Examples of bounded solutions with nonstationary limit profiles for semilinear heat equations on $\mathbb{R}$

P. Poláčik*

School of Mathematics, University of Minnesota

Minneapolis, MN 55455

Abstract. We consider bounded solutions of the semilinear heat equation $u_t = u_{xx} + f(u)$ on $\mathbb{R}$, where f is of a bistable type. We show that there always exist bounded solutions whose ω-limit set with respect to the locally uniform convergence contains functions which are not steady states. For balanced bistable nonlinearities, there are examples of such solutions with initial values $u(x, 0)$ converging to 0 as $|x| \to \infty$. Our example with an unbalanced bistable nonlinearity shows that bounded solutions whose ω-limit set do not consist of steady states occur for a robust class of nonlinearities f.

Key words: semilinear heat equation on the real line, asymptotic behavior, nonconvergent solutions

AMS Classification: 35K15, 35B40

Contents

*Supported in part by the NSF Grant DMS–1161923

1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and u_0 is a bounded continuous function on $\mathbb{R}$.

Problem (1.1), (1.2) has a unique (classical) solution u defined on a time interval $[0, T(u_0))$. We assume here that $T(u_0) \in (0, \infty]$ is maximal possible. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is *global*.

In this paper, we examine the large-time behavior of bounded solutions. For this purpose we introduce the ω-limit set, $\omega(u)$, of a bounded solution u as follows:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some } t_n \to \infty\}. \tag{1.3}$$

Here, the convergence is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence). Thus we consider the behavior of u, as $t \to \infty$, on arbitrarily large compact sets. By standard parabolic regularity estimates, the trajectory $\{u(\cdot, t) : t \geq 1\}$ of any bounded solution u is relatively compact in $L^\infty_{loc}(\Omega)$. Therefore,

$$\omega(u) \neq \emptyset \text{ and } \operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot, t), \omega(u)) \to 0 \text{ as } t \to \infty. \tag{1.4}$$

It is also well known that the set $\omega(u)$ is connected. By compactness, $\omega(u)$ consists of a single element φ if and only if u is *convergent:* $u(\cdot, t) \to \varphi$ in

$L^\infty_{loc}(\mathbb{R})$. Necessarily, φ is an equilibrium, or a steady state, of (1.1) in this case.

If equation (1.1) is considered on a bounded interval, instead of $\mathbb{R}$, and one of common boundary conditions, say Dirichlet, Neumann, Robin, or periodic, is assumed, then each bounded solution is convergent [7, 29, 40]. In contrast, bounded solutions (1.1) on $\mathbb{R}$ are not convergent in general even for the linear heat equation (that is, equation (1.1) with $f \equiv 0$). More specifically, if u_0 takes values 0 and 1 on suitably spaced long intervals with small transitions between them, then, as $t \to \infty$, $u(\cdot, t)$ will oscillate between 0 and 1, thus creating a continuum $\omega(u)$ which contains both 0, 1 (see [8]). In the case of the linear heat equation, it is easy to show that each bounded solution is *quasiconvergent:* its ω-limit set consists of steady states. This follows, for example, from the invariance property of the ω-limit set: $\omega(u)$ consists of *entire* solutions of (1.1), by which we mean solutions defined for all $t \in \mathbb{R}$. If u is bounded, then the entire solutions in $\omega(u)$ are bounded as well and, by the Liouville theorem for the linear heat equation, all such solutions are constant.

For the semilinear problem (1.1) with $f \not\equiv 0$, there are several general results describing the large time behavior of special classes of bounded solutions. For example, convergence to an equilibrium has been proved for bounded nonnegative solutions whose initial value u_0 has compact support [10, 13, 14, 41] and for the solutions, not necessarily nonnegative, which are localized in the sense that they decay to zero at $x = \pm\infty$ uniformly in $t > 0$ (see [16]; in this result, it is also assumed that $f'(0) < 0$). Recently, a quasiconvergence result for positive bounded solutions whose initial values u_0 decay to 0 at $x = \pm\infty$ has been proved in [30].

With no extra assumptions on u_0, assuming just that u is bounded, it has been proved in [22, 23] that $\omega(u)$ necessarily contains an equilibrium. Unlike in the linear heat equation, this result cannot in general be improved so as to say that u is quasiconvergent. Indeed, a construction of [12] (see, also [38] and Proposition 4.1 below) yields an example of a bounded solution of (1.1) with $f(u) = u(1-u^2)$ whose ω-limit set contains the constant equilibria -1 and 1 as well as some nonequilibrium solutions. Similarly to the example of [8] for the heat equation, the initial value of the solution in [12] oscillates between the constant equilibria, being identical or close to one of them on larger and larger intervals.

The example of [12] seems to be the only known example of a bounded solution which is not quasiconvergent (in $L^\infty_{loc}(\mathbb{R})$) for problems of the form

(1.1), (1.2). Considering the special form of the nonlinearity f and the initial datum u_0, two natural questions arise. The first one concerns the oscillatory character of u_0. Is it true that the solutions with initial values which do not exhibit large oscillations, say for initial values u_0 which converge to a zero of f as $|x| \to \infty$, the corresponding solution must be quasiconvergent if bounded? The answer is "yes" for the linear heat equation (the solution converges to $u_0(\infty)$) and also in the semilinear case if $u_0(x) \geq u_0(\infty)$ for all $x \in \mathbb{R}$ (see [30]). However, in general, the answer is negative as we prove in Theorem 2.1 below. See also Remark 2.2 where another example of a non-quasiconvergent solution is mentioned.

The second question concerns the structure of the nonlinearity. In the example of [12], as well as in our Theorem 2.1, the nonlinearity is of the balanced bistable type: there are two zeros $\alpha < \gamma$ of f such that $f'(\alpha) < 0$, $f'(\gamma) < 0$, and the function $F(u) = \int_0^u f(s)\,ds$ satisfies

$$F(u) < F(\alpha) = F(\gamma) \quad (u \in (\alpha, \gamma)). \tag{1.5}$$

This is obviously not a robust condition: it is easily broken by an arbitrarily small perturbation of f. Thus our next concern is whether bounded solutions which are not quasiconvergent occur only for a meager class of nonlinearities, or, in other words, whether the quasiconvergence of all bounded solutions is a generic property of the nonlinearity f. We answer this question in the negative as well. More specifically, we prove that there is a function f such that (1.1) has a bounded solution which is not quasiconvergent, and the same is true for any small C^1 perturbation of f, see Theorem 2.3 and the remarks following condition (C2) in Section 2.

In this paper, our main goal has been to give examples of bounded solutions which are not quasiconvergent; one with an initial value which has a finite limit as $|x| \to \infty$ and another one with a robust nonlinearity. We do not give a detailed description of the ω-limit sets of these and or other bounded solutions in general. As mentioned above, the ω-limit set always consists of bounded entire solutions. There is a vast variety of entire solutions, including spatially periodic heteroclinic orbits between steady states (see [17, 18] and references therein), traveling waves, and many types of "nonlinear superpositions" of traveling waves and other entire solutions (see [4, 5, 24, 26, 31, 32] and references therein). It is not clear which of these entire solutions can actually occur in the ω-limit set of a bounded solution.

We emphasize that having defined the ω-limit set with respect to the convergence in $L^\infty_{loc}(\mathbb{R})$, we examine the behavior of solutions in compact sets only. This captures one aspect of the behavior of the solutions. Additional information can be gathered by considering moving coordinate frames. Several classes of solutions have been examined from this point of view. For example, the classical papers [19, 20] contain fundamental theorems on convergence to a single traveling front. Examples of solutions approaching a family of traveling waves, while not converging to any single one, can be found in [25, 27, 37, 39].

Let us add a few remarks on related results for the higher-dimensional version of (1.1):

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{1.6}$$

We define the ω-limit set of a bounded solution u as in (1.3), with convergence in $L^\infty_{loc}(\mathbb{R}^N)$. The result of [22, 23] stating that $\omega(u)$ always contains an equilibrium remains valid if $N = 2$, but it is not known if it is valid for $N > 2$. There are also several convergence results concerning positive bounded solutions for nonlinearities satisfying $f(0) = 0$. Under the additional condition $f'(0) < 0$, the convergence in $L^\infty(\mathbb{R}^N)$ for bounded solutions in the energy space was proved in [3, 9, 15] and for localized solutions in [21]. The convergence in $L^\infty_{loc}(\mathbb{R}^N)$ for solutions with compact initial support was established in [11]. For initial data which do not have compact support, bounded positive solutions, even the localized ones, can behave in a much more complicated manner. For $N \geq 11$ and $f(u) = u^p$ with a sufficiently large p, examples of nonconvergent localized solutions were given in [34, 35]. The ω-limit sets of these solutions are formed by radially symmetric equilibria and their translations. In the more recent paper [36], positive, bounded, localized solutions which are not even quasiconvergent are found for any $N \geq 3$ and $f(u) = u^p$ with a suitable exponent p. It is not known if such solution exist for some nonlinearity f if $N = 2$. They do not exist if $N = 1$ (see [30]).

The remainder of the paper is organized as follows. We formulate our main theorems in the next section. Section 3, contains basic ingredients of our proofs: intersection comparison properties, continuous dependence in $L^\infty_{loc}(\mathbb{R})$ of solutions on their initial data, and some special solutions. The proofs of our main theorems are given is Sections 5 and 6. They both follow a similar scheme. The initial value for a solution is constructed recursively on an increasing sequence of intervals covering the whole of $\mathbb{R}$. The definition of u_0 on any of these intervals, say I, guarantees a certain behavior of the solution on a large time interval, regardless of the values of u_0 outside I.

This is where the continuous dependence on initial data in $L^\infty_{loc}(\mathbb{R})$ plays a crucial role. In Section 4, we revisit the example of [12]. As an illustration of our method in a simpler setting, we prove the existence of a bounded non-quasiconvergent solution similar to the one in [12].

2 Main results

In our first theorem, we consider a balanced bistable nonlinearity f. More specifically, f is assumed to satisfy the following conditions.

(C1) For some $\alpha < 0 < \gamma$ one has $f(\alpha) = f(0) = f(\gamma) = 0$, $f'(\alpha) < 0$, $f'(\gamma) < 0$, $f < 0$ in $(\alpha, 0)$, $f > 0$ in $(0, \gamma)$, and

$$F(\alpha) = F(\gamma). \tag{2.1}$$

Here and below,

$$F(u) = \int_0^u f(s)\,ds.$$

Note that (2.1), in conjunction with the other conditions in (C1), implies (1.5).

Recall that $C_0(\mathbb{R})$ stands for the space of continuous functions on $\mathbb{R}$ converging to 0 at $x = \pm\infty$.

Theorem 2.1. *Let f be a C^1 function satisfying* (C1)*. Then there exists a function $u_0 \in C_0(\mathbb{R})$ with $\alpha \le u_0 \le \gamma$ such that the ω-limit set of the solution of* (1.1), (1.2) *contains the equilibria α, γ, as well as some functions which are not equilibria of* (1.1)*.*

Note that, by the comparison principle, the relations $\alpha \le u_0 \le \gamma$ imply that the solution u satisfies $\alpha \le u(\cdot, t) \le \gamma$ for all t. In particular, the solution is bounded.

Remark 2.2. One can also find initial data $u_0 \in C_0(\mathbb{R})$ such that $\omega(u)$ contains an equilibrium which is increasing in x, another one which is decreasing in x, as well as some functions which are not equilibria of (1.1). We elaborate on this at the end of Section 5.

In our second theorem, the following conditions are assumed.

(C2) For some $\alpha < \beta < \gamma$ one has $f(\alpha) = f(\beta) = f(\gamma) = 0$, $f'(\alpha) < 0$, $f'(\gamma) < 0$, $f < 0$ in (α, β), $f > 0$ in (β, γ), and

$$F(\gamma) > F(\alpha). \tag{2.2}$$

Here, too, f is of the bistable type. However, unlike (2.1), condition (2.2) is robust. Consequently, if f satisfies (C2) and in addition $f'(\beta) > 0$, then any any small C^1 perturbation $\tilde{f}$ of f satisfies (C2) with some perturbed zeros $\tilde{\alpha} \approx \alpha$, $\tilde{\beta} \approx \beta$, $\tilde{\gamma} \approx \gamma$.

It is well known (and easily proved by an elementary phase plane analysis, cp. Section 3) that if f satisfies (C2), then the equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}, \tag{2.3}$$

has a solution v such that $v > \alpha$ and $v - \alpha \in C_0(\mathbb{R})$. We refer to any such solution as a *ground state* of (2.3) at level α. In one space dimension considered here, the ground state is unique up to translations [2]. Moreover, if its point of maximum is placed at the origin, then it is even in x and decreasing with increasing $|x|$.

Theorem 2.3. *Let f be a C^1 function satisfying* (C2)*. Then there exists $u_0 \in C(\mathbb{R})$ with $\alpha \le u_0 \le \gamma$ such that the ω-limit set of the solution of* (1.1), (1.2) *contains the equilibrium α, a ground state of* (2.3) *at level α, and some functions which are not equilibria of* (1.1).

The proofs of the theorems will be given below after some preparations.

We shall frequently use the following notation. By $u(x, t, u_0)$ we denote the (maximally defined) classical solution of (1.1), (1.2). We use the following abbreviated notation

$$\omega(u_0) := \omega(u(\cdot, \cdot, u_0))$$

for its ω-limit set (if the solution is bounded). As a rule, we take the ω-limit set with respect to the topology of $L^\infty_{loc}(\mathbb{R})$. However, if the trajectory $\{u(\cdot, , u_0) : t \ge 1\}$ happens to be relatively compact in $L^\infty(\mathbb{R})$, then $\omega(u_0)$ is also the ω-limit set with respect to the topology of $L^\infty(\mathbb{R})$, that is, the uniform convergence on $\mathbb{R}$ can be assumed in (1.3).

If $\alpha < \gamma$ are two zeros of f, we denote by $\mathcal{B}_{\alpha,\gamma}$ the space all continuous functions on $\mathbb{R}$ taking values in $[\alpha, \gamma]$. We equip $\mathcal{B}_{\alpha,\gamma}$ with the metric given by the weighted sup norm

$$\|v\|_w \equiv \sup_{x \in \mathbb{R}} w(x)|v(x)|, \tag{2.4}$$

where $w(x) := 1/(1+|x|^2)$. The topology on $\mathcal{B}_{\alpha,\gamma}$ generated by this metric is the same as the topology induced from $L^\infty_{loc}(\mathbb{R})$.

3 Preliminaries

We start this section with a brief discussion of equilibria of (1.1). Then we exhibit some special time-dependent solutions which are used in our constructions. Other technical tools recalled in the section include the continuity in L^∞_{loc} of solutions with respect to their initial data and the zero number functional.

Throughout the section, we assume that $f \in C^1(\mathbb{R})$.

3.1 Equilibria

The equilibria of (1.1) are solutions of the equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}. \tag{3.1}$$

The first-order system associated with (3.1),

$$v_x = w, \quad w_x = -f(v), \tag{3.2}$$

is a Hamiltonian system with respect to the energy

$$H(v,w) := w^2/2 + F(v).$$

Thus the trajectories of (3.2) are contained in the level sets of H. Note that these level sets are symmetric about the v axis. The following results are all well known and easily proved by phase plane analysis of system (3.2).

System (3.2) has only four types of orbits: equilibria (stationary solutions), all of them on the v axis, nonconstant periodic orbits, homoclinic orbits, and heteroclinic orbits. If v is a periodic nonconstant solution of (3.1), then it is even about each of its critical points. If v is a solution of (3.1) corresponding to a heteroclinic orbit of (3.2), then $|v_x| > 0$ on $\mathbb{R}$. If v is a solution of (3.2) corresponding to a homoclinic orbit of (3.2), then v has a unique critical point a and is symmetric about a.

For bistable nonlinearities, the following lemmas describe the equilibria of (1.1) in $\mathcal{B}_{\alpha,\gamma}$ (see Figure 1).

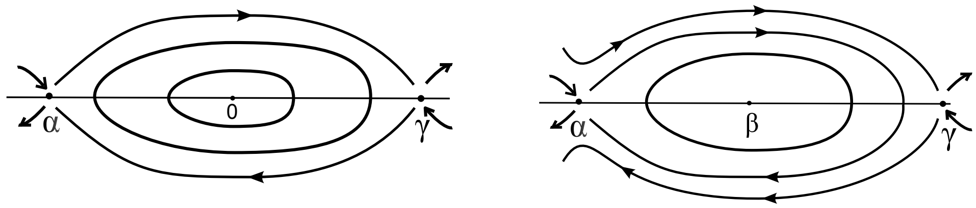

Figure 1: The phase diagrams of system (3.2) in the bistable case; balanced on the left, unbalanced on the right

Lemma 3.1. *Assume* (C1). *If* $v \in \mathcal{B}_{\alpha,\gamma}$ *is a solution of* (3.1)*, then it is a periodic solution or a monotone solution corresponding to a heteroclinic orbit of* (3.2) *between the equilibria* $(\alpha, 0)$ *and* $(\gamma, 0)$*. For each* $\xi \in (\alpha, 0) \cup (0, \gamma)$*, the solution* v *of* (3.1) *with* $v(0) = \xi$*,* $v'(0) = 0$ *is a nonconstant periodic, even function with infinitely many zeros, and* $v, v' \to 0$ *in* $L^\infty(\mathbb{R})$*, as* $\xi \to 0$.

Of course, since $f(0) = 0$, the zeros of nonconstant periodic solutions must all be simple by uniqueness for the initial value problem. The same applies to the zeros of the function $v - \beta$ in the next lemma.

Lemma 3.2. *Assume* (C2). *If* $v \in \mathcal{B}_{\alpha,\gamma}$ *is a solution of* (3.1)*, then it is a periodic solution or a solution corresponding to a homoclinic orbit of* (3.2) *to the equilibrium* $(\alpha, 0)$*. For each* $\xi \in (\alpha, \beta) \cup (\beta, \gamma)$ *sufficiently close to* β*, the solution* v *of* (3.1) *with* $v(0) = \xi$*,* $v'(0) = 0$ *is a nonconstant periodic, even function,* $v - \beta$ *has infinitely many zeros, and* $v, v' \to \beta$ *in* $L^\infty(\mathbb{R})$*, as* $\xi \to \beta$.

3.2 Some time-dependent solutions

In this subsection, we deal with solutions whose initial values are identical to a constant outside a compact interval.

Lemma 3.3. *Let* (C1) *or* (C2) *hold. Assume that* $u_0 \in \mathcal{B}_{\alpha,\gamma}$ *and* $\operatorname{spt}(u_0 - \eta)$ *is compact for some* $\eta \in [\alpha, \beta) \cup (\beta, \gamma]$*, where* $\beta = 0$ *in the case of* (C1)*. Let* $u := u(\cdot, \cdot, u_0)$*. Then for each finite* $T > 0$ *one has*

$$\limsup_{|x|\to\infty,\, t\in[0,T]} u(x,t) \le \eta \quad (\textit{if } \eta \in [\alpha, \beta)), \tag{3.3a}$$

$$\liminf_{|x|\to\infty,\, t\in[0,T]} u(x,t) \ge \eta \quad (\textit{if } \eta \in (\beta, \gamma]). \tag{3.3b}$$

Note that since $\alpha \le u(x,t) \le \gamma$ for all $t \ge 0$, in case $\eta \in \{\alpha, \gamma\}$ relations (3.3) mean that

$$u(x,t) \to \eta, \text{ as } |x| \to \infty, \text{ uniformly in } t \in [0,T]. \tag{3.4}$$

Proof of Lemma 3.3. For $\eta = \alpha$ a proof of (3.4) can be found in [10, Lemma 2.2]. We treat the case $\eta \in [\alpha, \beta)$ by similar arguments. To derive an upper estimate on u, we may assume that $u_0 \ge \eta$ (otherwise, we replace $u_0(x)$ with $\max\{u_0(x), \eta\}$ and use the comparison principle). Let $y(t)$ be the solution of $\dot{y} = f(y)$ with $y(0) = \eta$. By the comparison principle, the function $v(x,t) := u(x,t) - y(t)$ is nonnegative. Since u is bounded, we have $f(u(x,t)) - f(y(t)) \le Mv(x,t)$ for some constant $M \ge 0$. Hence, by comparison,

$$v(x,t) \le e^{Mt}\bar{v}(x,t) \tag{3.5}$$

where $\bar{v}$ is the solution of $\bar{v}_t = \bar{v}_{xx}$ with $\bar{v}(\cdot,0) = v(\cdot,0) = u_0 - \eta$. Since $\bar{v}(\cdot,0)$ has compact support, we have $\bar{v}(x,t) \to 0$, as $|x| \to \infty$, uniformly in $t \in [0,T]$. This and (3.5) give

$$\limsup_{|x|\to\infty,\, t\in[0,T]} (u(x,t) - y(t)) \le 0. \tag{3.6}$$

Since $y(t) \le y(0) = \eta$ (as $f \le 0$ in $[\alpha, \beta)$), (3.3b) follows from (3.6). The case $\eta \in (\beta, \gamma]$ is analogous. $\square$

Lemma 3.4. *Assume* (C1). *If* $u_0 \in \mathcal{B}_{\alpha,\gamma}$, $\eta \in [\alpha, 0) \cup (0, \gamma]$, *and* $\mathrm{spt}(u_0 - \eta)$ *is compact, then*

$$\lim_{t\to\infty} \|u(\cdot,\cdot,u_0) - \xi\|_{L^\infty(\mathbb{R})} = 0, \tag{3.7}$$

where $\xi = \alpha$ *if* $\eta \in [\alpha, 0)$ *and* $\xi = \gamma$ *if* $\eta \in (0, \gamma]$.

Proof. We only treat the case $\eta \in [\alpha, 0)$, the case $\eta \in (0, \gamma]$ is analogous. We derive the result from [19].

Take continuous functions $u_0^+, u_0^- \in \mathcal{B}_{\alpha,\gamma}$ such that $u_0^\pm \ge u_0$, and

$$u_0^+(-\infty) = \eta < \gamma = u_0^+(\infty),$$
$$u_0^-(-\infty) = \gamma > \eta = u_0^+(\infty).$$

Set $u^\pm := u(\cdot,\cdot,u_0^\pm)$. By comparison, $u^\pm \ge u(\cdot,\cdot,u_0)$. Theorem 3.1 of [19] shows that, as $t \to \infty$, $u^\pm(\cdot,t) \to \phi^\pm$ in $L^\infty(\mathbb{R})$, where ϕ^+ is an increasing solution (a standing wave) of (2.3) with $\phi^+(-\infty) = \alpha$, $\phi^+(\infty) = \gamma$; and

$\phi^-(x) = \phi^+(x_0 - x)$ for some x_0. (To get the convergence result for u^-, one applies Theorem 3.1 of [19] to the solution $u^-(-x,t)$.) From this, we obtain two conclusions. First, by the previous comparison and convergence properties, the following holds.

Given $\varepsilon > 0$, there are $\tau > 0$ and $R > 0$ such that

$$0 \le u(x,t,u_0) - \alpha \le \varepsilon \quad (|x| > R, t > \tau). \tag{3.8}$$

Second, for each $w \in \omega(u_0)$ one has

$$w(x) \le \psi(x) := \min\{\phi^+(x), \phi^-(x)\}. \tag{3.9}$$

Now, ψ is a time-independent supersolution of (1.1), and, as $t \to \infty$, the solution $u(\cdot,t,\psi)$ converges monotonically to α (the only equilibrium between α and ψ). We now use the well-known invariance property of $\omega(u_0)$: for any $t > 0$ and $w \in \omega(u_0)$, one has $w = u(\cdot,t,\tilde{w})$ for some $\tilde{w} \in \omega(u_0)$. Since $\tilde{w} \le \psi$, the comparison principle gives $w \le u(\cdot,t,\psi)$. Since $t > 0$ is arbitrary, we have $w = \alpha$, proving that $\omega(u_0) = \{\alpha\}$. Thus $u(\cdot,t,u_0) \to \alpha$, as $t \to \infty$, in $L^\infty_{loc}(\mathbb{R})$. Using this in conjunction with (3.8), we see that the convergence also takes place in $L^\infty(\mathbb{R})$, which proves (3.7) with $\xi = \alpha$. $\square$

3.3 Continuity with respect to initial data

Fix two zeros $\alpha < \gamma$ of f and set $\mathcal{B} := \mathcal{B}_{\alpha,\gamma}$. In the following lemma, we employ the norm defined in (2.4).

Lemma 3.5. *Given any $T > 0$ and any two solutions u, $\tilde{u}$ of (1.1) with $u(\cdot,0), \tilde{u}(\cdot,0) \in \mathcal{B}$, one has*

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_w \le L(T)\|u(\cdot,0) - \tilde{u}(\cdot,0)\|_w \quad (t \in [0,T]), \tag{3.10}$$

where $L(T)$ is a constant depending on T (and on α, β, f), but not on the solutions.

This continuity result is proved easily by considering the linear parabolic equation satisfied by $v(x,t) := w(x)(u(x,t) - \tilde{u}(x,t))$, where w is as in (2.4). As one verifies by a simple computation, the linear equation has bounded coefficients, hence (3.10) follows by standard parabolic estimates (see [16, Lemma 6.2] for details).

Corollary 3.6. *Given any $u_0 \in \mathcal{B}$, $T > t_0 > 0$, $R > 0$, and $\epsilon > 0$, there exist $\rho \geq R$ and $\delta > 0$ with the following property. If $\tilde{u}_0 \in \mathcal{B}$ satisfies*

$$\sup_{x \in [-\rho, \rho]} |u_0(x) - \tilde{u}_0(x)| < \delta, \tag{3.11}$$

then

$$\sup_{x \in [-R,R],\ t \in [0,T]} |u(x,t,u_0) - u(x,t,\tilde{u}_0)| < \epsilon, \tag{3.12}$$

$$\sup_{x \in [-R,R],\ t \in [t_0,T]} |u_x(x,t,u_0) - u_x(x,t,\tilde{u}_0)| < \epsilon, \tag{3.13}$$

$$\sup_{x \in [-R,R],\ t \in [t_0,T]} |u_t(x,t,u_0) - u_t(x,t,\tilde{u}_0)| < \epsilon. \tag{3.14}$$

Proof. Let us ignore (3.13), (3.14) for a while. The statement then follows directly from Lemma 3.5 and the fact that by choosing ρ sufficiently large and $\delta > 0$ sufficiently small, one can make $\|u_0 - \tilde{u}_0\|_w$ arbitrarily small (regardless of the values of $\tilde{u}_0(x)$ for $|x| > \rho$, as long as $\tilde{u}_0 \in \mathcal{B}$).

To prove that (3.13), (3.14) hold as well, possibly with smaller $\delta > 0$ and larger ρ, one uses the statement just proved with R replaced by $R + 1$ and then applies standard parabolic regularity estimates [28]. □

3.4 Zero number

If $v = u - \tilde{u}$ or $v = u_t$, where u, $\tilde{u}$ are global solutions of (1.1), then v is a solution of a linear equation

$$v_t = v_{xx} + c(x,t)v, \quad x \in \mathbb{R},\ t > 0, \tag{3.15}$$

where c is a continuous function on $\mathbb{R} \times [0, \infty)$. Specifically,

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(\tilde{u}(x,t) - u(x,t)))\, ds$$

if $v = u - \tilde{u}$, and $c(x,t) = f'(u(x,t))$ if $v = u_t$. We will use properties of the zero-number functional for solutions of such linear equations.

For an interval $I = (a,b)$, with $-\infty \leq a < b \leq \infty$, we define $z_I(v(\cdot,t))$ as the number of zeros, possibly infinite, of the function $x \to v(x,t)$ in I. If $I = \mathbb{R}$, we usually omit the subscript I:

$$z(v(\cdot,t)) := z_{\mathbb{R}}(v(\cdot,t)).$$

The following intersection-comparison principle holds (see [1, 6]).

Lemma 3.7. *Let v be a solution of* (3.15)*. Assume that for some interval $[\tau, T) \subset [0, \infty)$ the following conditions are satisfied:*

(c1) *if $b < \infty$, then $v(b,t) \neq 0$ for all $t \in [\tau, T)$,*

(c2) *if $a > -\infty$, then $v(a,t) \neq 0$ for all $t \in [\tau, T)$.*

Then the following statements hold true:

(i) *If $a > -\infty$ and $b < \infty$, then $z_I(v(\cdot, t)) < \infty$ for all $t \in (\tau, T)$.*

(ii) *$t \mapsto z_I(v(\cdot, t))$ is a monotone nonincreasing function on $[\tau, T)$ with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (\tau, T)$, the function $v(\cdot, t_0)$ has a multiple zero in I and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in [\tau, T)$ with $t_1 < t_0 < t_2$ one has*

$$z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_0)) \geq z_I(v(\cdot, t_2)). \tag{3.16}$$

If (3.16) holds, we say that $z_I(v(\cdot, t))$ drops in the interval (t_1, t_2) or that $v(\cdot, t)$ drops a zero in the time interval (t_1, t_2). If this holds for all t_1, t_2 with $t_1 < t_0 < t_2$, we also say that $z_I(v(\cdot, t))$ drops at t_0.

Remark 3.8. It is clear that if the assumptions of Lemma 3.7 are satisfied and $z_I(v(\cdot, \tau)) < \infty$, then $z_I(v(\cdot, t))$ can drop at most finitely many times in (τ, T), and if it is constant on an interval $[\tau_1, \tau_2] \subset (\tau, T)$, then $v(\cdot, t)$ has only simple zeros in I for each $t \in (\tau_1, \tau_2]$.

4 Initial data with large oscillations in the balanced bistable case

In this section, we assume that f satisfies (C1) (see Section 2) and let $\mathcal{B} := \mathcal{B}_{\alpha,\gamma}$.

As already noted above, if $u_0 \in \mathcal{B}$, then the corresponding solution satisfies $u(\cdot, t, u_0) \in \mathcal{B}$ for all $t \geq 0$. In particular, the solution is bounded.

The next proposition yields a solution oscillating between the constant equilibria α and γ, as $t \to \infty$. The existence of such a solution is known by [12], we prove it as an illustration of our techniques in a simple setting and provide additional information on the solution (see Proposition 4.1 and Remark 4.3 below).

Proposition 4.1. *There exists $u_0 \in \mathcal{B}$ such that $\omega(u_0)$ contains the constant equilibria α, γ and no other equilibria.*

In the following lemma, which details our recursive construction, we set

$$d := \min\{|\alpha|, \gamma\}$$

and let $\psi \not\equiv 0$ be a periodic solution of (2.3) with

$$|\psi|, |\psi'| < d/2 \tag{4.1}$$

(cp. Lemma 3.1).

Lemma 4.2. *There exist $t_0 \in (0,1)$ and a sequence $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots,$ in $\mathcal{B} \times (0,\infty)^4$ such that the statements* (i)-(iv) *below are valid for all $k = 1, 2, \dots,$ and statements* (v), (vi) *are valid for all $k = 2, 3 \dots$.*

(i) $t_k > 1$, $\rho_k > R_k > 2$,

(ii) *u_k is piecewise linear, even, and $\operatorname{spt}(u_k - \eta_k) \subset (-R_k, R_k)$, where*

$$\eta_k := \begin{cases} \alpha, & \text{if } k \text{ is odd,} \\ \gamma, & \text{if } k \text{ is even.} \end{cases}$$

(iii) *The solution $u(\cdot,\cdot,u_k)$ satisfies the following relations for both $\varphi \equiv 0$ and $\varphi \equiv \psi$:*

$$|u(x,t,u_k) - \varphi(x)| > 0 \quad (x \in \mathbb{R} \setminus (-R_k, R_k),\ t \in [0, t_k]), \tag{4.2}$$

$$\|u(\cdot, t_k, u_k) - \eta_k\|_{L^\infty(\mathbb{R})} < \frac{1}{k}, \tag{4.3}$$

$$u(\cdot, t_{k-1}, u_k) - \varphi \ \text{ has exactly two zeros, both simple,} \tag{4.4}$$

$$(-1)^k (u(x, t_k, u_k) - \varphi(x)) > 0 \quad (x \in \mathbb{R}). \tag{4.5}$$

(iv) *For each $u_0 \in \mathcal{B}$ with $\|u_k - u_0\|_{L^\infty(-\rho_k, \rho_k)} < \delta_k$, the following relations hold for both $\varphi \equiv 0$ and $\varphi \equiv \psi$:*

$$|u(x,t,u_0) - \varphi(x)| > 0 \quad (x = \pm R_k,\ t \in [0, t_k]), \tag{4.6}$$

$$\|u(\cdot, t_k, u_0) - \eta_k\|_{L^\infty(-R_k, R_k)} < \frac{2}{k}, \tag{4.7}$$

$$u(\cdot, t_{k-1}, u_0) - \varphi \ \text{ has exactly two zeros in } (-R_k, R_k), \text{ both simple,} \tag{4.8}$$

$$z_{(-R_k, R_k)}(u(\cdot, t, u_0) - \varphi) \le 2 \quad (t \in [t_{k-1}, t_k]), \tag{4.9}$$

$$(-1)^k (u(x, t_k, u_0) - \varphi(x)) > 0 \quad (x \in [-R_k, R_k]). \tag{4.10}$$

(v) $t_k > t_{k-1} + 1,\ R_k > \rho_{k-1} + 1$.

(vi) $u_k \equiv u_{k-1}$ *on* $[-\rho_{k-1}, \rho_{k-1}]$.

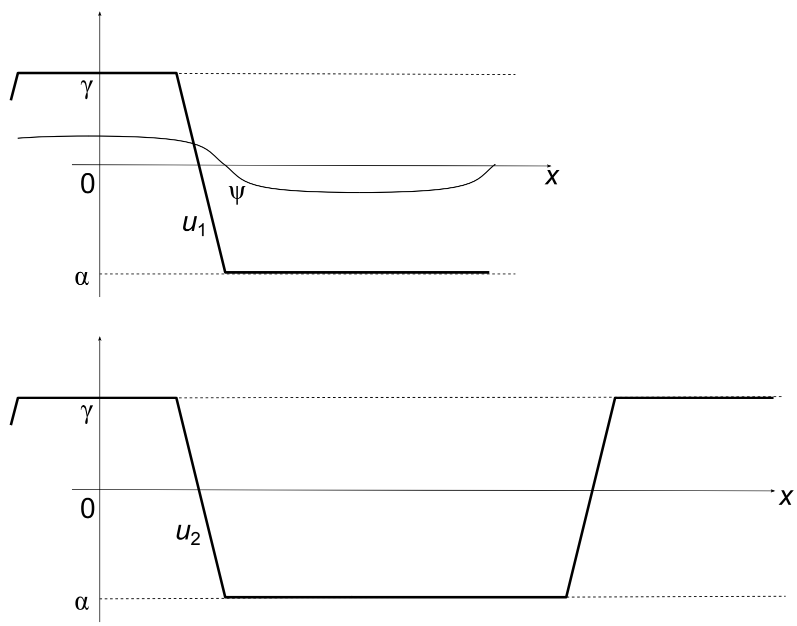

Figure 2: The graphs of u_1, ψ (top figure), and u_2 (bottom figure)

Proof. STEP 1. Set $k = 1$ and define an even piecewise linear function u_1 as follows (see Figure 2):

$$u_1(x) = \begin{cases} \gamma & (x \in [0,1]), \\ (\gamma - \alpha)(1 - x) + \gamma & (x \in [1,2]), \\ \alpha & (x \geq 2), \\ u_1(-x) & (x < 0). \end{cases}$$

Obviously, $u_1 \in \mathcal{B}$ and statement (ii) holds, provided $R_1 > 2$. (A specific choice of $R_1 > 2$ will be made below.) By Lemma 3.4,

$$\lim_{t\to\infty} \|u(\cdot, \cdot, u_1) - \alpha\|_{L^\infty(\mathbb{R})} = 0. \tag{4.11}$$

In the following, $\varphi \equiv 0$ or $\varphi \equiv \psi$. In either case,

$$|\varphi|, |\varphi'| < d/2 < \frac{1}{2} \min\{|\alpha|, \gamma\}. \tag{4.12}$$

Therefore, (4.11) implies that for each sufficiently large $t_1 > 1$ relations (4.3) and (4.5) hold. We fix such t_1, and then use Lemma 3.3 to pick $R_1 > 2$ so large that (4.2) holds. Using the definition of u_1 and (4.12), and comparing the slopes of u_1 and φ, we see that $z(u(\cdot,t,u_1)-\varphi) \geq 2 = z(u(\cdot,0,u_1)-\varphi)$ for all small $t \geq 0$. Hence, $z(u(\cdot,t,u_k)-\varphi) = 2$ for all small $t \geq 0$, by the monotonicity of the zero number (see Lemma 3.7). Therefore, by Remark 3.8, there is $t_0 \in (0,1)$ such that (4.4) holds.

In view of (4.2)-(4.5), Corollary 3.6 clearly implies the existence of constants $\rho_1 > R_1$ and $\delta_1 > 0$ such that statement (iv) holds, possibly with the exception of (4.9). Relation (4.9) follows from (4.8) by an application of Lemma 3.7(ii), which is legitimate by (4.6).

STEP 2 (the induction argument). Suppose that for some $n \geq 1$,

$$(u_k, R_k, \rho_k, t_k, \delta_k) \in \mathcal{B} \times (0,\infty)^4, \quad k = 1, \dots, n,$$

have been defined such that statements (i)-(iv) hold for all $k = 1, \dots, n$, and, in case $n \geq 2$, statement (v), (vi) hold for all $k = 2, \dots, n$. We need to define $(u_{n+1}, R_{n+1}, \rho_{n+1}, t_{n+1}, \delta_{n+1})$ in such a way that statements (i)-(vi) are valid for $k = n+1$. We give the definition assuming n is odd; the case of n even is analogous.

Set $k = n+1$ and define an even piecewise linear function u_{n+1} as follows (see Figure 2):

$$u_{n+1}(x) = \begin{cases} u_n(x) & (x \in [0,\rho_n]), \\ (\gamma-\alpha)(x-\rho_n)+\alpha & (x \in [\rho_n, \rho_n+1]), \\ \gamma & (x \geq \rho_n+1), \\ u_{n+1}(-x) & (x < 0). \end{cases}$$

As $\operatorname{spt}(u_n - \alpha) \subset (-\rho_n, \rho_n)$, one has $u_n \in \mathcal{B}$ and $\operatorname{spt}(u_{n+1}-\gamma) \subset (-\rho_n - 1, \rho_n+1)$. Thus a choice of $R_{n+1} > \rho_n + 1$, to be made specific below, will guarantee that (ii) holds. As in STEP 1, Lemma 3.4 implies that for each sufficiently large $t_{n+1} > t_n$ relations (4.3), (4.5) hold. Fix such t_{n+1}.

Since $u_{n+1} \equiv u_n$ on $[-\rho_n, \rho_n]$, statement (iv) with $k = n$ applies to $u_0 = u_{n+1}$. Thus, (4.10), (4.6) give

$$u(x, t_n, u_{n+1}) - \varphi(x) < 0 \quad (x \in [-R_n, R_n]), \tag{4.13}$$

$$u(\pm R_n, t, u_{n+1}) - \varphi(\pm R_n) < 0 \quad (t \in [0, t_n]). \tag{4.14}$$

We also know, by Lemma 3.3, that $u(\infty, t, u_{n+1}) = \gamma > \varphi$, so $u(\cdot, t, u_{n+1}) - \varphi$ has at least one zero in (R_n, ∞) for all $t \in [0, t_n]$. By the definition of u_{n+1} and (4.12), the zero is unique at $t = 0$, hence it is unique and simple for all $t \in [0, t_n]$, by virtue of Lemma 3.7 (the application of Lemma 3.7 on (R_n, ∞) is justified by (4.14)). This, combined with (4.13) and the evenness in x, implies that (4.4) holds. Using Lemma 3.3, we find $R_{n+1} > \rho_n + 1$ such that (4.2) holds.

Having verified (4.2)-(4.5), we use Corollary 3.6 to find $\rho_{n+1} > R_{n+1}$ and $\delta_{n+1} > 0$ such that statement (iv) holds (to verify (4.9) one uses 3.7(ii) as in STEP 1).

Relations (v) and (vi) hold by construction.

This completes STEP 2, and thereby the construction of the sequence $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots$, with the given properties. $\square$

Proof of Proposition 4.1. Let $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots$, be as in Lemma 4.2. Take any function $u_0 \in \mathcal{B}$ with

$$\|u_k - u_0\|_{L^\infty(-\rho_k, \rho_k)} < \delta_k \quad (k = 1, 2, \dots). \tag{4.15}$$

For example, we can define u_0 by

$$u_0(x) \equiv u_k(x) \quad (|x| \leq \rho_k, \ k = 1, 2, \dots),$$

which is legitimate by (vi). The function thus defined is in $\mathcal{B}$, as $u_k \in \mathcal{B}$ and $R_k \to \infty$ (see (i) and (v)).

By (4.15), statement (iv) applies to u_0 for each k. Since $t_k, R_k \to \infty$ by (i), (v), from (4.7) we obtain that the constant equilibria α, γ are contained in $\omega(u_0)$.

We now show that no other equilibrium is contained in $\omega(u_0)$. For that we use the following direct consequence of relations (4.9), (i), and (v): for each $b \in (0, \infty)$ there is $\tau_b > 0$ such that

$$z_{(-b,b)}(u(\cdot, t, u_0) - \varphi) \leq 2 \quad (t \geq \tau_b, \ \varphi \in \{\psi, 0\}). \tag{4.16}$$

We now go by contradiction. Assume $\phi \in \omega(u_0) \setminus \{\alpha, \gamma\}$ and ϕ is an equilibrium of (1.1). Obviously, $\phi \in \mathcal{B}$ and there is a sequence $s_k \to \infty$ such that $u(\cdot, s_k) \to \phi$ in $L^\infty_{loc}(\mathbb{R})$. Take $\varphi \equiv 0$ if $\phi \not\equiv 0$ and $\varphi \equiv \psi$ if $\phi \equiv 0$. In either case, by Lemma 3.1, $\varphi - \phi$ has infinitely many zeros, all of them simple. Consequently, if $b > 0$ is sufficiently large, then there is k_0 such that

$$z_{(-b,b)}(u(\cdot, s_k, u_0) - \varphi) \geq 3 \quad (k = k_0, k_0 + 1, \dots).$$

This clearly contradicts (4.16).

The proof is now complete. $\square$

Remark 4.3. (i) There is some flexibility in choosing u_0, see (4.15). In particular, u_0 can be chosen smooth.

(ii) Relations (4.8)-(4.10) show that in each time interval (t_{k-1}, t_k) the solution $u(\cdot, t, u_0)$ has initially two zeros in $(-R_k, R_k)$ (and a "hump" between them) and it looses both of them as t increases to t_k. This corresponds to the annihilation of kinks studied in more detail in [12, 38].

(iii) With a more careful construction, choosing the slopes of the nonconstant parts of the u_k sufficiently steep and controlling better how close $u(\cdot, \cdot, u_0)$ is to η_k on $(-R_k, R_k)\times\{t_k\}$ and $\{\pm R_k\}\times[0, t_k]$ one can achieve that the above statement concerning (4.16) is valid for each equilibrium $\varphi \in \mathcal{B} \setminus \{\alpha, \gamma\}$. Using this and the properties of the zero number, one shows easily that for each such equilibrium φ and any entire solution $q(\cdot, t)$ in $\omega(u_0)$ one has

$$z(q(\cdot, t) - \varphi) \leq 2 \quad (t \in \mathbb{R}). \tag{4.17}$$

This in turn implies that q is either one of the equilibria α, γ or a heteroclinic connection between these two equilibria (we sketch the argument for this below). Most likely, these heteroclinic solutions are the two-front entire solutions studied in detail in [5].

Let us indicate how it follows from (4.17) that q is a heteroclinic solution, if it is not an equilibrium. Determined readers will have no difficulty to fill in the details.

Assume that u_0 is as in the above construction and that (4.17) holds for each entire solution $q(\cdot, t)$ in $\omega(u_0)$. First note that u_0 being even, all elements of $\omega(u_0)$ are even. Also, by the strong comparison principle, $\alpha < q < \beta$. We next prove that $t \mapsto q(0, t)$ is a strictly monotone function on $\mathbb{R}$. If not, then $q(0, \cdot)$ has a local maximum or local maximum at some t_0. Set $\beta = q(0, t_0)$ and let φ be the equilibrium of (1.1) with $\varphi(0) = \beta$, $\varphi'(0) = 0$ (φ is an even periodic function, see Lemma 3.1). Then $v := q - \varphi$ is a solution of a linear equation, which is even in x and has a multiple zero at $x = 0$ for $t = t_0$. By Lemma 3.7, v drops a zero at $t = t_0$. By (4.17) and the evenness of v, we have

$$z(q(\cdot, t_1) - \varphi) = 2, \quad z(q(\cdot, t_2) - \varphi) = 0,$$

for any $t_1 < t_0 < t_2$. It is then easy to verify that $q(0,t_1) - \beta$ and $q(0,t_2) - \beta$ must have opposite signs, if t_1, t_2 are close to t_0, contradicting the fact that β is an extremal value of $q(0,\cdot)$.

Thus, the strict monotonicity of $t \mapsto q(0,t)$ has been established. In particular, the limits $\xi^\pm := q(0,\pm\infty) \in [\alpha,\gamma]$ exist. Obviously then, any function in $\omega(q)$ takes the value ξ^+ at $x = 0$, so for every entire solution $\tilde{q}$ in $\omega(q)$, one has $\tilde{q}(0,t) = \xi^+$ for all t. Since $\omega(q) \subset \omega(u_0)$, $\tilde{q}$ must be an equilibrium, by the strict monotonicity result just proved. Hence, we have showed that $\omega(q) \subset \{\alpha,\gamma\}$, for these are the only equilibria in $\omega(u_0)$. The connectedness of $\omega(q)$ gives $\omega(q) = \{\beta^+\}$, where $\beta^+ \in \{\alpha,\gamma\}$. In other words $q(\cdot,t) \to \beta^+$ in $L^\infty_{loc}(\mathbb{R})$, as $t \to \infty$. In a similar way, considering the α-limit set in place of the ω-limit set, one shows that $q(\cdot,t) \to \beta^- \in \{\alpha,\gamma\}$ in $L^\infty_{loc}(\mathbb{R})$, as $t \to -\infty$. In view of the strict monotonicity of $q(0,t)$, one has $\beta^- \neq \beta^+$. So q is a heteroclinic solution between the equilibria α and γ.

5 Proof of Theorem 2.1

As in the previous section, we assume that f satisfies (C1) and take $\mathcal{B} := \mathcal{B}_{\alpha,\gamma}$. Our goal here is to find $u_0 \in \mathcal{B} \cap C_0(\mathbb{R})$ such that $\omega(u_0)$ contains the constant equilibria α, γ, but it does not contain any nonzero equilibrium of (1.1). Since $\omega(u_0)$ is connected, the conclusion of Theorem 2.1 is then valid: $\omega(u_0)$ contains some functions which are not equilibria of (1.1).

Basic solutions for the construction here are solutions with initial data identical to a constant $\epsilon \in (\alpha,0) \cup (0,\gamma)$ outside large intervals. By Lemma 3.4, each such solution converges to one of the stable constant steady states α, γ.

Lemma 5.1. *Let ϵ_k, $k = 1,2,\dots$ be a sequence in $(\alpha,0) \cup (0,\gamma)$ such that $|\epsilon_k| \searrow 0$ and $(-1)^k\epsilon_k > 0$ ($k = 1,2,\dots$). There exist $t_0 \in (0,1)$ and a sequence $(u_k,R_k,\rho_k,t_k,\delta_k)$, $k = 1,2,\dots$, in $\mathcal{B} \times (0,\infty)^4$ such that the statements* (i)-(iv) *below are valid for all $k = 1,2,\dots$, and statements* (v), (vi) *are valid for all $k = 2,3\dots$.*

(i) $t_k > 1$, $\rho_k > R_k > 2$,

(ii) *u_k is piecewise linear, even, and* $\operatorname{spt}(u_k - \epsilon_k) \subset (-R_k, R_k)$.

(iii) *Setting*

$$\eta_k := \begin{cases} \alpha, & \textit{if } k \textit{ is odd,} \\ \gamma, & \textit{if } k \textit{ is even.} \end{cases}$$

the solution $u(\cdot,\cdot,u_k)$ satisfies the following relations:

$$|u(x,t,u_k)| > 0 \quad (x \in \mathbb{R} \setminus (-R_k, R_k),\ t \in [0,t_k]), \tag{5.1}$$

$$\|u(\cdot,t_k,u_k) - \eta_k\|_{L^\infty(\mathbb{R})} < \frac{1}{k}, \tag{5.2}$$

$$u(\cdot,t_{k-1},u_k) \textit{ has exactly two zeros, both simple,} \tag{5.3}$$

$$(-1)^k u(x,t_k,u_k) > 0 \quad (x \in \mathbb{R}). \tag{5.4}$$

(iv) *For each $u_0 \in \mathcal{B}$ with $\|u_k - u_0\|_{L^\infty(-\rho_k,\rho_k)} < \delta_k$, the following relations hold:*

$$|u(x,t,u_0)| > 0 \quad (x = \pm R_k,\ t \in [0,t_k]), \tag{5.5}$$

$$\|u(\cdot,t_k,u_0) - \eta_k\|_{L^\infty(-R_k,R_k)} < \frac{2}{k}, \tag{5.6}$$

$$u(\cdot,t_{k-1},u_0) \textit{ has exactly two zeros in } (-R_k,R_k), \textit{ both simple,} \tag{5.7}$$

$$z_{(-R_k,R_k)}(u(\cdot,t,u_0)) \le 2 \quad (t \in [t_{k-1},t_k]), \tag{5.8}$$

$$(-1)^k u(x,t_k,u_0) > 0 \quad (x \in [-R_k,R_k]). \tag{5.9}$$

(v) $t_k > t_{k-1} + 1,\ R_k > \rho_{k-1} + 1,\ \delta_{k+1} < \delta_k/2.$

(vi) $u_k \equiv u_{k-1}$ *on* $[-\rho_{k-1},\rho_{k-1}]$ *and* $|u_k| \le |\epsilon_{k-1}|$ *on* $\mathbb{R} \setminus [-\rho_{k-1},\rho_{k-1}]$.

The proof of this lemma follows a similar scheme as the proof of Lemma 4.2, with some modifications. Note that this time the constant “tails” of the functions u_k have to converge to zero because the resulting initial value u_0 (see (5.12) below) is to be in $C_0(\mathbb{R})$. This means that if $\varphi \in \mathcal{B}$ is a fixed nonconstant equilibrium, then $z(\varphi - u_k) = \infty$ for all sufficiently large k. For this reason and unlike Lemma 4.2, Lemma 5.1 does not provide any information on the zero number of $u(\cdot,\cdot,u_k) - \varphi$. Consequently, this construction does not rule out the possibility that $0 \in \omega(u_0)$.

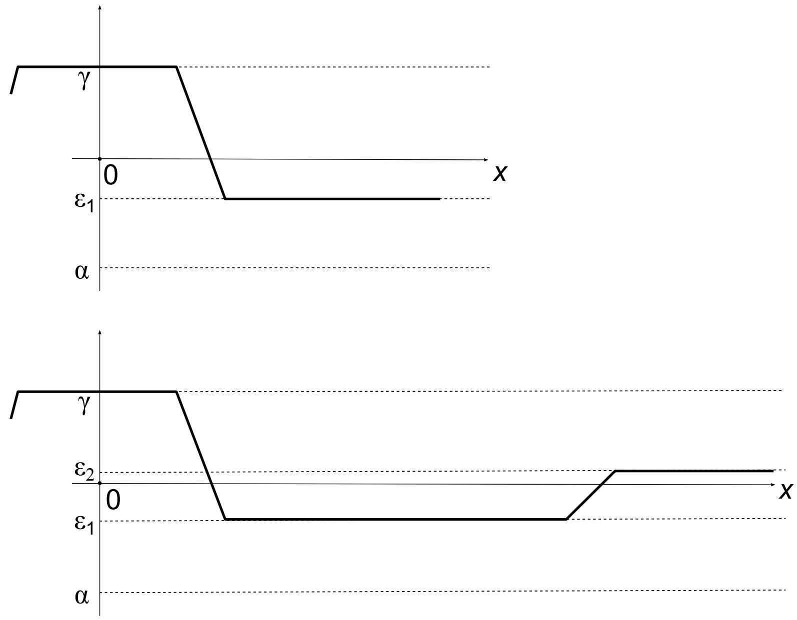

Figure 3: The graphs of u_1 (top figure) and u_2 (bottom figure)

Proof of Lemma 5.1. STEP 1. Set $k = 1$ and define a piecewise linear function u_1 as follows (see Figure 3):

$$u_1(x) = \begin{cases} \gamma & (x \in [0,1]), \\ (\gamma - \epsilon_1)(1-x) + \gamma & (x \in [1,2]), \\ \epsilon_1 & (x \geq 2), \\ u_1(-x) & (x < 0), \end{cases}$$

Obviously, $u_1 \in \mathcal{B}$ and any choice of $R_1 > 2$ will guarantee that statement (ii) holds. By Lemma 3.4,

$$\lim_{t\to\infty} \|u(\cdot, t, u_1) - \alpha\|_{L^\infty(\mathbb{R})} = 0,$$

hence there is $t_1 > 1$ such that relations (5.2), (5.4) hold. We fix such t_1, and then use Lemma 3.3 to pick $R_1 > 2$ so large that (5.1) holds. By the definition of u_1 and monotonicity of the zero number, $z(u(\cdot, t, u_1)) = 2 = z(u(\cdot, 0, u_1))$ for all small $t \geq 0$. Therefore, by Remark 3.8, there is $t_0 \in (0, 1)$ such that (5.3) holds.

In view of (5.1)-(5.4), Corollary 3.6 clearly implies the existence of constants $\rho_1 > R_1$ and $\delta_1 > 0$ such that statement (iv) holds with (5.8) excluded.

Relation (5.8) follows from (5.7) by an application of Lemma 3.7(ii), which is legitimate by (5.5).

STEP 2 (the induction argument). Suppose that for some $n \geq 1$,

$$(u_k, R_k, \rho_k, t_k, \delta_k) \in \mathcal{B} \times (0, \infty)^4, \quad k = 1, \dots, n,$$

have been defined such that statements (i)-(iv) hold for all $k = 1, \dots, n$, and, in case $n \geq 2$, statements (v), (vi) hold for all $k = 2, \dots, n$. Assuming n is odd, we define $(u_{n+1}, R_{n+1}, \rho_{n+1}, t_{n+1}, \delta_{n+1})$ in such a way that statements (i)-(vi) are valid for $k = n + 1$ (the case of n even is analogous).

Set $k = n+1$ and define an even piecewise linear function u_{n+1} as follows (see Figure 3):

$$u_{n+1}(x) = \begin{cases} u_n(x) & (x \in [0, \rho_n]), \\ (\epsilon_{n+1} - \epsilon_n)(x - \rho_n) + \epsilon_n & (x \in [\rho_n, \rho_n + 1]), \\ \epsilon_{n+1} & (x \geq \rho_n + 1), \\ u_{n+1}(-x) & (x < 0). \end{cases}$$

Since $\mathrm{spt}(u_n - \epsilon_n) \subset (-R_n, R_n) \subset (-\rho_n, \rho_n)$, we have $u_{n+1} \in \mathcal{B}$. Below, we choose $R_{n+1} > \rho_n + 1$, which makes (ii) valid. As in STEP 1, Lemma 3.4 implies that for each sufficiently large $t_{n+1} > t_n + 1$ relations (5.2), (5.4) hold. Fix such t_{n+1}.

Since $u_{n+1} \equiv u_n$ on $[-\rho_n, \rho_n]$, statement (iv) with $k = n$ applies to $u_0 = u_{n+1}$. Thus, (5.9), (5.5) give

$$u(x, t_n, u_{n+1}) < 0 \quad (x \in [-R_n, R_n]), \tag{5.10}$$

$$u(\pm R_n, t, u_{n+1}) < 0 \quad (t \in [0, t_n]). \tag{5.11}$$

This and Lemma 3.3 imply that $u(\cdot, t, u_{n+1})$ has at least one zero in (R_n, ∞) for all $t \in [0, t_n]$. By the definition of u_{n+1}, the zero is unique at $t = 0$, hence it is unique and simple for all $t \in [0, t_n]$, by virtue of Lemma 3.7 (the application of Lemma 3.7 on (R_n, ∞) is justified by (5.11)). This, combined with (5.10) and the evenness in x, implies that (5.3) holds (with $k = n + 1$). Using Lemma 3.3, we find $R_{n+1} > \rho_n + 1$ such that (5.1) holds.

Having verified (5.1)-(5.4), we use Corollary 3.6 to find $\rho_{n+1} > R_{n+1}$ and $\delta_{n+1} > 0$ such that statement (iv) holds (to verify (5.8) one uses Lemma 3.7(ii) as in STEP 1). Of course, making $\delta_{n+1} > 0$ smaller has no effect on this conclusion, thus we take $\delta_{n+1} < \delta_n/2$.

Relations (v) and (vi) hold by construction (and the fact that the sequence $|\epsilon_k|$ is decreasing).

This completes STEP 2, and thereby the construction of the sequence $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots$, with the given properties. $\square$

Completion of the proof of Theorem 2.1. With ϵ_k and $(u_k, R_k, \rho_k, t_k, \delta_k)$ as in Lemma 5.1, take any $u_0 \in \mathcal{B}$ with

$$\|u_k - u_0\|_{L^\infty(-\rho_k,\rho_k)} < \delta_k \quad (k = 1, 2, \dots). \tag{5.12}$$

An example is

$$u_0(x) \equiv u_k(x) \quad (|x| \le \rho_k,\ k = 1, 2, \dots),$$

which is a correctly defined function, in view of (vi), and it is in $\mathcal{B}$, as $u_k \in \mathcal{B}$. Clearly, since $\epsilon_n, \delta_n \to 0$ and $\rho_n \to \infty$ (see statements (i) and (v)), relations (5.12) and (vi) imply that $u_0 \in C_0(\mathbb{R})$.

By (5.12), statement (iv) applies to u_0 for each k. Since $R_k, t_k \to \infty$ by (i) and (v), from (5.6) we obtain that the constant equilibria α, γ are contained in $\omega(u_0)$.

We claim that if $\phi \in \mathcal{B} \setminus \{\alpha, \gamma, 0\}$ is an equilibrium of (1.1), then it is not contained in $\omega(u_0)$. To prove this, one first uses relations (5.7), (i), and (v) to show that for each b there is $\tau_b > 0$ such that

$$z_{(-b,b)}(u(\cdot, t, u_0)) \le 2 \quad (t \ge \tau_b). \tag{5.13}$$

Now one merely repeats the arguments given in the proof of Proposition 4.1 with $\varphi \equiv 0$. Thus, our claim is proved, and the connectedness of $\omega(u_0)$ implies that it contains functions which are not equilibria of (1.1). The theorem is proved. $\square$

The ideas of the above constructions in the balanced bistable case can also be used with different classes of basic solutions. We indicate how one can find initial data $u_0 \in C_0(\mathbb{R}) \cap \mathcal{B}$ such that $\omega(u_0)$ contains standing waves (strictly monotone equilibria of (1.1)), as well as some functions which are not equilibria of (1.1) (cp. Remark 2.2).

Assume for simplicity that f is odd, so that solutions with odd initial values are odd in x for each t. In particular, if $u_0^-, u_0^+ \in \mathcal{B}$ are odd and $u_0^\pm(\mp\infty) < 0 < u_0^\pm(\pm\infty)$, then [19, Theorem 3.1] implies that $u(\cdot, t, u_0^\pm)$ approaches the unique standing wave $\phi^\pm \in \mathcal{B}$ with $\pm(\phi^\pm)' > 0$ and $\phi^\pm(0) = 0$. Let us now consider a sequence of odd, piecewise linear functions $u_k \in \mathcal{B}$ such that $u_1 \equiv 0$ and for $k = 2, 3, \dots$ one has

(a) $u_{k+1} \equiv u_k$ on $[-\rho_k, \rho_k]$,

(b) $u_{k+1} \equiv (-1)^k \epsilon_{k+1}$ on $[\rho_k + 1, \infty)$,

(c) $|u_{k+1}| \leq \epsilon_k$ on $[\rho_k, \rho_k + 1]$,

where $\rho_k \nearrow \infty$ and $\epsilon_k \searrow 0$ are suitable sequences of positive numbers. By the above remarks, $u(\cdot, t, u_k)$ converges to ϕ^+ if k is odd and to ϕ^- if k is even. Similarly as in the constructions in Section 4 and in the proof of Theorem 2.1, the continuity with respect to initial data implies that if ρ_{k+1} are chosen suitably, then for the function defined by

$$u_0(x) \equiv u_k(x) \quad (|x| \leq \rho_k,\ k = 1, 2, \dots)$$

one has $\phi^{\pm} \in \omega(u_0)$. Using a more precise construction, similar to the one in the proof of Lemma 5.1, one can also control the zero number of the solutions $u(\cdot, t, u_k)$ and consequently the zero number of the solution $u(\cdot, t, u_0)$ in large bounded intervals. In analogy to (5.13), one then shows that for each $b \in (0, \infty)$ there is $\tau_b > 0$ such that

$$z_{(-b,b)}(u(\cdot, t, u_0)) \leq 3 \quad (t \geq \tau_b).$$

(Note that in this case the pertinent zero numbers are always odd and $x = 0$ is always a zero of the solutions at hand.) As in the proof of Theorem 2.1, this implies that $\omega(u_0)$ contains no nonconstant equilibria. We omit the details of this construction.

Since $\omega(u_0)$ is connected and consists of odd functions, with ϕ^+ and ϕ^- it necessarily contains some nonequilibrium solutions.

6 Proof of Theorem 2.3

In this section, we assume that f satisfies (C2). Without affecting quasiconvergence properties of the solutions, we replace f by a translation so that $\alpha = 0$. Thus, condition (C2) reads

(C2) For some $\gamma > \beta > 0$ one has $f(0) = f(\beta) = f(\gamma) = 0$, $f'(0) < 0$, $f'(\gamma) < 0$, $f < 0$ in $(0, \beta)$, $f > 0$ in (β, γ), and

$$F(\gamma) > 0. \tag{6.1}$$

A ground state of (2.3) now refers to a positive solution of (2.3) contained in $C_0(\mathbb{R})$ (and the corresponding solution of (3.2) is a homoclinic solution to $(0,0)$). We set $\mathcal{B} = \mathcal{B}_{0,\gamma}$.

The basic building blocks of our construction in this section are threshold solutions, as considered in the following lemma. (In different setting, but for a similar purpose, threshold solutions were used in [36].)

Lemma 6.1. *For each $\theta \in (\beta, \gamma)$ the following statements are valid.*

(i) *There exists $\ell = \ell(\theta)$ such that if $u_0 \in \mathcal{B}$ and $u_0 \geq \theta$ on an interval of length ℓ, then $u(\cdot, t, u_0) \to \gamma$ in $L^\infty_{loc}(\mathbb{R})$.*

(ii) *Let ℓ be as in* (i) *and let ψ_μ, $\mu \in [0,1]$, be a family of functions in $\mathcal{B}$ with the following properties*

(a1) *For each $\mu \in [0,1]$, ψ_μ has compact support, $\psi_1 \geq \theta$ on an interval of length ℓ, and*

$$\lim_{t\to\infty} u(\cdot, t, \psi_0) = 0 \text{ in } L^\infty(\mathbb{R}). \tag{6.2}$$

(a2) *The function $\mu \to \psi_\mu : [0,1] \to L^1(\mathbb{R})$ is continuous and monotone increasing in the sense that if $\mu < \nu$, then $\psi_\lambda \leq \psi_\nu$ everywhere, with the strict inequality on a nonempty (open) set.*

Then there exists a unique $\mu^* \in (0,1)$ with the following properties:

(t1) *If $u_0 = \psi_\mu$ with $\mu \in (0, \mu^*)$, then $\lim_{t\to\infty} u(\cdot, t, u_0) = 0$ in $L^\infty(\mathbb{R})$.*

(t2) *If $u_0 = \psi_\mu$ with $\mu \in (\mu^*, 1]$, then $\lim_{t\to\infty} u(\cdot, t, u_0) = \gamma$ in $L^\infty_{loc}(\mathbb{R})$.*

(t3) *If $u_0 = \psi_{\mu^*}$, then $\lim_{t\to\infty} u(\cdot, t, u_0) \to v$ in $L^\infty(\mathbb{R})$ for some ground state v of* (2.3).

We refer to μ^* as the threshold value (relative to the family ψ_μ, $\mu \in [0,1]$), to the solution in (t3) as the *threshold solution*, and to the solutions in (t1) as *subthreshold solutions*.

Statement (i) of Lemma 6.1 is due to [19] (see also [10, Lemma 4.2], [11, Lemma 2.4], [16, Lemma 6.3], or [33, Lemma 3.5]). Statement (ii) is proved in [10] (an earlier result for specific families was proved in [41]). Strictly speaking, the relevant result, Theorem 1.3 of [10], does not apply in our situation directly, as it has the assumption that $\psi_0 = 0$ a.e. instead of (6.2).

However, if this is not satisfied, then one can extend the family ψ_μ, $\mu \in [0,1]$ by defining $\psi_\mu := (1+\mu)\psi_0$ for $\mu \in [-1,0)$. To this extended family [10, Theorem 1.3] does apply and (6.2) implies that the unique threshold value for the extended family is positive, thus it is a threshold value for the original family.

Lemma 6.2. *Let $\phi \in \mathcal{B}$ be the (unique) ground state of* (2.3) *with maximum at $x = 0$. There exist $t_0 \in (0,1)$ and a sequence $(u_k, R_k, \rho_k, t_k, \delta_k)$, $k = 1, 2, \dots,$ in $\mathcal{B} \times (0,\infty)^4$ such that the statements* (i)-(iv) *below are valid for all $k = 1, 2, \dots,$ and statements* (v), (vi) *are valid for all $k = 2, 3 \dots$.*

(i) $t_{2k} > t_{2k-1} > 1$, $\rho_k > R_k > 2$,

(ii) *u_k is piecewise linear, even, and* $\operatorname{spt}(u_k) \subset (-R_k, R_k)$.

(iii) *The solution $u(\cdot,\cdot,u_k)$ has the following properties:*

$$\lim_{t\to\infty} \|u(\cdot,t,u_k)\|_{L^\infty(\mathbb{R})} = 0, \tag{6.3}$$

$$u(x,t,u_k) < \beta \quad (x \in \mathbb{R} \setminus (-R_k, R_k),\ t \in [0, t_{2k}]), \tag{6.4}$$

$$\|u(\cdot,t_{2k-1},u_k) - \phi\|_{L^\infty(\mathbb{R})},\ \|u(\cdot,t_{2k},u_k)\|_{L^\infty(\mathbb{R})} < \frac{1}{k}, \tag{6.5}$$

$$u(\cdot,t_{2k-2},u_k) - \beta \ \textit{ exactly four zeros, all of them simple,} \tag{6.6}$$

$$u(x,t_{2k},u_k) < \beta \quad (x \in \mathbb{R}). \tag{6.7}$$

(iv) *For each $u_0 \in \mathcal{B}$ with $\|u_k - u_0\|_{L^\infty(-\rho_k,\rho_k)} < \delta_k$, the following relations hold:*

$$u(x,t,u_0) < \beta \quad (x = \pm R_k,\ t \in [0, t_{2k}]), \tag{6.8}$$

$$\|u(\cdot,t_{2k-1},u_0) - \phi\|_{L^\infty(-R_k,R_k)},\ \|u(\cdot,t_{2k},u_0)\|_{L^\infty(-R_k,R_k)} < \frac{2}{k}, \tag{6.9}$$

$$u(\cdot,t_{2k-2},u_0) - \beta \ \textit{ has exactly four zeros in } (-R_k,R_k), \textit{ all of them simple,} \tag{6.10}$$

$$z_{(-R_k,R_k)}(u(\cdot,t,u_0) - \beta) \le 4 \quad (t \in [t_{2k-2}, t_{2k}]), \tag{6.11}$$

$$u(x,t_{2k},u_0) < \beta \quad (x \in [-R_k, R_k]). \tag{6.12}$$

(v) $t_{2k-1} > t_{2k-2} + 1$, $R_k > R_{k-1} + 1$.

(vi) $u_k \equiv u_{k-1}$ *on* $[-\rho_{k-1}, \rho_{k-1}]$.

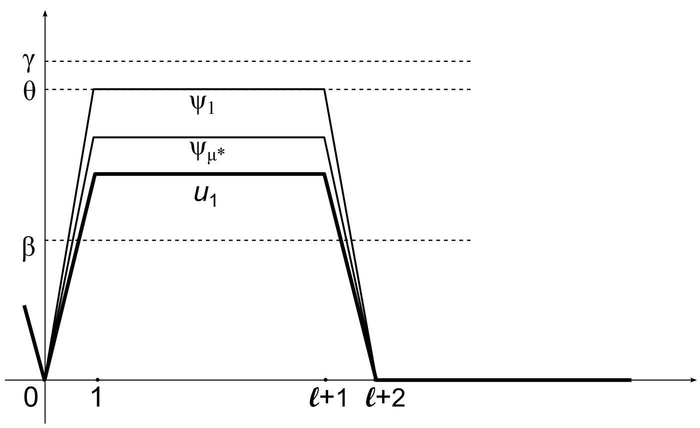

Figure 4: The graphs of ψ_1, ψ_{μ^*}, and u_1. One has $u_1 = \psi_\mu$, for some $\mu < \mu^*$, $\mu \approx \mu^*$.

Proof. For the whole proof, θ is a fixed number in (β, γ) and $\ell = \ell(\theta)$ is as in Lemma 6.1.

STEP 1. Set $k = 1$ and for each $\mu \in [0, 1]$ define an even piecewise linear function ψ_μ as follows (see Figure 4):

$$\psi_\mu(x) = \begin{cases} \mu\theta x & (x \in [0,1]), \\ \mu\theta & (x \in [1, \ell+1]), \\ \mu\theta(\ell + 2 - x) & (x \in [\ell+1, \ell+2]), \\ 0 & (x \geq \ell + 2), \\ \psi_\mu(-x) & (x < 0). \end{cases}$$

Obviously, the family ψ_μ, $\mu \in [0, 1]$, satisfies the assumptions of Lemma 6.1(ii). Let $\mu^* \in (0, 1)$ be as in that lemma. Since the ψ_μ are all even in x, the limit ground state of the threshold solution $u(\cdot, \cdot, \psi_{\mu^*})$ is ϕ. Also $\mu^*\theta > \beta$, for otherwise $\psi_{\mu^*} \leq \beta$ and then, by the comparison principle, $u(\cdot, \cdot, \psi_{\mu^*}) \leq \beta$, hence it cannot converge to ϕ. Now, by the continuity of the solutions with respect to initial data (in $L^\infty(\mathbb{R})$), if $\mu < \mu^*$ is close to μ^*, then the subthreshold solution $u(\cdot, \cdot, \psi_\mu)$ gets close to ϕ at a large time t_1 and then approaches zero in $L^\infty(\mathbb{R})$, as $t \to \infty$. Thus we can choose $\mu \in (\beta/\theta, \mu^*)$ and times $t_2 > t_1 > 1$ such that for $u_1 := \psi_\mu$ relations (6.5), (6.7) are valid. Next, using Lemma 3.3, we pick $R_1 > \ell + 2$ so that (6.4) holds. Then also $\mathrm{spt}(u_1) \subset (-\ell - 2, \ell + 2) \subset (-R_1, R_1)$ and statement (ii) holds. Finally, since $\mu\theta > \beta$, the definition of u_1 implies that $z(u(\cdot, t, u_1) - \beta) \geq 4 = z(u(\cdot, 0, u_1) - \beta)$ for all small $t \geq 0$. By the monotonicity of the zero number, the equality must hold there. Using Remark 3.8, we pick $t_0 \in (0,1)$ such that (6.6) holds.

Relations (6.4)-(6.7) and Corollary 3.6 imply the existence of constants $\rho_1 > R_1$ and $\delta_1 > 0$ such that statement (iv) holds with (6.11) excluded. Relation (6.11) follows by an application of Lemma 3.7(ii) with $I = (-R_1, R_1)$, which is legitimate by (6.8).

STEP 2 (the induction argument). Suppose that for some $n \geq 1$,

$$(u_k, R_k, \rho_k, t_k, \delta_k) \in \mathcal{B} \times (0,\infty)^4, \quad k = 1, \dots, n,$$

have been defined such that statements (i)-(iv) hold for all $k = 1, \dots, n$, and, in case $n \geq 2$, statements (v), (vi) hold for all $k = 2, \dots, n$. We need to define $(u_{n+1}, R_{n+1}, \rho_{n+1}, t_{n+1}, \delta_{n+1})$ in such a way that statements (i)-(vi) are valid for $k = n+1$.

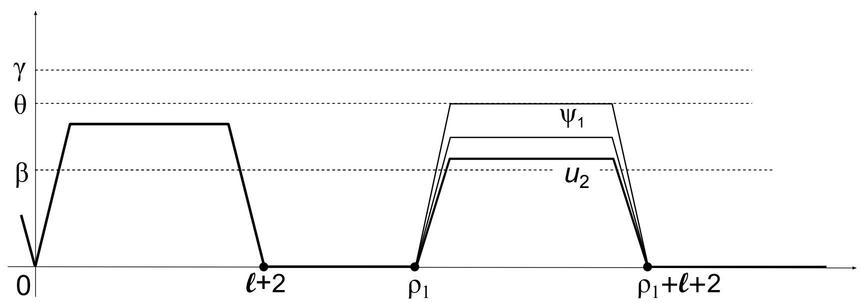

Figure 5: The graphs of ψ_1 and u_2, with the graph of ψ_{μ^*} in between. They all coincide with the graph of u_1 on $[0, \rho_1]$.

For each $\mu \in [0,1]$ define an even piecewise linear function ψ_μ as follows (see Figure 5):

$$\psi_\mu(x) = \begin{cases} u_n(x) & (x \in [0, \rho_n]), \\ \mu\theta(x - \rho_n) & (x \in [\rho_n, \rho_n + 1]), \\ \mu\theta & (x \in [\rho_n + 1, \rho_n + \ell + 1]), \\ \mu\theta(\rho_n + \ell + 2 - x) & (x \in [\rho_n + \ell + 1, \rho_n + \ell + 2]), \\ 0 & (x \geq \rho_n + \ell + 2), \\ \psi_\mu(-x) & (x < 0). \end{cases}$$

Since $\mathrm{spt}(u_n) \subset (-R_n, R_n) \subset (-\rho_n, \rho_n)$, this definition implies that $\psi_\mu \in \mathcal{B}$. Also, $\psi_0 \equiv u_n$. The latter and (6.3) (with $k = n$) imply that condition

(6.2) of Lemma 6.1(ii) holds. It is obvious that the family ψ_μ, $\mu \in [0,1]$ also satisfies all the other assumptions of Lemma 6.1(ii). Let $\mu^* \in (0,1)$ be as in the conclusion of that lemma. Again, by the evenness of ψ_μ, the limit ground state of the threshold solution $u(\cdot,\cdot,\psi_{\mu^*})$ must be ϕ.

Since $\psi_\mu \equiv u_n$ on $[-\rho_n, \rho_n]$, statement (iv) with $k = n$ applies to $u_0 = \psi_\mu$ for each $\mu \in [0,1]$. We first intend to use this property to show that

$$\mu^* \theta > \beta \tag{6.13}$$

and that for each $\mu < \mu^*$ sufficiently close to μ^*

$$u(\cdot, t_{2n}, \psi_\mu) - \beta \text{ has exactly four zeros, all of them simple.} \tag{6.14}$$

We start by using (6.8) (with $k = n$) to obtain

$$u(R_n, t, \psi_\mu) < \beta \quad (t \in [0, t_{2n}],\ \mu \in [0,1]). \tag{6.15}$$

Next, we show that for each $t \in [0, t_{2n}]$

$$u(\cdot, t, \psi_{\mu^*}) > \beta \text{ somewhere in } (R_n, \infty). \tag{6.16}$$

Thus we have to rule out the following possibility

$$u(\cdot, \tau, \psi_{\mu^*}) \le \beta \text{ on } (R_n, \infty) \text{ at some } \tau \in [0, t_{2n}]. \tag{6.17}$$

Assume (6.17) is true. Then, using (6.15) and the comparison principle, we obtain $u(x,t,\psi_{\mu^*}) \le \beta$ for all $(x,t) \in (R_0, \infty) \times [\tau, t_{2n}]$. Combining this with (6.12) (with $k = n$) and the evenness in x, we obtain $u(\cdot, t_{2n}, \psi_{\mu^*}) \le \beta$. But then the comparison principle shows that $u(\cdot,\cdot,\psi_{\mu^*})$ cannot converge to ϕ, in contradiction to the fact that it is a threshold solution.

Hence, we have proved that (6.16) holds for each $t \in [0, t_{2n}]$. This in particular implies that (6.13) holds (otherwise (6.17) would hold with $\tau = 0$, due to $\mathrm{spt}(u_n) \subset (-R_n, R_n)$). Also, in view of (6.15) and Lemma 3.3, relation (6.16) means that

$$z_{(R_n,\infty)}(u(\cdot, t, \psi_{\mu^*}) - \beta) \ge 2 \quad (t \in [0, t_{2n}]). \tag{6.18}$$

Now, by (6.15) and Lemma 3.7,

$$z_{(R_n,\infty)}(u(\cdot, t, \psi_\mu) - \beta) \le z_{(R_n,\infty)}(\psi_\mu - \beta) \le 2 \quad (t \in [0, t_{2n}],\ \mu \in [0,1]), \tag{6.19}$$

where the last inequality is by the definition of ψ_μ. Hence, the equality holds in (6.18). Therefore, by Remark 3.8, $u(\cdot, t_{2n}, \psi_{\mu^*}) - \beta$ has exactly two zeros in (R_n, ∞) both of them simple. The same is then true for all $\mu \approx \mu^*$, by (6.19) and the continuity with respect to the initial data. This, the evenness in x, and (6.12) (with $k = n$) imply (6.14). Thus, (6.13)-(6.14) are proved.

As in STEP 1, we use the continuity with respect to initial data in $L^\infty(\mathbb{R})$, to find $\mu \in (\beta/\theta, \mu^*)$ so close to μ^* that (6.14) holds and that the subthreshold solution $u(\cdot, \cdot, \psi_\mu)$ is close to ϕ at a large time $t_{2n+1} > t_{2n} + 1$ and close to 0 at a later time t_{2n+2}: relations (6.5), (6.7) are valid with $u_{n+1} := \psi_\mu$ and $k = n + 1$. Of course, (6.3) is valid for the subthreshold solution.

Having defined u_{n+1} and $t_{2n+2} > t_{2n+1}$, we use Lemma 3.3 and pick $R_{n+1} > \rho_n + \ell + 2$ large enough so that (6.4) holds with $k = n + 1$. Note that $R_{n+1} > \rho_n + \ell + 2$ guarantees that $\operatorname{spt} u_{n+1} \subset (-R_{n+1}, R_{n+1})$). Hence, statements (i)-(iii) hold with $k = n + 1$.

We next use Corollary 3.6 to find $\rho_{n+1} > R_{n+1}$ and $\delta_{n+1} > 0$ such that statement (iv) holds (to verify (6.11) one uses Lemma 3.7 and (6.8)).

Relations (v) and (vi) hold by construction.

This completes the induction argument and thereby the proof of Lemma 6.2. $\square$

Completion of the proof of Theorem 2.3. With ϕ and $(u_k, R_k, \rho_k, t_k, \delta_k)$ as in Lemma 6.2, take any $u_0 \in \mathcal{B}$ with

$$\|u_k - u_0\|_{L^\infty(-\rho_k, \rho_k)} < \delta_k \quad (k = 1, 2, \dots), \tag{6.20}$$

for example

$$u_0(x) \equiv u_k(x) \quad (|x| \le \rho_k,\ k = 1, 2, \dots), \tag{6.21}$$

(cp. statements (i),(v), (vi)).

By (6.20), statement (iv) applies to u_0 for each k. Since $R_k, t_k \to \infty$, from (6.9) we obtain that the equilibria 0 and ϕ are contained in $\omega(u_0)$. We next show that if $\varphi \not\equiv \phi$ is a nonconstant equilibrium of (1.1), then $\varphi \notin \omega(u_0)$. By the connectedness of $\omega(u_0)$, this will complete the proof of Theorem 2.3.

First of all, no ground state other than ϕ can be contained in $\omega(u_0)$, by the evenness of u_0. If $\varphi \in \mathcal{B}$ is a nonconstant equilibrium of (1.1) which is not a ground state, then it is periodic and $\varphi - \beta$ has infinitely many zeros, all of them simple (cp. Lemma 3.2). On the other hand, relations (6.11), (i), and (v) imply that for each $b > 0$ there is $\tau_b > 0$ such

$$z_{(-b,b)}(u(\cdot, t, u_0) - \beta) \le 4 \quad (t \ge \tau_b). \tag{6.22}$$

Thus arguments similar to those given in the proof of Proposition 4.1 show that $\varphi \notin \omega(u_0)$. $\square$

References

[1] S. Angenent, *The zeroset of a solution of a parabolic equation*, J. reine angew. Math. **390** (1988), 79–96.

[2] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[3] J. Busca, M.-A. Jendoubi, and P. Poláčik, *Convergence to equilibrium for semilinear parabolic problems in $\mathbb{R}^N$*, Comm. Partial Differential Equations **27** (2002), 1793–1814.

[4] X. Chen and J.-S. Guo, *Existence and uniqueness of entire solutions for a reaction-diffusion equation*, J. Differential Equations **212** (2005), 62–84.

[5] X. Chen, J.-S. Guo, and H. Ninomiya, *Entire solutions of reaction-diffusion equations with balanced bistable nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 1207–1237.

[6] X.-Y. Chen, *A strong unique continuation theorem for parabolic equations*, Math. Ann. **311** (1998), 603–630.

[7] X.-Y. Chen and H. Matano, *Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations*, J. Differential Equations **78** (1989), 160–190.

[8] P. Collet and J.-P. Eckmann, *Space-time behaviour in problems of hydrodynamic type: a case study*, Nonlinearity **5** (1992), 1265–1302.

[9] C. Cortázar, M. del Pino, and M. Elgueta, *The problem of uniqueness of the limit in a semilinear heat equation*, Comm. Partial Differential Equations **24** (1999), 2147–2172.

[10] Y. Du and H. Matano, *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*, J. European Math. Soc. **12** (2010), no. 2, 279–312.

[11] Y. Du and P. Poláčik, *Locally uniform convergence to an equilibrium for nonlinear parabolic equations on* $\mathbb{R}^N$, Indiana Univ. Math. J. (to appear).

[12] J.-P. Eckmann and J. Rougemont, *Coarsening by Ginzburg-Landau dynamics*, Comm. Math. Phys. **199** (1998), no. 2, 441–470.

[13] E. Fašangová, *Asymptotic analysis for a nonlinear parabolic equation on* $\mathbb{R}$, Comment. Math. Univ. Carolinae **39** (1998), 525–544.

[14] E. Feireisl, *On the long time behavior of solutions to nonlinear diffusion equations on* $\mathbb{R}^N$, NoDEA Nonlinear Differential Equations Appl. **4** (1997), 43–60.

[15] E. Feireisl and H. Petzeltová, *Convergence to a ground state as a threshold phenomenon in nonlinear parabolic equations*, Differential Integral Equations **10** (1997), 181–196.

[16] E. Feireisl and P. Poláčik, *Structure of periodic solutions and asymptotic behavior for time-periodic reaction-diffusion equations on* $\mathbb{R}$, Adv. Differential Equations **5** (2000), 583–622.

[17] B. Fiedler and P. Brunovský, *Connections in scalar reaction diffusion equations with Neumann boundary conditions*, Equadiff 6 (Brno, 1985), Lecture Notes in Math., vol. 1192, Springer, Berlin, 1986, pp. 123–128.

[18] B. Fiedler and C. Rocha, *Heteroclinic orbits of semilinear parabolic equations*, J. Differential Equations **125** (1996), 239–281.

[19] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335–361.

[20] P. C. Fife and J. B. McLeod, *A phase plane discussion of convergence to travelling fronts for nonlinear diffusion*, Arch. Rational Mech. Anal. **75** (1980/81), no. 4, 281–314.

[21] J. Földes and P. Poláčik, *Convergence to a steady state for asymptotically autonomous semilinear heat equations on* R^N, J. Differential Equations **251** (2011), 1903–1922.

[22] T. Gallay and S. Slijepčević, *Energy flow in extended gradient partial differential equations*, J. Dynam. Differential Equations **13** (2001), 757–789.

[23] ______, *Distribution of energy and convergence to equilibria in extended dissipative systems*, J. Dynam. Differential Equations, to appear.

[24] J.-S. Guo and Y. Morita, *Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations*, Discrete Contin. Dynam. Systems **12** (2005), 193–212.

[25] F. Hamel, R. Monneau, and J-M. Roquejoffre, *Stability of travelling waves in a model for conical flames in two space dimensions*, Ann. Sci. École Norm. Sup. (4) **37** (2004), 469–506.

[26] F. Hamel and N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math. **52** (1999), no. 10, 1255–1276.

[27] F. Hamel and Y. Sire, *Spreading speeds for some reaction-diffusion equations with general initial conditions*, SIAM J. Math. Anal. **42** (2010), 2872–2911.

[28] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Urall'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968, Russian Original: "Nauka", Moscow 1967.

[29] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18** (1978), 221–227.

[30] H. Matano and P. Poláčik, Dynamics of nonnegative solutions of one-dimensional reaction-diffusion equations with localized initial data, in preparation.

[31] Y. Morita and H. Ninomiya, *Entire solutions with merging fronts to reaction-diffusion equations*, J. Dynam. Differential Equations **18** (2006), 841–861.

[32] ______, *Traveling wave solutions and entire solutions to reaction-diffusion equations*, Sugaku Expositions **23** (2010), 213–233.

[33] P. Poláčik, *Threshold solutions and sharp transitions for nonautonomous parabolic equations on* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **199** (2011), 69–97, Addendum: www.math.umn.edu/$\sim$polacik/Publications.

[34] P. Poláčik and E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann. **327** (2003), 745–771.

[35] ______, *Nonstabilizing solutions and grow-up set for a supercritical semilinear diffusion equation*, Differential Integral Equations **17** (2004), 535–548.

[36] ______, *Localized solutions of a semilinear parabolic equation with a recurrent nonstationary asymptotics*, SIAM J. Math. Anal, to appear.

[37] J.-M. Roquejoffre and V. Roussier-Michon, *Nontrivial large-time behaviour in bistable reaction-diffusion equations*, Ann. Mat. Pura Appl. (4) **188** (2009), 207–233.

[38] J. Rougemont, *Dynamics of kinks in the Ginzburg-Landau equation: approach to a metastable shape and collapse of embedded pairs of kinks*, Nonlinearity **12** (1999), 539–554.

[39] E. Yanagida, *Irregular behavior of solutions for Fisher's equation*, J. Dynam. Differential Equations **19** (2007), 895–914.

[40] T. I. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4** (1968), 17–22.

[41] A. Zlatoš, *Sharp transition between extinction and propagation of reaction*, J. Amer. Math. Soc. **19** (2006), 251–263.