gamma_k\|_{L^p(B_1(x)\times(t-1,t)} \quad (t \geq 2,\ x \in \mathbb{R}^N).$$

The $W_p^{2,1}$-norm controls the sup-norm of ∇u, by the Sobolev imbedding theorem [20], and on the bounded domain $(B_1(x)\times(t-1,t))$ the L^p-norm of u is controlled by its sup-norm. This implies estimate (3.48) for ∇u. Next, each of the functions u_{x_i} also satisfies an equation $w_t = \Delta w + c(x,t)w$ with a bounded coefficient c. Thus using the interior L^p-estimates again, we see that the estimate (3.48) for $D^2 u(x,t)$ follows from the estimate for ∇u.

Assume now that $|x| \leq 2ct < c_0 t$. Then for each $(y,s) \in B_1(x)\times(t-1,t)$, we have

$$|x| - 1 \leq |y| \leq |x| + 1 \leq 2ct + 1 < 2cs + 2c + 1 < c_0 s,$$

provided t is sufficiently large. Therefore, (3.47), (3.48) imply that if t is sufficiently large and $|x| \leq 2ct$, then

$$\begin{aligned} |u(x,t) - \gamma_k|,\ |\nabla u(x,t)|,\ |D^2 u(x,t)| &\leq M_2 \max\{e^{-\sigma(t-1)},\ e^{-\sigma_1(|x|-1)}\} \\ &\leq M_3 \max\{e^{-\sigma t},\ e^{-\sigma_1 |x|}\}, \end{aligned}$$

where $M_3 = M_2 \max\{e^{\sigma},\ e^{\sigma_1}\}$. In the relevant range $ct \leq |x| \leq 2ct$, with t large, this gives

$$|u(x,t) - \gamma_k|,\ |\nabla u(x,t)|,\ |D^2 u(x,t)| \leq M_3 e^{-\varepsilon t}, \tag{3.49}$$

if we choose $0 < \varepsilon \leq \min\{\sigma,\, c\sigma_1\}$. Next, denoting by L the Lipschitz constant of f, we obtain, in the same relevant range,

$$|f(u(x,t))| = |f(u(x,t)) - f(\gamma_k)| \leq LM_3 e^{-\varepsilon t}$$

and

$$\begin{aligned} &|f(u(x,t)\varrho(x,t) + \gamma_k(1-\varrho(x,t)) + Me^{-\sigma t}\varrho(x,t))| \\ &\qquad = |f(\gamma_k + (u(x,t)-\gamma_k)\varrho(x,t) + Me^{-\sigma t}\varrho(x,t)) - f(\gamma_k)| \\ &\qquad \leq L(|u(x,t)-\gamma_k| + Me^{-\sigma t})) \leq 2LM_3 e^{-\varepsilon t}. \end{aligned}$$

Using the estimates on $u - \gamma_k$ and ∇u, one can similarly bound the x-derivatives of the functions

$$f(u),\ f(\gamma_k + (u-\gamma_k)\varrho + Me^{-\sigma t}\varrho).$$

We have thus established the desired exponential estimates on the functions in (3.46) and their x-derivatives in the range $ct < |x| < 2ct$, with t sufficiently large. Hence the functions in (3.45) are bounded if $\epsilon > 0$ is sufficiently small, which implies that statement (piii) holds true.

Since we have verified all statements (pi)-(piii), the proof of Theorem 1.2 is complete. $\square$

4 The one-dimensional problem

If $N \geq 2$, then Theorem 1.1 follows from Theorem 1.2. The remainder of this section is devoted to the case $N = 1$.

Proof of Theorem 1.1 for $N = 1$. We assume that the standing hypotheses (SH) hold and the solution u of (1.1), (1.2) is bounded. We make a similar modification of f as in the previous section; hence we may assume that there is an element of $\tilde{\Gamma}$ which is an upper bound on u.

By [7],

$$\lim_{t\to\infty} u(\cdot,t) = \varphi \ \text{ in } L^\infty_{loc}(\mathbb{R}), \tag{4.1}$$

where φ is a steady state of (1.1), and there is a zero γ of f such that $\varphi \equiv \gamma$ or φ is a ground state of (1.9) based at γ. We need to prove that $\gamma \in \tilde{\Gamma}$.

We go by contradiction. Assume that $\gamma \notin \tilde{\Gamma}$. In particular, $\gamma > 0$. Let $\underline{\gamma}$, $\overline{\gamma}$, be, respectively, the maximal element of $\tilde{\Gamma} \cap [0, \gamma)$ and the minimal element of $\tilde{\Gamma} \cap (\gamma, \infty)$. Since both these sets are nonempty and $\tilde{\Gamma}$ is closed, $\underline{\gamma}$, $\overline{\gamma}$ are well defined. Clearly,

$$F(\underline{\gamma}) \leq F(\overline{\gamma}).$$

Moreover,

$$F(\xi) < F(\overline{\gamma}) \quad (\xi \in (\underline{\gamma}, \overline{\gamma})).$$

Indeed, if not, then some $\xi \in (\underline{\gamma}, \overline{\gamma})$ is a maximizer of F in $[\underline{\gamma}, \overline{\gamma}]$. Necessarily, such ξ satisfies $f(\xi) = 0$ and it also maximizes F in $[0, \xi)$, as $\underline{\gamma} \in \tilde{\Gamma}$. Thus $\xi \in \tilde{\Gamma}$, but this is impossible by the definition of $\underline{\gamma}$, $\overline{\gamma}$.

Furthermore, we have $F(\gamma) < F(\underline{\gamma})$. Otherwise $F(\gamma) \geq F(\underline{\gamma})$ and we deduce $F(\xi) < F(\gamma)$ for $\xi \in (\underline{\gamma}, \gamma)$ by repeating the above argument with $\overline{\gamma}$ replaced by γ. This implies that γ is a maximizer of F over $[0, \gamma]$, contradicting the assumption that $\gamma \notin \tilde{\Gamma}$.

We claim that for each $\theta \in (\gamma, \overline{\gamma})$ the set

$$D_\theta := \{y \in \mathbb{R} : u(y, t) \geq \theta \text{ for some } t > 0\}$$

is bounded. Indeed, if it is unbounded, then there is a sequence $(y_j, t_j) \in \mathbb{R} \times (0, \infty)$ such that $|y_j| \to \infty$ and $u(y_j, t_j) \geq \theta$, $j = 1, 2, \dots$. Using Lemma 2.2, by passing to a subsequence of t_j, we find a sequence ρ_j, such that $\rho_j \to \infty$ and $u(\cdot, t_j) > \theta$ on $(-\rho_j, \rho_j)$. Passing to a further subsequence, we may assume that $t_j \to t_\infty$ for some finite or infinite t_∞. If $t_\infty < \infty$, then we obtain $u(\cdot, t_\infty) \geq \theta$, which is impossible (recall that (1.6) holds for each solution with compact initial support). If $t_\infty = \infty$, then

$$\liminf_{j \to \infty} u(x, t_j) \geq \theta \quad (x \in \mathbb{R}),$$

which is impossible by (4.1) and $\gamma < \theta$.

We next derive a contradiction separately in each of the following cases:

$$\text{(c1)} \quad F(\underline{\gamma}) = F(\overline{\gamma}), \qquad\qquad \text{(c2)} \quad F(\underline{\gamma}) < F(\overline{\gamma}).$$

If (c1) holds, Lemma 2.3(iii) yields a solution v of (2.3) such that $v(r) \to \underline{\gamma}$, as $r \to \infty$, and $v(r) \to \overline{\gamma}$, as $r \to -\infty$. Replacing v with its translation, if necessarily, we may assume that $\theta := v(0) > \gamma$. Fix $q > 0$ large enough so that $\operatorname{spt} u_0 \cup D_\theta \subset (-q, q)$. Then

$$\begin{aligned} u(x, 0) &\equiv 0 < v(|x| - q) \quad (|x| \geq q) \\ u(x, t) &< \theta = v(0) \quad (|x| = q). \end{aligned}$$

Using a comparison argument on $\{(x,t) : |x| \geq q,\ t > 0\}$, we now obtain

$$u(x,t) \leq v(|x| - q) \quad (|x| \geq q,\ t > 0). \tag{4.2}$$

This implies that u is localized at level $\underline{\gamma}$, which is a contradiction to (4.1) and $\varphi \geq \gamma$.

Next assume that (c2) holds. Then there is $\beta^* \in (\gamma, \overline{\gamma})$ such that

$$F(\beta^*) = F(\underline{\gamma}) > F(\xi) \quad (\xi \in (\underline{\gamma}, \beta^*)).$$

Necessarily $f(\beta^*) \neq 0$, for otherwise β^* would be an element of $\tilde{\Gamma}$, in contradiction to the minimality of $\overline{\gamma}$. We now use Lemma 2.3(ii) to find a solution v of (2.3) such that $v(0) = \beta^*$ and $v(r) \to \underline{\gamma}$, as $r \to \infty$. Taking $\theta := \beta^*$ and repeating the comparison argument from case (c1), we arrive at the same contradiction (4.2).

Thus the assumption $\gamma \notin \tilde{\Gamma}$ always leads to a contradiction, which completes the proof of Theorem 1.1. $\square$

Finally we give the example that justifies Remark 1.3(i).

Example 4.1. *Let f be a C^1-function on $\mathbb{R}$ with the following properties* (*see* Fig. 4) :

$$f^{-1}\{0\} = \{0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2\})$$
$$f'(k) < 0 \quad (k \in \{0,1,2\}),$$
$$\int_0^1 f(s)\,ds = 0, \qquad \int_1^2 f(s)\,ds > 0.$$

Clearly $\tilde{\Gamma} = \{0,\ 1,\ 2\}$ and $\Gamma = \{0,\ 2\}$. One shows easily that $\gamma = 1$ is the unique zero of f such that equation (1.9) *with $N = 1$ has a ground state based at γ. As we show below, such a ground state is the locally uniform limit of the solution u of* (1.1), (1.2)*, for some continuous, nonnegative function u_0 with compact support.*

Note that the assumptions of Lemma 2.4 are satisfied with $\beta = 3/2$, $\hat{\gamma} = 2$. Pick $\theta \in (3/2, 2)$ and let $R := 2R(\theta)$, where $R(\theta)$ is as in Corollary 2.5. Thus, if u is a positive global solution of (1.1) such that

$$\text{for some } t_1 > 0 \text{ one has } \ u(x,t_1) > \theta \ \ (x \in [-R/2, R/2]), \tag{4.3}$$

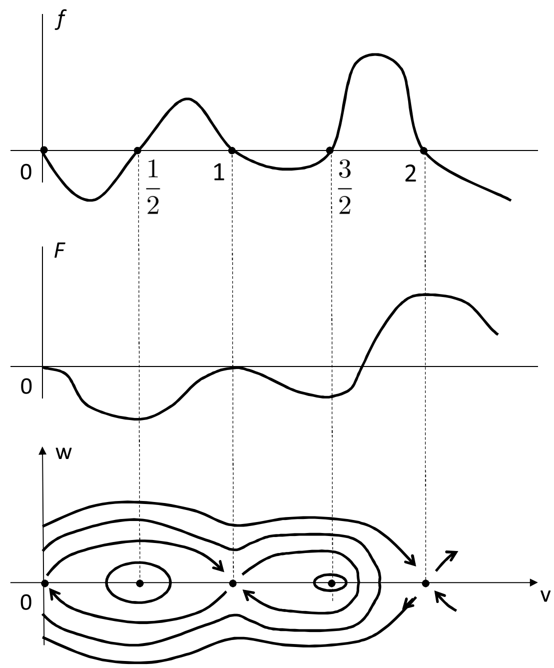

Figure 4: The functions f, F from Example 4.1 and trajectories of the corresponding system (2.4).

then $\liminf_{t\to\infty} u(x,t) \geq 2$, uniformly for x in compact sets.

Choose any $\psi \in C(\mathbb{R})$ such that $0 \leq \psi < 2$ everywhere, $\psi \equiv 2$ on $[-R/2, R/2]$, and $\psi \equiv 0$ on $\mathbb{R}\setminus[-R,R]$. We consider the family $\lambda\psi$, $\lambda \in [0,1]$, of initial data and denote by u^λ the solution of (1.1), (1.2) with $u_0 = \lambda\psi$.

We have $0 \leq u^\lambda \leq 2$ everywhere and, by Lemma 2.1(ii),

$$xu_x^\lambda(x,t) < 0 \quad (|x| \geq R). \tag{4.4}$$

Consider the following statement:

$$\text{there is } t_2 > 0 \text{ such that } \; u^\lambda(x,t_2) < 3/2 \quad (x \in [-R,R]). \tag{4.5}$$

Observe that if (4.5) holds, then (4.4) implies that $u^\lambda(\cdot,t_2) < 3/2$ on $\mathbb{R}$, hence $u^\lambda < 3/2$ on $\mathbb{R}\times[t_2,\infty)$ by the comparison principle.

It is obvious that (4.5) holds for $\lambda \approx 0$, whereas if $\lambda \approx 1$, then $u = u^\lambda$ satisfies (4.3). Continuity with respect to u_0 shows that there are λ_0, λ_1 such the set of all $\lambda \in [0, 1]$ for which (4.5) holds is the interval $[0, \lambda_0)$ and the set of all λ for which (4.3) holds is the interval $(\lambda_1, 1]$. Moreover, the large-time behavior of the corresponding solutions, as discussed above, shows that $[0, \lambda_0) \cap (\lambda_1, 1] = \emptyset$. Hence there exists $\lambda \in (0, 1)$ for which neither (4.5) nor (4.3) holds. Let φ be the locally uniform limit of $u^\lambda(\cdot, t)$, as $t \to \infty$. Then there are $x_0, x_1 \in [-R, R]$ such that

$$\varphi(x_0) \geq 3/2, \quad \varphi(x_1) \leq \theta.$$

Since the only zero of f in $[3/2, \theta]$ is $3/2$ and it is not in $\tilde{\Gamma}$, φ is nonconstant. Thus φ is a ground state based at some $\gamma \geq 0$. As remarked above, the only possibility is $\gamma = 1 \in \tilde{\Gamma} \setminus \Gamma$.

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