The principal Floquet bundle and exponential separation for linear parabolic equations

Juraj Húska, Peter Poláčik*
School of Mathematics, University of Minnesota
Minneapolis, MN 55455

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Abstract

We consider linear nonautonomous second order parabolic equations on bounded domains subject to Dirichlet boundary condition. Under mild regularity assumptions on the coefficients and the domain, we establish the existence of a principal Floquet bundle exponentially separated from a complementary invariant bundle. Our main theorem extends in a natural way standard results on principal eigenvalues and eigenfunctions of elliptic and time-periodic parabolic equations. Similar theorems were earlier available only for smooth domains and coefficients. As a corollary of our main result, we obtain the uniqueness of positive entire solutions of the equations in question.

Keywords: Nonautonomous parabolic equations, principal Floquet bundle, exponential separation, positive entire solutions.

1 Introduction

Consider the problem

$$\begin{aligned} u_t + \mathcal{A}(t)u = 0 \quad &\text{in} \quad \Omega \times J, \\ u = 0 \quad &\text{on} \quad \partial\Omega \times J, \end{aligned} \tag{1.1}$$

*email addresses: huska@math.umn.edu, polacik@math.umn.edu

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, J is an open interval in $\mathbb{R}$, and $\mathcal{A}(t)$ is a time-dependent elliptic operator of the form

$$\mathcal{A}(t)u = -\partial_i(a_{ij}(x,t)\partial_j u + a_i(x,t)u) + b_i(x,t)\partial_i u + c_0(x,t)u \tag{1.2}$$

with real valued coefficients $a_{ij}, a_i, b_i, c_0 \in L^\infty(\Omega \times \mathbb{R})$, $i, j = 1, \dots, N$ (we use the summation convention and the notation $\partial_i = \partial/\partial x_i$).

Without assuming any special dependence (like periodicity or almost periodicity) of the coefficients on t, we examine solutions of problem (1.1) that exhibit properties analogous to standard features of principal eigenfunctions of time-independent or time-periodic parabolic problems. To state our results, we fix a basic notation and formulate our standing hypotheses on the operator $\mathcal{A}$ and its coefficients.

We shall always assume that $\mathcal{A}$ is uniformly strongly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad ((x,t) \in \bar{\Omega} \times \mathbb{R},\ \xi \in \mathbb{R}^N). \tag{1.3}$$

Fix a constant $d_0 > 0$ and let $\mathcal{B}$ be a subset of $L^\infty(\Omega \times \mathbb{R})$ with the following properties

B1) $\|f\|_{L^\infty(\Omega\times\mathbb{R})} \leq d_0$ for all $f \in \mathcal{B}$.

B2) $\mathcal{B}$ is invariant under time-translations: if $f \in \mathcal{B}$ then the function $\tau_s f$ defined by

$$\tau_s f(x,t) = f(x, t+s) \quad ((x,t) \in \Omega \times \mathbb{R})$$

belongs to $\mathcal{B}$ for each $s \in \mathbb{R}$.

B3) $\mathcal{B}$ is sequentially compact with respect to convergence almost everywhere: any sequence in $\mathcal{B}$ has a subsequence that converges almost everywhere on $\Omega \times \mathbb{R}$ to a function in $\mathcal{B}$.

As an example of $\mathcal{B}$ one can take the set of all continuous functions f satisfying B1 which have the modulus of continuity bounded above by a fixed function ω_0 with $\omega_0(h) \to 0$ as $h \to 0^+$. See the end of the introduction for an example of a larger class $\mathcal{B}$, including discontinuous functions, and a discussion of the hypotheses.

We shall assume that $\mathcal{A}$ has all its coefficients in $\mathcal{B}$:

$$a_{ij}, a_i, b_i, c_0 \in \mathcal{B} \quad (i, j = 1, \dots, N). \tag{1.4}$$

We equip $\mathcal{B}$ with the weak* topology of $L^\infty(\Omega \times \mathbb{R})$. By B1, $\mathcal{B}$ is a compact metrizable space; it is not difficult to verify that B1, B3 imply that the convergence in $\mathcal{B}$ is equivalent to the convergence almost everywhere.

Further, with $\alpha_0 > 0$ fixed, let $\mathcal{B}_{\alpha_0}$ be the set of all matrix functions $(a_{ij})_{i,j=1}^N \in (\mathcal{B})^{N^2}$ satisfying (1.3) We equip $\mathcal{B}_{\alpha_0}$ with the induced topology from the product space $\mathcal{B}^{N^2}$ ($\mathcal{B}_{\alpha_0}$ is compact in this topology).

We now introduce the adjoint problem to (1.1)

$$\begin{aligned} -v_t + \mathcal{A}^*(t)v = 0 \quad &\text{in} \quad \Omega \times J, \\ v = 0 \quad &\text{on} \quad \partial\Omega \times J, \end{aligned} \tag{1.5}$$

where $\mathcal{A}^*$ is defined by

$$\mathcal{A}^*(t)v = -\partial_j(a_{ij}(x,t)\partial_i v + b_j(x,t)v) + a_j(x,t)\partial_j v + c_0(x,t)v. \tag{1.6}$$

By a solution of (1.1) or (1.5) we always mean a weak solution (see Section 2 for a precise definition); by an entire solution we mean a solution defined for each $t \in \mathbb{R}$.

Let $X = L^2(\Omega)$ with the standard norm. Our main result reads as follows.

Theorem 1.1. *Let $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \dots, N$. Then there exist positive entire solutions $\varphi_\mathcal{A}$, $\psi_\mathcal{A}$ of* (1.1), (1.5), *respectively, with the following properties.*

(i) For each $t \in \mathbb{R}$, the functions $\varphi_\mathcal{A}(\cdot,t) \in X$, $\psi_\mathcal{A}(\cdot,t) \in X$ depend continuously (in the $L^2(\Omega)$ norm) on the coefficients $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \dots, N$.

(ii) Set

$$\begin{aligned} X^1_\mathcal{A}(t) &:= \operatorname{span}\{\varphi_\mathcal{A}(\cdot,t)\}, \\ X^2_\mathcal{A}(t) &:= \{v \in X : \int_\Omega \psi_\mathcal{A}(x,t)v(x)\,dx = 0\} \quad (t \in \mathbb{R}). \end{aligned}$$

These spaces are invariant under (1.1) *in the following sense: if $i \in \{1,2\}$, $u_0 \in X^i_\mathcal{A}(s)$, then $u(\cdot,t;s,u_0) \in X^i_\mathcal{A}(t)$ ($t \geq s$), where $u(\cdot,t;s,u_0)$ denotes the solution of* (1.1) *with the initial condition $u(\cdot,s) = u_0$. Moreover, $X^1_\mathcal{A}(t)$, $X^2_\mathcal{A}(t)$ are complementary subspaces of X:*

$$X = X^1_\mathcal{A}(t) \oplus X^2_\mathcal{A}(t) \quad (t \in \mathbb{R}). \tag{1.7}$$

(iii) There are constants $C, \gamma > 0$ such that for any $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \dots, N$, $t > s$, and any $u_0 \in X^2_{\mathcal{A}}(s)$ one has

$$\frac{||u(\cdot, t; s, u_0)||_{L^2(\Omega)}}{||\varphi_{\mathcal{A}}(\cdot, t)||_{L^2(\Omega)}} \leq Ce^{-\gamma(t-s)} \frac{||u_0||_{L^2(\Omega)}}{||\varphi_{\mathcal{A}}(\cdot, s)||_{L^2(\Omega)}}. \tag{1.8}$$

We refer to the collection of the one-dimensional spaces $X^1_{\mathcal{A}}(t)$, $t \in \mathbb{R}$, as the *principal Floquet bundle* of (1.1) and to property (*iii*) as an *exponential separation.*

To draw a connection of the above results to principal eigenvalues, assume for a while that the coefficients of $\mathcal{A}(t)$ are independent of t: $\mathcal{A}(t) = \mathcal{A}$. Consider the eigenvalue problem

$$\begin{aligned} \mathcal{A}\phi &= \lambda\phi, &&\text{on } \Omega,\\ \phi &= 0, &&\text{on } \partial\Omega. \end{aligned} \tag{1.9}$$

It is well known (see [2, 3] for example) that there is a unique eigenvalue λ_1 (the principal eigenvalue) with a positive eigenfunction (the principal eigenfunction); it is real, algebraically simple and smaller than the real part of any other eigenvalue. Also λ_1 is the principal eigenvalue of the adjoint problem

$$\begin{aligned} \mathcal{A}^*\phi^* &= \lambda\phi^*, &&\text{on } \Omega,\\ \phi^* &= 0, &&\text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

Denoting by ϕ_1 and ϕ_1^* the principal eigenfunctions of (1.9), (1.10), it is easy to verify that

$$\varphi_{\mathcal{A}}(x,t) = \phi_1(x)e^{-\lambda_1 t}, \quad \psi_{\mathcal{A}}(x,t) = \phi_1^*(x)e^{\lambda_1 t}$$

are positive entire solutions of (1.1) and (1.5), respectively, that have all the properties stated in Theorem 1.1. Note that the exponential separation property (*iii*) follows from the fact that λ_1 is smaller than the real part of any other eigenvalue.

Similarly, if the coefficients of $\mathcal{A}(t)$ are periodic in t with a common period τ, then the period (Poincaré) map of (1.1) has a unique Floquet multiplier with a positive eigenfunction. The multiplier is real, simple and greater than the modulus of any other multiplier, and it is also a Floquet multiplier of the adjoint problem (1.5) with a positive eigenfunction (see [7]). Denote the positive eigenfunctions of the period maps of (1.1), (1.5) by ϕ_1 and ϕ_1^*, respectively, and take them as initial conditions for (1.1), (1.5) at $t = 0$. The resulting solutions are positive entire solutions $u = \varphi_{\mathcal{A}}$, $v = \psi_{\mathcal{A}}$ with properties as in Theorem 1.1.

If the coefficients of $\mathcal{A}$ and the boundary of Ω are sufficiently regular, the above properties of elliptic and time-periodic equations can be derived from the Krein-Rutman theorem on positive operators (see [14, 9], for example). Without the periodicity assumption there is no such easy derivation of Theorem 1.1, even in the smooth case (that is, when the coefficients and $\partial\Omega$ are smooth). An abstract exponential separation theorem, dealing with positive vector bundle maps was proved in [29] (a finite-dimensional predecessor of this result can be found in [30]). This abstract result in particular implies Theorem 1.1 in the smooth case (see [20, 31]). For a class of second order parabolic operators the smooth case was also treated in [18], and more recently, with a different method, in [27]. The regularity assumptions in these papers are essential for the techniques used there. These techniques generally do not apply under the present assumptions, as we explain below.

The principal Floquet bundle with the exponential separation property has been used, as a key ingredient, in several results on nonlinear parabolic equations (see [28, 15, 19, 31]); applications in linear equations can be found in [28, 20, 21, 22, 23, 31, 16]. Let us also mention that in one space dimension one can establish Floquet bundles corresponding to any nodal number, not just to the nodal number zero (the principal Floquet bundle) as in the multidimensional case (see [4, 5, 32]). These results and corresponding exponential separation theorems extend the Sturm-Liouville theory of second order ordinary differential equations in a similar way Theorem 1.1 extends Krein-Rutman type results for elliptic and parabolic equations.

Let us explicitly mention one application in linear equations. As was shown in [20, 21], the exponential separation property implies the uniqueness (up to scalar multiples) of positive entire solutions of (1.1) (see also [26] for earlier uniqueness results obtained by different methods). It was later pointed out in [27] that the uniqueness follows from a simpler exponential growth estimate of expressions involving solutions of (1.1) and (1.5). Both the new exponential estimate and its corollary on the uniqueness are interesting at their own rate (we formulate them in Theorem 1.2 and Proposition 1.3 below), but, as shown in [27], one can also use them as basic ingredients of an alternative proof of the exponential separation theorem. We prove these results in our more general setting and employ them in the proof of Theorem 1.1 in a similar way as in [27].

Denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $L^2(\Omega)$. From now on, we always assume that the coefficients of $\mathcal{A}$ in the equation (1.1) satisfy the assumptions as stated in Theorem 1.1.

Theorem 1.2. *Let $v_0 \in L^2(\Omega)$ be nonnegative and nontrivial. If $u(\cdot,t)$ is a nontrivial solution of* (1.1) *on $(-\infty, t_0]$, $v(\cdot,t)$ is the solution of* (1.5) *with $v(\cdot,t_0) = v_0$, and*

$$\langle u(\cdot,t_0), v_0\rangle = 0,$$

then the function

$$\xi(t) := \langle |u(\cdot,t)|, v(\cdot,t)\rangle$$

is decreasing and grows exponentially as $t \to -\infty$: there are positive constants C and γ such that

$$\xi(t) \le Ce^{-\gamma(t-s)}\xi(s) \qquad (t_0 > t > s). \tag{1.11}$$

Let us show, following [27], how this theorem implies the uniqueness of positive entire solutions:

Proposition 1.3. *If u_1 and u_2 are positive entire solutions of* (1.1)*, then there is a constant q such that $u_1 \equiv qu_2$.*

Proof. Choose a nontrivial continuous function $v_0 \ge 0$ and let v be the solution of (1.5) with $v(\cdot,t_0) = v_0$. There is a constant q such that

$$\langle u_1(\cdot,t_0) - qu_2(\cdot,t_0), v_0\rangle = 0. \tag{1.12}$$

Set $u = u_1 - qu_2$. By positivity of u_1, u_2, v,

$$\langle |u(\cdot,t)|, v(\cdot,t)\rangle \le \langle u_1(\cdot,t), v(\cdot,t)\rangle + |q|\langle u_2(\cdot,t), v(\cdot,t)\rangle. \tag{1.13}$$

It is not difficult to verify (cf. (3.1) below) that if $u(t)$ and $v(t)$ are any solutions of (1.1) and (1.5) on the same interval, then $\langle u(\cdot,t), v(\cdot,t)\rangle$ is constant on that interval. Thus (1.13) implies $\langle |u(\cdot,t)|, v(\cdot,t)\rangle$ is bounded. By Theorem 1.2 this is possible only if $u \equiv 0$. □

As mentioned above all previous results on the principal Floquet bundle and exponential separation were proved under stronger regularity assumptions. A key point is that if $\partial\Omega$ and the coefficients of $\mathcal{A}$ are sufficiently regular then the Hopf boundary lemma implies a strong positivity property of the evolution operator of (1.1). Namely, the evolution operator takes any

$0 \not\equiv u_0 \in L^2(\Omega)$, $u_0 \geq 0$, into the interior of the positive cone in a functional space Y, for example,

$$Y := \{v \in C^1(\bar{\Omega}) : v(x) = 0 \text{ on } \partial\Omega\}$$

with the standard C^1 topology. In our present setting, the Hopf boundary Lemma no longer applies (see the discussion following [13, Lemma 3.4]) thus the strong positivity property is not available. To circumvent this difficulty, we will appeal to more basic properties of solutions (in particular positive ones) of linear parabolic equations. For this purpose we will use several forms of Harnack type estimates proved by Fabes and Safonov [12]. We have collected them in Appendix B, where we also derive several consequences useful for our arguments. In fact, that appendix constitutes a substantial part of our proof. Our paper closely follows the approach used in [27] and many results are stated in an almost identical form. However, as our assumptions are far less restrictive, many proofs are completely different.

The paper is organized as follows. In Section 2 we give a precise definition of (entire) solutions of (1.1) and (1.5). We also list several properties of the evolution operator associated with equation (1.1). Sections 3, 4 contain, respectively, the proofs of the exponential growth estimate of Theorem 1.2 and the existence of positive entire solutions of (1.1) and (1.5). The proof of Theorem 1.1 is completed in Section 5. In addition to Appendix B on Harnack type results, we have included Appendix A, where we give a perturbation result from [6].

We finish the introduction with a few comments on our assumptions B1-B3. As already mentioned, an example of $\mathcal{B}$ is the set of all functions f satisfying B1 which have the modulus of continuity bounded above by a fixed function ω_0. More generally, fix any positive and increasing function ω_0 with $\omega_0(h) \to 0$ as $h \to 0^+$. Let $\mathcal{B}$ be the set of all functions f satisfying B1 and the following condition

$$\sup_{\substack{|y|+|s|\leq h \\ y\in\mathbb{R}^N,\, s\in\mathbb{R}}} \int_{\Omega\times(T-1,T)} |f(x+y,t+s) - f(x,t)|\, dxdt \leq \omega_0(h) \quad (h > 0,\, T \in \mathbb{R})$$

(it is understood here that $f(x+y,t+s) = 0$ if $x+y \notin \Omega$). It is clear that $\mathcal{B}$ satisfies B2. By a well-known compactness criterion and a diagonalization procedure, any sequence in $\mathcal{B}$ has a subsequence convergent in $L^1_{loc}(\Omega \times \mathbb{R})$ and hence also a subsequence convergent almost everywhere. This shows that B3 is satisfied.

Observe that B2 and B3 in particular imply that for $f \in \mathcal{B}$ and for any sequence $s_n \in \mathbb{R}$ the sequence of time translations $\tau_{s_n} f$ has a subsequence that converges almost everywhere to a function $\tilde{f} \in \mathcal{B}$. It is the nature of some of our proofs that requires this compactness condition, since they depend on a limiting contradiction argument. For it to work, we need the coefficients of a sequence of operators (1.2) to have at least pointwise limits. The reason is a perturbation argument, formulated in Appendix A, which uses the pointwise convergence. This result can be improved slightly by allowing the limits of the lowest order coefficients to be merely weak* limits (thus no extra condition, other than boundedness, on c_0 is actually necessary), but it is not clear whether weak* limits are sufficient for the higher order coefficients.

After this paper was completed, we have learned from Mikhail Safonov about new elliptic-type Harnack inequalities for quotients of positive solutions of (1.1). Based on these estimates, a different approach to the problem considered here is possible. It does not rely on limiting arguments and thus does not need any compactness assumption like B3. Note that we do not need B3 to establish the existence of the entire solutions $\varphi_{\mathcal{A}}$, $\psi_{\mathcal{A}}$ either. With the new approach one can thus improve the theorems presented here. The results will be presented elsewhere.

2 Preliminaries

Consider the following initial value problem

$$\begin{aligned} u_t + \mathcal{A}(t)u = 0 \quad &\text{in} \quad \Omega \times (s,T), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (s,T), \\ u = u_0 \quad &\text{in} \quad \Omega \times \{s\}, \end{aligned} \tag{2.1}$$

where $s, T \in \mathbb{R}$, $s < T$, $\mathcal{A}$ is defined in (1.2) with coefficients satisfying (1.3) and (1.4). The quadratic form corresponding to (2.1) is defined in a usual way

$$a(t; u, v) := \int_\Omega [(a_{ij}\partial_j u + a_i u)\partial_i v + (b_i \partial_i u + c_0 u)v]\, dx$$

for all $u, v \in W_0^{1,2}(\Omega)$ (we have omitted the arguments of u, v, a_{ij} etc.). Set $V := W_0^{1,2}(\Omega)$. Below $\mathcal{D}([s,T))$ stands for the space of smooth functions with compact support in $[s,T)$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$. Also, if there is no danger of confusion, we will often suppress the spatial argument and write $u(t)$ for a solution of (1.1).

Definition 2.1. *Assume $u_0 \in L^2(\Omega)$. A function u is called a weak solution of problem* (2.1) *on $\Omega \times [s,T]$, if $u \in L^2((s,T);V)$ and*

$$-\int_s^T \langle u(t), v\rangle \varphi'(t)\,dt + \int_s^T a(t;u(t),v)\varphi(t)\,dt - \langle u_0, v\rangle \varphi(s) = 0$$

for all $v \in V$ and $\varphi \in \mathcal{D}([s,T))$. A function u is called a weak solution of (1.1) *on an open time interval $J \subseteq \mathbb{R}$ if it is a weak solution of* (2.1) *with $u_0 = u(s)$ on $\Omega \times [s,T]$ for all $s,T \in \mathbb{R}$ such that $[s,T] \subseteq J$. In particular, u is called an entire solution if it is a weak solution of* (1.1) *on $J = \mathbb{R}$.*

We will use the terms weak solution and solution interchangeably throughout this paper. Under our assumptions on the coefficients and the domain, a weak solution of the initial boundary value problem (2.1) from Definition 2.1 is also a weak solution of the equation $u_t + \mathcal{A}(t)u = 0$ in $\Omega \times (s,T)$ in the usual sense (see [17, Chapter III]). Moreover, for any $u_0 \in L^2(\Omega)$, and $s < T$, the weak solution of (2.1) always exists, it is unique and can be (uniquely) extended to a solution on (s,∞). Denote the solution by $U(t,s)u_0$, $t \geq s$. Let $\|\cdot\|_{p,q}$ stand for the operator norm of the space $\mathcal{L}(L^p(\Omega), L^q(\Omega))$ of bounded linear operators from $L^p(\Omega)$ to $L^q(\Omega)$. It is well known that the evolution operator $U(t,s)$, $t \geq s$, satisfies the following $L^p - L^q$ estimates (see [8], for example).

Proposition 2.2. *For all $1 \leq p \leq q \leq \infty$, $t,s \in \mathbb{R}$, $t > s$, one has $U(t,s) \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ and*

$$\|U(t,s)\|_{p,q} \leq M(t-s)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} e^{\beta(t-s)},$$

where $M \geq 1$ and $\beta \in \mathbb{R}$ are constants depending only on the $L^\infty(\Omega \times \mathbb{R})$ bound of the coefficients of $\mathcal{A}$ and the constant α_0 in (1.3)*. Moreover, for any $u_0 \in L^2(\Omega)$ and $T \geq s$ one has $U(\cdot,s)u_0 \in C([s,T]; L^2(\Omega))$.*

Another property of the evolution operator $U(t,s)$ we will use is positivity. For any $p \in (1,\infty)$ and $u_0 \in L^p(\Omega)$, $u_0 \geq 0$, we have $U(t,s)u_0 \geq 0$ for all $t \geq s$ (see [8], for example). This can be improved on: nonnegative nontrivial solutions are strictly positive. Since we use this fact frequently, we formulate it in the following lemma. It is a direct consequence of the Harnack inequality [25, 1, 12].

Lemma 2.3. *If u is a nonnegative and nontrivial solution of* (1.1) *on $\Omega \times (s,\infty)$ then it is strictly positive in $\Omega \times (s,\infty)$.*

Besides positivity, the evolution operator has the smoothing property. In this regard, we mention the following standard regularity result [17, Chapter III, Theorem 10.1]. We use the usual notation for the parabolic Hölder spaces $C^{\alpha,\frac{\alpha}{2}}(\Omega \times (s,T))$.

Theorem 2.4. *Let u be a (weak) solution of (2.1) with $u_0 \in L^2(\Omega)$. Suppose that the coefficients a_{ij}, a_i, b_i, c_0 $(i,j = 1,\ldots,N)$ have the $L^\infty(\Omega \times \mathbb{R})$-norm bounded above by a constant d. Then for any $T > s$ we have $u \in C^{\alpha,\frac{\alpha}{2}}_{loc}(\bar{\Omega} \times (s,T])$, and for any $\delta > 0$ $(\delta < T-s)$ the norm $\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[s+\delta,T])}$ is estimated from above by a constant K depending only on N, d, $\operatorname*{ess\,sup}_{\Omega\times(s,T)} |u|$, α_0 in (1.3), $|T-s|$, δ, and Ω. The exponent $\alpha > 0$ is determined only by N, d, α_0, and Ω.*

Remark 2.5. Below it will be useful to have noted that in the above theorem, the exponent α and the constant K depend on Ω only via its diameter and the regularity of its boundary. In particular, α and K can be chosen uniformly for a class of domains Ω contained in a fixed ball and such that we can choose fixed numbers r_0 and m characterizing their Lipschitz properties (see Section B.1).

Let us now turn our attention to (1.5). As before, define the corresponding bilinear form associated with (1.5) by

$$a^*(t;u,v) := \int_\Omega [(a_{ij}\partial_i u + b_j u)\partial_j v + (a_j \partial_j u + c_0 u)v]\,dx$$

for all $u, v \in W_0^{1,2}(\Omega)$ (omitting the arguments). A weak solution of (1.5) with $v(\cdot,t_0) = v_0 \in L^2(\Omega)$ can now be defined using $a^*(\cdot,\cdot,\cdot)$ analogously as in Definition 2.1. One can prove (see [8]) that there is a well defined evolution operator, henceforth denoted by $U^*(t,s)$, $t \le s$, for the adjoint problem (1.5). Reversing time, we obtain

$$U^*(t,s) = \tilde{U}(-t,-s) \qquad (t \le s),$$

where $\tilde{U}(t,s)$, $t \ge s$, is the ("forward") evolution operator for the problem

$$\begin{aligned} w_t + \mathcal{A}^*(-t)w &= 0 \quad x \in \Omega, \\ w &= 0 \quad x \in \partial\Omega. \end{aligned}$$

This problem is of the same form as (2.1) and thus $U^*(t,s)$ has the same smoothing and positivity properties as $U(t,s)$. We will use this fact frequently without notice. In particular, in Appendix B, when proving several general properties of positive solutions of (2.1) and (1.5), we will restrict ourselves to just (2.1). Finally, let us note that

$$\langle U(t,s)u, v\rangle = \langle u, U^*(s,t)v\rangle \qquad (u, v \in L^2(\Omega), t \geq s), \tag{2.2}$$

a fact that can be proved in a standard way by applying the Fubini theorem to the integral representations of the solutions $U(t,s)u$, $U^*(s,t)v$ using the weak Green's functions (see [1, Theorem 9 (i), (vi)]).

3 Proof of Theorem 1.2

Let u be a solution of (1.1) on $(-\infty, t_0)$ and let v be the solution of (1.5) with $v(\cdot, t_0) = v_0 \in L^2(\Omega)$. One can easily verify that then

$$\langle u(\cdot,t), v(\cdot,t)\rangle := \int_\Omega u(x,t)v(x,t)\,dx \equiv const. \tag{3.1}$$

is a constant function of t. Indeed, the fact that the time derivative of this expression equals zero follows from the definition of $\mathcal{A}$, $\mathcal{A}^*$ and integration by parts in case the coefficients are smooth. Using this and a standard approximation procedure (see [6]), one proves (3.1) in the general case.

Let us now prove an analogue of [27, Lemma 3.1].

Lemma 3.1. *Assume that u is a nontrivial solution of* (1.1) *on J and v is a positive solution of* (1.5) *on the same interval J such that* $\langle u(t), v(t)\rangle = 0$ *for some (hence every) $t \in J$. Then $\xi(t) := \langle |u(t)|, v(t)\rangle$ is a nonincreasing function on J. More precisely, $\xi(t)$ is nonincreasing on J and it is (strictly) decreasing at all times $t \in J$ for which $u(t) \not\equiv 0$.*

Remark 3.2. Under our assumptions on the regularity of the coefficients the unique continuation property of linear parabolic equations does not hold in general. This means that if u is a nontrivial solution of (1.1) on an open interval J, it may happen that it becomes identically zero at some $t \in J$ (and then continues as a zero solution). This is the reason for the seemingly awkward formulation of the above lemma (cp. [27, Lemma 3.1]). We refer the reader to [24] for an example of non-unique continuation in the case of Neumann boundary conditions and to [10] for what seems to be a sharp condition on smoothness of coefficients under which one still has unique continuation.

Proof of Lemma 3.1. Let $s, t \in J$, $t > s$. Using the positivity of the evolution operator $U(t,s)$ associated with (1.1), one easily gets (see [27] for details)

$$|u(t)| \leq U(t,s)|u(s)|. \tag{3.2}$$

If, moreover, $u(t)$ changes sign in Ω then the inequality is strict on a subdomain $\tilde{\Omega}$:

$$|u(x,t)| < (U(t,s)|u(s)|)(x) \quad (x \in \tilde{\Omega}).$$

Since $v(\cdot,t) > 0$ in Ω for $t \in J$ and since $\langle u(t), v(t)\rangle = 0$, we must have that either $u(\cdot,t)$ changes sign or is identically zero. Suppose that the former happens, that is, $u(\cdot,t)$ changes sign. Then (3.2) holds on Ω and is strict on a nonempty subdomain. Therefore

$$\xi(t) < \langle U(t,s)|u(s)|, v(t)\rangle. \tag{3.3}$$

Applying (3.1) to the solution $\bar{u}(t) = U(t,s)|u(s)|$, we see that the right hand side of (3.3) is independent of t. Taking $t = s$, we obtain

$$\xi(t) < \langle |u(s)|, v(s)\rangle = \xi(s).$$

Thus $\xi(t)$ decreases strictly up to the time when u becomes identically zero (if at all). □

Assume now that the hypotheses of Theorem 1.2 are satisfied and that $u(t) \not\equiv 0$ for $t \in (-\infty, t_0]$ (this causes no loss of generality). Lemma 3.1 implies that there is a constant $\rho \leq 1$ such that

$$\frac{\xi(t+2)}{\xi(t)} \leq \rho \qquad (t \leq t_0 - 2). \tag{3.4}$$

If $\rho < 1$ in this inequality, then this fact and the monotonicity of ξ imply

$$\xi(t) \leq Ce^{-\gamma(t-s)}\xi(s) \qquad (s < t < t_0)$$

for $\gamma = -\frac{\log \rho}{2}$ and $C = \rho^{-1}$. Thus Theorem 1.2 follows from the next assertion.

Lemma 3.3. *Under the assumptions of Theorem 1.2, inequality* (3.4) *holds for some* $\rho < 1$.

Proof. We first derive a useful estimate for the quotient $\xi(t+1)/\xi(t)$. It will later help us prove that certain limiting solution which we obtain by a contradiction argument is nontrivial.

Fix $t \le t_0 - 2$. By Lemma B.8 applied to the solution $\bar{u}(\tau) = U(\tau, t)|u(t)|$ we have

$$U(t+1,t)|u(t)|(x) \ge Cd(x)^\theta \| U(t+1,t)|u(t)| \|_{L^2(\Omega)}.$$

This inequality and (3.2) (replace there t by $t+1$ and s by t, respectively) imply

$$U(t+1,t)|u(t)|(x) \ge Cd(x)^\theta \|u(t+1)\|_{L^2(\Omega)}. \tag{3.5}$$

For $r > 0$ set

$$\Omega^r := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > r\}$$

and fix $r = r_0/2$, where r_0 is as in the definition of a Lipschitz domain at the beginning of Section B.1. Then an application of Corollary B.7 to v gives

$$\begin{aligned}
&\frac{\xi(t+1)}{\|u(t+1)\|_{L^2(\Omega)}\|v(t+1)\|_{L^2(\Omega)}} = \\
&\int_{\Omega\setminus\Omega^r} \frac{|u(t+1)|}{\|u(t+1)\|_{L^2(\Omega)}} \frac{v(t+1)}{\|v(t+1)\|_{L^2(\Omega)}}\, dx + \int_{\Omega^r} \frac{|u(t+1)|}{\|u(t+1)\|_{L^2(\Omega)}} \frac{v(t+1)}{\|v(t+1)\|_{L^2(\Omega)}}\, dx \\
&\qquad \le a(t+1) + \int_{\Omega^r} \frac{|u(t+1)|}{\|u(t+1)\|_{L^2(\Omega)}}\, dx, \qquad (3.6)
\end{aligned}$$

where

$$a(t) := \int_{\Omega\setminus\Omega^r} \frac{|u(t)|}{\|u(t)\|_{L^2(\Omega)}} \frac{v(t)}{\|v(t)\|_{L^2(\Omega)}}\, dx$$

and C is a positive constant (independent of v). We know that (cf. (3.1))

$$\xi(t) = \int_\Omega |u(t)|v(t)\, dx = \int_\Omega (U(t+1,t)|u(t)|)v(t+1)\, dx.$$

Splitting this integral into a sum of two integrals over $\Omega \setminus \Omega^r$ and Ω^r, respectively, and applying inequality (3.2) (with "$t = t+1$" and "$s = t$") to the first one, we obtain

$$\frac{\xi(t)}{\|u(t+1)\|_{L^2(\Omega)}\|v(t+1)\|_{L^2(\Omega)}} \ge a(t+1) + \tilde{C}. \tag{3.7}$$

By (3.5) and Lemma B.8 (applied to v), $\tilde{C}$ is bounded below by a positive constant independent of u and v. Combining (3.6) and (3.7) implies

$$\frac{\xi(t+1)}{\xi(t)} \leq \frac{a(t+1) + C\int_{\Omega^r} \frac{|u(t+1)|}{\|u(t+1)\|_{L^2(\Omega)}}\,dx}{a(t+1)+\tilde{C}} \quad (t \leq t_0 - 2). \tag{3.8}$$

Notice that by the Hölder inequality and positivity of v, we have $0 \leq a(t) \leq 1$ $(t \leq t_0 - 1)$.

We now proceed by contradiction. Suppose the statement of Lemma 3.3 is false. Then there is a sequence $t_n \to -\infty$ such that

$$1 \geq \frac{\xi(t_n+s)}{\xi(t_n)} \geq \frac{\xi(t_n+2)}{\xi(t_n)} \to 1 \qquad (s \in [0,2]) \tag{3.9}$$

as $n \to \infty$. Set

$$\begin{aligned} u_n(\tau) : &= \frac{u(t_n+1+\tau)}{\|u(t_n+1)\|_{L^2(\Omega)}}, \ \mathcal{A}_n(\tau) := \mathcal{A}(t_n+1+\tau) \quad , \\ v_n(\tau) : &= \frac{v(t_n+1+\tau)}{\|v(t_n+1)\|_{L^2(\Omega)}}, \ \mathcal{A}_n^*(\tau) := \mathcal{A}^*(t_n+1+\tau) \quad (\tau < t_0 - t_n - 1), \end{aligned}$$

where $\mathcal{A}$, $\mathcal{A}^*$ are as in (1.2), (1.6), respectively. Clearly, u_n is a solution of

$$\begin{aligned} u_\tau + \mathcal{A}_n(\tau)u = 0 \quad &\text{in} \quad \Omega \times (-\infty, t_0 - t_n - 1), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (-\infty, t_0 - t_n - 1). \end{aligned}$$

Similarly, v_n is a positive solution of

$$\begin{aligned} -v_\tau + \mathcal{A}_n^*(\tau)v = 0 \quad &\text{in} \quad \Omega \times (-\infty, t_0 - t_n - 1), \\ v = 0 \quad &\text{on} \quad \partial\Omega \times (-\infty, t_0 - t_n - 1). \end{aligned}$$

In addition, we have $v_n(\tau) \geq 0$ and

$$\|u_n(0)\|_{L^2(\Omega)} = \|v_n(0)\|_{L^2(\Omega)} = 1, \ \langle u_n(\tau), v_n(\tau)\rangle = 0 \quad (\tau < t_0 - t_n - 1). \tag{3.10}$$

Using our assumptions on the coefficients of $\mathcal{A}$, we obtain, passing to subsequences if necessary, that the coefficients of $\mathcal{A}_n$ converge almost everywhere (in $\Omega\times\mathbb{R}$) to the coefficients of some uniformly elliptic operator $\tilde{\mathcal{A}}$. By (3.10), the sequence $u_n(0)$ contains a subsequence, still denoted by $u_n(0)$, converging to some $\tilde{u}_0 \in L^2(\Omega)$ weakly in $L^2(\Omega)$. Thus all assumptions of Lemma A.1

are satisfied and we can conclude that (a subsequence of) u_n converges to $\tilde{u}$ in $C([\delta, T]; L^2(\Omega))$ for all $T, \delta > 0$, $\delta < T$, where $\tilde{u}$ is the solution of

$$\begin{aligned} u_\tau + \tilde{\mathcal{A}}(\tau)u = 0 \quad &\text{in} \quad \Omega \times (0, \infty), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (0, \infty), \\ u = \tilde{u}_0 \quad &\text{in} \quad \Omega \times \{0\}. \end{aligned} \tag{3.11}$$

Next, it can be shown (see (B.2.8) and the argument preceding it) that there exists $C > 1$ such that

$$\frac{1}{C} \leq \|v_n\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times [-1,1])} \leq C,$$

where α is as in Theorem 2.4. Hence there is a subsequence of v_n, still denoted by v_n, converging uniformly on $\bar{\Omega} \times [-1, 1]$ to some $\tilde{v} \in C(\bar{\Omega} \times [-1, 1])$. Passing to another subsequence, we can assume that the coefficients of $\mathcal{A}_n^*$ converge almost everywhere (in $\Omega \times \mathbb{R}$) to the coefficients of the operator $\tilde{\mathcal{A}}^*$ adjoint to $\tilde{\mathcal{A}}$. From Lemma A.1 we deduce that $\tilde{v}$ is a solution of

$$\begin{aligned} -v_\tau + \tilde{\mathcal{A}}^*(\tau)v = 0 \quad &\text{in} \quad \Omega \times (-1, 1), \\ v = 0 \quad &\text{on} \quad \partial\Omega \times (-1, 1). \end{aligned}$$

Now, since $v_n \geq 0$ and $\|v_n(0)\|_{L^2(\Omega)} = 1$, we must have $\tilde{v} \geq 0$ and $\|\tilde{v}(0)\|_{L^2(\Omega)} = 1$. This shows that $\tilde{v}$ is nontrivial and nonnegative, hence by the Harnack inequality it is positive on $\Omega \times (-1, 1)$.

By (3.10), we have

$$\langle \tilde{u}(\tau), \tilde{v}(\tau) \rangle = 0 \ \ (\tau \in (0, 1)).$$

We claim that $\tilde{u}$ is a nontrivial solution of (3.11) such that $\tilde{u}(t) \not\equiv 0$ for $t \in [0, \delta_0]$, where δ_0 is some positive number to be determined later (cf. Remark 3.2). Indeed, taking $t = t_n$ in inequality (3.8), noting that we assume (3.9), and sending n to infinity, one immediately obtains

$$\int_{\Omega^r} |\tilde{u}(0)| \, dx > 0,$$

which implies that $\|\tilde{u}(0)\|_{L^2(\Omega)} > 0$. This and the continuity of $t \to \tilde{u}(t) \in L^2(\Omega)$ (cf. Proposition 2.2) imply that, for some positive δ_0 ($\delta_0 < 1$), we have $\tilde{u}(t) \not\equiv 0$ whenever $t \in [0, \delta_0]$, as claimed.

An application of Lemma 3.1 to the solutions $\tilde{u}(\tau)$, $\tilde{v}(\tau)$ gives that $\tau \mapsto \langle|\tilde{u}(\tau)|, \tilde{v}(\tau)\rangle$ is a strictly decreasing function on $(0, \delta_0)$. On the other hand, by (3.9), we have for $a, b \in (0, \delta_0)$

$$\frac{\langle|\tilde{u}(a)|, \tilde{v}(a)\rangle}{\langle|\tilde{u}(b)|, \tilde{v}(b)\rangle} = \lim_{n\to\infty} \frac{\langle \frac{|u(t_n+1+a)|}{\|u(t_n+1)\|_{L^2(\Omega)}}, \frac{v(t_n+1+a)}{\|v(t_n+1)\|_{L^2(\Omega)}}\rangle}{\langle \frac{|u(t_n+1+b)|}{\|u(t_n+1)\|_{L^2(\Omega)}}, \frac{v(t_n+1+b)}{\|v(t_n+1)\|_{L^2(\Omega)}}\rangle} = \lim_{n\to\infty} \frac{\xi(t_n+1+a)}{\xi(t_n+1+b)} = 1,$$

in contradiction to the decreasing property of $\langle|\tilde{u}(\tau)|, \tilde{v}(\tau)\rangle$. The proof of Lemma 3.3 is complete. □

4 Existence of positive entire solutions

In this section we prove the existence and continuity properties of positive entire solutions of (1.1), (1.5). In a continuity statement below, we consider sequences $(a_{ij}^{(n)})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i^{(n)}, b_i^{(n)}, c_0^{(n)} \in \mathcal{B}$, $i = 1, \dots, N$, of coefficients and denote by $\mathcal{A}_n$ the operator defined as in (1.2) with natural replacements of the coefficients.

Lemma 4.1. *Let $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \dots, N$ be as in Theorem 1.1. Then there exist unique positive entire solutions $\varphi_{\mathcal{A}}$, $\psi_{\mathcal{A}}$ of* (1.1), (1.5), *respectively such that $\|\varphi_{\mathcal{A}}(0)\|_{L^2(\Omega)} = \|\psi_{\mathcal{A}}(0)\|_{L^2(\Omega)} = 1$. Moreover, if $(a_{ij}^{(n)})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$ converges to $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$ and $a_i^{(n)}, b_i^{(n)}, c_0^{(n)} \in \mathcal{B}$, $i = 1, \dots, N$ converge to a_i, b_i, $c_0 \in \mathcal{B}$, in $\mathcal{B}_{\alpha_0}$ and $\mathcal{B}$, respectively, then one has for each $t \in \mathbb{R}$*

$$\varphi_{\mathcal{A}_n}(t) \to \varphi_{\mathcal{A}}(t), \;\; \psi_{\mathcal{A}_n}(t) \to \psi_{\mathcal{A}}(t),$$

with convergence in $C^\delta(\bar{\Omega})$ for any $0 < \delta < \alpha$, where α is as in Theorem 2.4.

Proof. We will first prove the existence using results on principal eigenvalues for a periodic-parabolic problem [7]. Let a_{ij}, a_i, b_i, c_0 be as above. For each $k \in \mathbb{N}$ let $\tilde{\mathcal{A}}_k$ be the operator of the form (1.2) with coefficients $\tilde{a}_{ij}^{(k)}$, $\tilde{a}_i^{(k)}, \tilde{b}_i^{(k)}, \tilde{c}_0^{(k)}$ identical to the coefficients a_{ij}, a_i, b_i, c_0, respectively on $\bar{\Omega} \times [-k, k]$ and periodically extended (with period $2k$) to $\bar{\Omega} \times \mathbb{R}$. Using the same approximation procedure as in [7], we get sequences of smooth coefficients $\tilde{a}_{ij}^{(k)n}$, $\tilde{a}_i^{(k)n}, \tilde{b}_i^{(k)n}, \tilde{c}_0^{(k)n}$, corresponding operators $\tilde{\mathcal{A}}_k^n$, and smooth domains $\Omega_n \subset \Omega$, $n \in \mathbb{N}$ such that the assumptions of Lemma A.1 are satisfied. Moreover, the approximating domains Ω_n can be chosen in such a way that the constants r_0 and m in the definition of a Lipschitz domain (see (B.1.1)) are independent of $n \in \mathbb{N}$.

Now, for each n and k, we find a positive entire solution u_k^n of (1.1), with $\mathcal{A}$, Ω replaced by $\tilde{\mathcal{A}}_k^n$, Ω_n, respectively, satisfying $\|u_k^n(0)\|_{L^2(\Omega_n)} = 1$. For this part, since the coefficients are smooth, we may assume the symmetry of the principal part of $\tilde{\mathcal{A}}_k^n$ and also that $\tilde{a}_i^{(k)n} = 0$, $i = 1, \ldots, N$; indeed, this is achieved by redefining the coefficients as follows: $\overline{a}_{ij}^{(k)n} := (\tilde{a}_{ij}^{(k)n} + \tilde{a}_{ji}^{(k)n})/2$, $\overline{a}_i^{(k)n} = 0$, $\overline{b}_i^{(k)n} = \tilde{b}_i^{(k)n} - \tilde{a}_i^{(k)n}$, $i, j = 1, \ldots, N$, and $\overline{c}_0^{(k)n} = \tilde{c}_0^{(k)n} - \partial_i \tilde{a}_i^{(k)n}$. For a while assume that, in addition, $\tilde{c}_0^{(k)n} \geq 0$. Then we can use Theorem 2.2 from [7] for equation (1.1) with $\mathcal{A}$ replaced by $\tilde{\mathcal{A}}_k^n$ (set $m \equiv 1$ in that theorem). It implies that there are φ_k^n, λ_k^n such that φ_k^n is a positive solution of

$$
\begin{aligned}
u_t + \tilde{\mathcal{A}}_k^n(t)u = \lambda_k^n\, u \quad &\text{in} \quad \Omega_n \times \mathbb{R},\\
u = 0 \qquad &\text{on} \quad \partial\Omega_n \times \mathbb{R},\\
u(\cdot,\cdot) = u(\cdot, \cdot + 2k) \quad &\text{in} \quad \Omega_n \times \mathbb{R}.
\end{aligned}
$$

Set

$$u_k^n(t) := e^{-\lambda_k^n t}\varphi_k^n(t)/\|\varphi_k^n(0)\|_{L^2(\Omega_n)}.$$

Now, removing the above restriction and taking a general $\tilde{c}_0^{(k)n}$, we find $\xi = \xi(n,k) > 0$ such that $c_\xi = \tilde{c}_0^{(k)n} + \xi \geq 0$ and then apply the above arguments to $\mathcal{A}_\xi = \tilde{\mathcal{A}}_k^n + \xi I$. This yields a positive $2k$-periodic eigenfunction φ_ξ corresponding to an eigenvalue λ_ξ. We then set

$$u_k^n(t) := e^{-(\lambda_\xi - \xi)t}\varphi_\xi(t)/\|\varphi_\xi(0)\|_{L^2(\Omega_n)}.$$

In either case, we obtain a positive entire solution u_k^n as desired.

Fix now $k_0 \in \mathbb{N}$. By a repeated use of Corollary B.7 and Lemma B.5 (estimate (B.2.2)), we get that $u_{k_0}^n$ are bounded in $L^\infty(\Omega_n \times (-T,T))$ uniformly with respect to $n \in \mathbb{N}$ for each $T > 0$. Theorem 2.4 and Remark 2.5 imply that we also have a uniform bound in $C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_n \times [-T,T])$. The bound remains valid if the functions $u_{k_0}^n$ are extended by zero outside Ω_n. This and Lemma A.1 imply that for a suitable subsequence, again denoted by $u_{k_0}^n$, we have $u_{k_0}^n \to u_{k_0}$ uniformly on $\bar{\Omega} \times [-T,T]$ for each $T > 0$, where u_{k_0} is a positive entire solution of (1.1), with $\mathcal{A}$ replaced by $\tilde{\mathcal{A}}_{k_0}$, satisfying $\|u_{k_0}(0)\|_{L^2(\Omega)} = 1$. Now, just as we did above, applying Lemma A.1 to operators $\tilde{\mathcal{A}}_k$ with coefficients $\tilde{a}_{ij}^{(k)}$, $\tilde{a}_i^{(k)}, \tilde{b}_i^{(k)}, \tilde{c}_0^{(k)}$, and the solutions u_k, we find a positive entire solution of (1.1) as the limit of (a subsequence of) u_k.

One can proceed similarly to prove the existence of a positive entire solution of the adjoint equation (1.5) (see our discussion preceding (2.2)). This concludes the proof of existence part of Lemma 4.1.

The uniqueness statements follow from Proposition 1.3 because we have already proved Theorem 1.2, which is the only ingredient used in the proof of Proposition 1.3.

Let now $\varphi_{\mathcal{A}_n}$, $\psi_{\mathcal{A}_n}$ be as in the statement of lemma. By standard arguments similar to those given above one concludes that these sequences are bounded in $C^{\alpha,\frac{\alpha}{2}}(\bar\Omega \times [-T,T])$ uniformly with respect to $n \in \mathbb{N}$ for each $T > 0$. Using the compact imbedding $C^\alpha(\bar\Omega) \hookrightarrow\hookrightarrow C^\delta(\bar\Omega)$ for any $\delta > 0$, $\delta < \alpha$, and passing to subsequences (using a diagonalization procedure), we see that for any $t \in \mathbb{R}$ we have

$$\varphi_{\mathcal{A}_n}(t) \to \varphi(t), \;\; \psi_{\mathcal{A}_n}(t) \to \psi(t) \quad \text{in } C^\delta(\bar\Omega)$$

for each $\delta < \alpha$, where φ, ψ are entire solutions of (1.1), (1.5). They are necessarily nonnegative and nontrivial since $\|\varphi_{\mathcal{A}_n}(0)\|_{L^2(\Omega)} = \|\psi_{\mathcal{A}_n}(0)\|_{L^2(\Omega)} = 1$. By the Harnack inequality, they are positive in Ω for all $t \in \mathbb{R}$. By uniqueness, the limiting functions are always the same for every subsequence, namely, they must be equal to $\varphi_{\mathcal{A}}(t)$ and $\psi_{\mathcal{A}}(t)$, respectively. Thus any subsequence of $\varphi_{\mathcal{A}_n}(t)$ ($\psi_{\mathcal{A}_n}(t)$) must converge to $\varphi_{\mathcal{A}}(t)$ ($\psi_{\mathcal{A}}(t)$). This concludes the proof. □

5 Proof of Theorem 1.1

The proof of Theorem 1.1 will depend on the following two lemmas. The first one is analogous to Theorem 1.2.

Lemma 5.1. *There exist positive constants C, γ such that for all $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \dots, N$, as in Theorem 1.1 the following statement holds. If ψ is a positive entire solution of* (1.5) *and u is a solution of* (1.1) *on $[s_0, \infty)$ such that $\langle u(s_0), \psi(s_0)\rangle = 0$ (so that $\langle u(s), \psi(s)\rangle = 0$ for $s \ge s_0$) then*

$$\langle |u(t)|, \psi(t)\rangle \le Ce^{-\gamma(t-s)}\langle |u(s)|, \psi(s)\rangle \quad (t \ge s \ge s_0). \tag{5.1}$$

Proof. The argument is similar to the proof of Lemma 3.3. We will show how it can be modified to apply to the present case.

By Lemma 3.1, the left hand side of (5.1) is a decreasing function of t. It is therefore sufficient to show that for any $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \ldots, N$, and any solutions u and ψ as in Lemma 5.1, we have

$$\frac{\langle |u(t+2)|, \psi(t+2)\rangle}{\langle |u(t)|, \psi(t)\rangle} \leq \rho \qquad (t \geq s_0)$$

for some constant $\rho < 1$ independent of u, ψ, and $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \ldots, N$. Suppose this is not true, that is, there are sequences $(a_{ij}^n)_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i^{(n)}, b_i^{(n)}, c_0^{(n)} \in \mathcal{B}$, $i = 1, \ldots, N$, $t_n \in \mathbb{R}$, u_n, ψ_n such that, after the usual replacements of the coefficients, u_n is a solution of (1.1) on $[t_n, \infty)$, ψ_n is a positive entire solution of (1.5), $\langle u_n(t_n), \psi_n(t_n)\rangle = 0$ and

$$\frac{\langle |u_n(t_n+2)|, \psi_n(t_n+2)\rangle}{\langle |u_n(t_n)|, \psi_n(t_n)\rangle} \to 1.$$

Repeatedly passing to suitable subsequences similarly as we did in the proof of Lemma 3.3, we find nontrivial limit solutions u, ψ that violate the decreasing property of $\langle |u(t)|, \psi(t)\rangle$. This contradiction concludes the proof. □

Lemma 5.2. *Let u be a solution of* (1.1) *defined on* $[s_0, \infty)$ *and let φ, ψ be positive entire solutions of* (1.1) *and* (1.5), *respectively. Then there exists a constant $\tilde{C} > 0$ such that*

$$\tilde{C}\|\varphi(t)\|_{L^2(\Omega)} \leq \langle \varphi(t), \frac{\psi(t)}{\|\psi(t)\|_{L^2(\Omega)}}\rangle \qquad (t \in \mathbb{R}), \tag{5.2}$$

and for any $\delta > 0$ there is a constant $C(\delta) > 0$ such that

$$\langle |u(t)|, \frac{\psi(t)}{\|\psi(t)\|_{L^2(\Omega)}}\rangle \geq C(\delta)\|u(t+\delta)\|_{L^2(\Omega)} \qquad (t \geq s_0). \tag{5.3}$$

Moreover, the constants $\tilde{C}$, $C(\delta)$ are independent of u, φ, ψ, s_0.

Remark 5.3. Using the Hölder inequality in (5.2), we obtain that the expression $\langle \varphi(t), \frac{\psi(t)}{\|\psi(t)\|_{L^2(\Omega)}}\rangle$ is comparable to $\|\varphi(t)\|_{L^2(\Omega)}$ for each $t \in \mathbb{R}$. We will use this observation below.

Proof of Lemma 5.2. Applying estimate (B.2.4) from Lemma B.8 to φ and ψ, we get (5.2) immediately. To prove (5.3), fix $\delta > 0$ and $t \geq s_0$ and let

$\delta_0 = 2\delta$. Now using successively Lemma B.10, equality (2.2), inequality (3.2) (with $t+\delta_0$, t playing the role of t, s, respectively), and Proposition 2.2, we derive

$$\begin{aligned}
\langle |u(t)|, \frac{\psi(t)}{\|\psi(t)\|_{L^2(\Omega)}}\rangle &= \langle |u(t)|, \frac{\frac{\psi(t)}{\|\psi(t)\|_{L^2(\Omega)}}}{U^*(t,t+\delta_0)1} U^*(t,t+\delta_0)1\rangle \\
&\geq C(\delta_0)\langle |u(t)|, U^*(t,t+\delta_0)1\rangle = C(\delta_0)\langle U(t+\delta_0,t)|u(t)|, 1\rangle \\
&\geq C(\delta_0)\langle |u(t+\delta_0)|, 1\rangle = C(\delta_0)\|u(t+\delta_0)\|_{L^1(\Omega)} \\
&\geq C(\delta_0) M^{-1}\delta^{-\frac{N}{4}} e^{-\beta\delta}\|u(t+\delta)\|_{L^2(\Omega)}.
\end{aligned}$$

Here M, β, and $C(\delta_0)$ are the constants as in Proposition 2.2, and Lemma B.10, respectively; they are independent of u, φ, ψ, s_0. □

We are in a position to give

Proof of Theorem 1.1. Statement (i) follows directly from Lemma 4.1.

We prove (ii). The invariance of $X^2_{\mathcal{A}}(t)$, $t \in \mathbb{R}$, as stated in (ii), follows from (3.1). The invariance of $X^1_{\mathcal{A}}(t)$, $t \in \mathbb{R}$, is obvious. Since $\psi_{\mathcal{A}}(t) > 0$, the space $X^2_{\mathcal{A}}(t)$ contains no (nontrivial) nonnegative function. On the other, $X^1_{\mathcal{A}}(t)$ is spanned by a positive function, hence $X^1_{\mathcal{A}}(t) \cap X^2_{\mathcal{A}}(t) = \{0\}$ $(t \in \mathbb{R})$. This and a dimension-codimension count yield (1.7).

It remains to prove (iii). For brevity, set $\varphi = \varphi_{\mathcal{A}}$, $\psi = \psi_{\mathcal{A}}$ and $u(t) = u(\cdot, t; s, u_0)$, with u_0 as in (iii). We first apply Lemma 5.1 to these solutions, which yields (5.1). Since $\langle \varphi(t), \psi(t)\rangle$ is a constant function of $t \in \mathbb{R}$, (5.1) can be equivalently rewritten as

$$\frac{\langle |u(t)|, \psi(t)/\|\psi(t)\|_{L^2(\Omega)}\rangle}{\langle \varphi(t), \psi(t)/\|\psi(t)\|_{L^2(\Omega)}\rangle} \leq Ce^{-\gamma(t-s)}\frac{\langle |u(s)|, \psi(s)/\|\psi(s)\|_{L^2(\Omega)}\rangle}{\langle \varphi(s), \psi(s)/\|\psi(s)\|_{L^2(\Omega)}\rangle}. \qquad (5.4)$$

By Remark 5.3, we can replace the denominators in (5.4) by $\|\varphi(t)\|_{L^2(\Omega)}$ and $\|\varphi(s)\|_{L^2(\Omega)}$, respectively, enlarging the constant C if necessary. The numerator on the right hand side of (5.4) is bounded above by $\|u(s)\|_{L^2(\Omega)}$ by the Hölder inequality. Applying (5.3) (for some fixed $\delta > 0$, δ small) to the numerator on the left hand side of the above inequality, we obtain

$$\frac{\|u(t+\delta)\|_{L^2(\Omega)}}{\|\varphi(t)\|_{L^2(\Omega)}} \leq C(\delta)e^{-\gamma(t-s)}\frac{\|u(s)\|_{L^2(\Omega)}}{\|\varphi(s)\|_{L^2(\Omega)}} \quad (t \geq s). \qquad (5.5)$$

Using now the backward-in-time estimate (B.2.2) from Lemma B.5, we see that for all $r \in [0, \frac{r_0}{2}]$ (where r_0 is as in (B.1.1))

$$\frac{\|\varphi(t+2r^2)\|_{L^\infty(\Omega)}}{\|\varphi(t)\|_{L^\infty(\Omega)}} \geq C > 0. \tag{5.6}$$

By Corollary B.7, we can replace the norms in (5.6) by $L^2(\Omega)$ norms, making the constant C smaller if necessary. Taking $r = \sqrt{\frac{\delta}{2}}$ (we may assume $\delta \leq \frac{r_0^2}{2}$) and applying this modified estimate to (5.5), we obtain

$$\frac{\|u(t+\delta)\|_{L^2(\Omega)}}{\|\varphi(t+\delta)\|_{L^2(\Omega)}} \leq C(\delta) e^{-\gamma(t-s)} \frac{\|u(s)\|_{L^2(\Omega)}}{\|\varphi(s)\|_{L^2(\Omega)}}.$$

Thus, we have established (1.8) in (*iii*) of Theorem 1.1 for $t \geq s + \delta$.

Finally, using again (5.6) for all $r \in [0, \sqrt{\frac{\delta}{2}}]$ and Proposition 2.2 (with $p = q = 2$), we see that estimate (1.8) in (*iii*) is readily verified for $t \geq s$ such that $t - s \leq \delta$, with a constant C enjoying the same properties as in the statement of Theorem 1.1. This completes the proof. □

A Appendix: Perturbation of the initial value problem

Throughout the paper we use a perturbation result from [6]. We state it in a simplified form suitable for our purposes.

Lemma A.1. *Suppose that $\Omega_n \subset \Omega$ is a sequence of domains with $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Further suppose that $\mathcal{A}_n$ is of the form* (1.2) *with coefficients $a_{ij}^{(n)}$, $a_i^{(n)}$, $b_i^{(n)}$, $c_0^{(n)}$ bounded in $L^\infty(\Omega \times \mathbb{R})$ uniformly with respect to $n \in \mathbb{N}$, and converging pointwise almost everywhere to the corresponding coefficients of $\mathcal{A}$. Assume that the uniform ellipticity condition* (1.3) *is satisfied with a_{ij} replaced by $a_{ij}^{(n)}$, etc. (and with α_0 independent of $n \in \mathbb{N}$). Finally, fix $s \in \mathbb{R}$ and suppose that u_n is the solution of*

$$\begin{aligned} u_t + \mathcal{A}_n(t)u = 0 \quad & \text{in} \quad \Omega_n \times (s,\infty), \\ u = 0 \quad & \text{on} \quad \partial\Omega_n \times (s,\infty), \\ u = u_{0n} \quad & \text{in} \quad \Omega_n \times \{s\}, \end{aligned}$$

with u_{0n} converging to u_0 weakly in $L^2(\Omega)$ (when extended to zero outside Ω_n). Then, for all $T > s$ and $\delta \in (0, T-s]$, the function u_n converges to the solution u of (2.1) *in $C([s+\delta, T]; L^2(\Omega))$ as n tends to infinity. Moreover, if u_{0n} converges strongly in $L^2(\Omega)$ to u_0, then u_n converges to u in $C([s,T]; L^2(\Omega))$ for all $T > s$.*

Remark A.2. Much more general perturbation results can be found in [6].

B Appendix: Positive solutions of linear parabolic equations

In this section we derive several important properties of positive solutions of linear (nonautonomous) second order parabolic equations subject to Dirichlet boundary condition. As we have already mentioned in the introduction this will be achieved by using some of the results from [12] on the boundary behavior of such solutions. For the sake of completeness we include them in Subsection B.1.

B.1 Harnack inequalities and boundedness of quotients

Recall that our assumption is that Ω is a bounded *Lipschitz* domain in $\mathbb{R}^N$. This means there are positive constants r_0 and m such that for each $y \in \partial\Omega$, there is an orthonormal coordinate system centered at y in which

$$\Omega \cap B_{r_0}(y) = \{x = (x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > \phi(x'), |x| < r_0\} \qquad \text{(B.1.1)}$$

and $\|\nabla\phi\|_{L^\infty} \le m$. Here and below $B_r(x)$ denotes the ball in $\mathbb{R}^N$ of radius $r > 0$ and center x. For $X = (x,t) \in \mathbb{R}^{N+1}$, we define a "standard" parabolic cylinder to be

$$C_r(X) = C_r(x,t) \equiv B_r(x) \times (t - r^2, t + r^2).$$

Further, let us denote

$$Q = \Omega \times (0,\infty),\ \ S = \partial\Omega \times (0,\infty),\ \ Q_r(X) = Q \cap C_r(X).$$

For any constant $\delta > 0$, we set

$$\Omega^\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

According to the above Lipschitz criteria, for $y \in \partial\Omega$ there is an orthonormal system with y as the origin $(0,0)$ and $(0,r) \in \Omega$ for all $r \in (0, r_0]$. The new coordinates for $Y = (y,s) \in \mathbb{R}^{N+1}$ are $(0,0,s)$. In this coordinate system write

$$\overline{Y}_r = (0, r, s + 2r^2), \qquad \underline{Y}_r = (0, r, s - 2r^2).$$

Let us temporarily assume that the operator $\mathcal{A}$ in (1.2) contains only the coefficients of the highest order, that is, assume for now that $a_i = b_i = c_0 = 0$ for $i = 1, \dots, N$. As always, we still assume that the coefficients of $\mathcal{A}$ satisfy (1.3) and (1.4) although for the results in this subsection it is sufficient to require that they be bounded, measurable and that they satisfy (1.3). The next result is often referred to as a *boundary Harnack inequality.*

Theorem B.1. ([12, Theorem 2]) *Let $Y = (y,s) \in S$ and $0 < r \le \frac{1}{2}\min(r_0, \sqrt{s})$. Then for any nonnegative solution of $u_t + \mathcal{A}(t)u = 0$ in Q which continuously vanishes on $S \cap C_{2r}(Y)$, we have*

$$\sup_{Q_r(Y)} u \le C u(\overline{Y}_r).$$

The constant C depends only on N, m, α_0 (α_0 is as in (1.3)*), and the $L^\infty(Q)$ bound on the coefficients of $\mathcal{A}$.*

We will also need the following lemma which is used in the proof of the *backward Harnack inequality near the boundary.*

Theorem B.2. ([12, Lemma 1]) *Let u be a nonnegative solution of $u_t + \mathcal{A}(t)u = 0$ in Q. Take $Y = (y,s) \in S$ and $0 < r \le \frac{1}{2}\min(r_0, \sqrt{s})$. Then*

$$u(\underline{Y}_r) \le C r^\theta \inf_{Q_r(Y)} d^{-\theta} u,$$

where $d = d(x) \equiv \operatorname{dist}(x, \partial\Omega)$, and C, θ are positive constants depending only on N, m, α_0, and the $L^\infty(Q)$ bound on the coefficients of $\mathcal{A}$.

The next theorem states, roughly speaking, that the quotient of two positive solutions of a parabolic equation is bounded near the portion of the lateral boundary where each solution vanishes.

Theorem B.3. ([12, Theorem 5]) *Let $X_0 = (x_0, t_0) \in S$. Assume that u and v are two positive solutions of $u_t + \mathcal{A}(t)u = 0$ in Q which continuously vanish on $S \cap C_{2r}(X_0)$ with $0 < r \le \frac{1}{2}\min(r_0, \sqrt{t_0})$. Then*

$$\sup_{Q_r(X_0)} \frac{u}{v} \le C \frac{u(\overline{X_0}_r)}{v(\underline{X_0}_r)}.$$

The constant C depends only on N, m, α_0, and the $L^\infty(Q)$ bound on the coefficients of $\mathcal{A}$.

Remark B.4. Of course, the statements hold in exactly the same way if we consider nonnegative solutions on time intervals of the form (s, ∞) for arbitrary $s \in \mathbb{R}$.

So far we have assumed that the first- and zero-order coefficients of $\mathcal{A}$ are identically zero. Eventually we would like to apply the above theorems to positive solutions of (1.1) without this restriction. Let us indicate how this can be achieved using the method of "additional variable" (cf. [11]). Let

$$Lu = \partial_i(a_{ij}(x,t)\partial_j u + a_i(x,t)u) - b_i(x,t)\partial_i u - c_0(x,t)u,$$

and assume $u_t = Lu$ in $\Omega \times (0,T) \subset \mathbb{R}^{N+1}$. Defining $\Omega_0 = (-1,1) \times \Omega$ and setting

$$a^0_{ij}(x_0,x,t) = a_{ij}(x,t),\ a^0_{0i} = b_i,\ a^0_{i0} = a_i,\ a^0_{00}(x_0,x,t) = \kappa - c_0(x,t)$$

for all $i,j = 1,\dots,N$, one can see that the $(N+1)\times(N+1)$ matrix $(a^0_{ij})^N_{i,j=0}$ satisfies the ellipticity condition (1.3) with some constant $\nu_0 \in (0,1]$, provided the constant κ is sufficiently large. The function

$$U(x_0,x,t) = e^{x_0+\kappa t}u(x,t)$$

satisfies on $\Omega_0 \times (0,T)$ the following parabolic equation:

$$\begin{aligned} L_0 U = \sum_{i,j=0}^{N} \partial_i(a^0_{ij}(x_0,x,t)\partial_j U) &= e^{x_0+\kappa t}(Lu + c_0(x,t)u + (\kappa - c_0(x,t))u) \\ &= e^{x_0+\kappa t}(u_t + \kappa u) = U_t. \end{aligned}$$

We see that this equation no longer contains lower order terms. Therefore we can apply the above theorems to U provided the corresponding hypotheses are satisfied on the extended domain $\Omega_0 \times (0,\infty)$. This allows us to carry over all results to the case of general second order operators of the form (1.2). The only difference is that the constant C in Theorem B.1, Theorem B.2, and Theorem B.3 now depends also on the $L^\infty(Q)$ bound of the lower order coefficients a_i, b_i, c_0.

B.2 Uniform estimates at the boundary

Let us now use the results we have collected in the previous subsection to prove several properties of positive solutions of (1.1). These are used in the proof of Theorem 1.1. Assumptions here are the same as in the introduction and we use the notation as in the previous subsection. Let $y = (0, 0) \in \partial\Omega$ be as in that subsection. An easy geometric argument shows that for $r \in (0, r_0)$ we have

$$\operatorname{dist}((0, r), \partial\Omega) \in [c_1 r, r], \tag{B.2.1}$$

where $0 < c_1 < 1$ is a constant independent of $y \in \partial\Omega$ and $r \in (0, r_0)$. Let now u be a nonnegative solution of (1.1) on $\Omega \times (0, \infty)$ and fix a positive δ_0. It is easy to see that Theorem B.1 and (B.2.1) together imply the following estimate

$$\sup_{\substack{0 \le \operatorname{dist}(x,\partial\Omega) \le r \\ t - r^2 \le \tau \le t}} u(x, \tau) \le C \sup_{c_1 r \le \operatorname{dist}(x,\partial\Omega) \le r} u(x, t + 2r^2)$$

for any $t \ge \delta_0$ and $0 < r \le \frac{1}{2}\min(r_0, \sqrt{\delta_0})$, with a constant C enjoying the same properties as the constant in Theorem B.1. Combining this estimate with the usual Harnack inequality inside $\Omega \times (0, \infty)$ (see [25, 1, 12]), we get for any $t \ge \delta_0$ and $r \in (0, \frac{1}{2}\min(r_0, \sqrt{\delta_0})]$

$$\sup_{x \in \Omega} u(x, t) \le C(r) \sup_{x \in \overline{\Omega^{c_1 r}}} u(x, t + 2r^2),$$

where moreover $C(r) \le C$ as $r \to 0^+$. Using now the $L^\infty - L^\infty$ estimates from Proposition 2.2, we obtain (for $t \ge \delta_0$ and $r \in (0, \frac{1}{2}\min(r_0, \sqrt{\delta_0})]$)

$$\begin{aligned}
\|u(t + 2r^2)\|_{L^\infty(\Omega)} &\le C_1 \|u(t)\|_{L^\infty(\Omega)} \\
&\le C_2 \sup_{x \in \overline{\Omega^{c_1 r}}} u(x, t + 2r^2) = C_2 \|u(t + 2r^2)\|_{L^\infty(\Omega^{c_1 r})} \\
&\le C_2 \|u(t + 2r^2)\|_{L^\infty(\Omega)},
\end{aligned}$$

with constants C_1, C_2 uniformly bounded for $r \to 0^+$. This means that the supremum of any nonnegative solution at a given time is controlled by the supremum on a subdomain. We have thus proved

Lemma B.5. *Let $\delta_0 > 0$ and $0 < r \le \frac{1}{2}\min(r_0, \sqrt{\delta_0})$. Then there exists a constant $C = C(\delta_0)$ such that if u is a nonnegative solution of* (1.1) *on $\Omega \times (0, \infty)$, then the following estimates hold for $t \ge \delta_0$*

$$\|u(t)\|_{L^\infty(\Omega)} \le C(\delta_0) \|u(t + 2r^2)\|_{L^\infty(\Omega)}, \tag{B.2.2}$$

$$\|u(t + 2r^2)\|_{L^\infty(\Omega)} \le C(\delta_0) \|u(t + 2r^2)\|_{L^\infty(\Omega^{c_1 r})}, \tag{B.2.3}$$

where $\Omega^{c_1 r} = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) > c_1 r\}$, *with* $c_1 > 0$ *depending only on* $\partial\Omega$ *(cf.* (B.2.1)*).*

Remark B.6. By a time translation, it is obvious that the above lemma (and the results below) apply to positive solutions defined on $\Omega \times (s, \infty)$ for some $s \in \mathbb{R}$.

Corollary B.7. *Let* $0 < \delta_0 \le r_0^2$*. Then there exists* $C = C(\delta_0) > 0$ *such that for any nonnegative nontrivial solution* u *of* (1.1) *on* $\Omega \times (0, \infty)$ *we have*

$$\frac{\|u(t)\|_{L^2(\Omega)}}{\|u(t)\|_{L^\infty(\Omega)}} \ge C(\delta_0) \qquad\qquad (t \ge \delta_0).$$

Proof. For $0 < r \le \frac{1}{2}\sqrt{\delta_0}$ and $t \ge \delta_0$ write

$$\frac{\|u(t)\|_{L^2(\Omega)}}{\|u(t)\|_{L^\infty(\Omega)}} = \left(\frac{\|u(t)\|_{L^2(\Omega)}}{\|u(t+2r^2)\|_{L^\infty(\Omega)}} \right) \left(\frac{\|u(t+2r^2)\|_{L^\infty(\Omega)}}{\|u(t)\|_{L^\infty(\Omega)}} \right).$$

The first fraction on the right is estimated below by $M^{-1}(2r^2)^{\frac{N}{4}} e^{-\beta 2r^2}$, where M and β are as in Proposition 2.2. The second fraction is bounded from below by $1/C(\delta_0)$ by Lemma B.5. Hence for any $r \in (0, \frac{1}{2}\sqrt{\delta_0}]$

$$\frac{\|u(t)\|_{L^2(\Omega)}}{\|u(t)\|_{L^\infty(\Omega)}} \ge \frac{1}{MC(\delta_0)} (2r^2)^{\frac{N}{4}} e^{-\beta 2r^2}.$$

Setting $r = \frac{1}{2}\sqrt{\delta_0}$, we get the desired inequality (redefining $C(\delta_0)$). □

We will use Lemma B.5 and Corollary B.7 in the proof of the following important pointwise estimate, which is also of independent interest.

Lemma B.8. *For each* $\delta_0 > 0$ *there is a constant* $C(\delta_0)$ *with the following properties. For any* $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$*,* $a_i, b_i, c_0 \in \mathcal{B}$*,* $i = 1, \dots, N$ *and* $s \in \mathbb{R}$ *and any positive solution* u *of* (1.1) *on* $\Omega \times (s, \infty)$ *one has*

$$\frac{u(x,t)}{\|u(t)\|_{L^2(\Omega)}} \ge C(\delta_0) d(x)^\theta \quad ((x,t) \in \Omega \times [s + \delta_0, \infty)), \tag{B.2.4}$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$ *and* θ *is as in Theorem B.2.*

Proof. Let $\delta_0 > 0$ and let u be a positive solution of (1.1) on $\Omega \times (s, \infty)$. Fix $t \geq s + \delta_0$, $r \in (0, \frac{1}{2}\min(r_0, \sqrt{\delta_0})]$ and $x \in \Omega$ such that $d(x) = \operatorname{dist}(x, \partial\Omega) \leq c_1 r$, with c_1 as in (B.2.1). Then there exists $Y = (y, t) \in \partial\Omega \times \{t\}$ such that $(x, t) \in Q_r(Y)$. Moreover, we can find $\tilde{x} \in \Omega$ such that $\underline{Y}_r = (\tilde{x}, t - 2r^2)$. Using Theorem B.2, we see that

$$u(x, t) \geq C r^{-\theta} d(x)^{\theta} u(\tilde{x}, t - 2r^2).$$

Notice that by (B.2.1) we have $d(\tilde{x}) \in [c_1 r, r]$. Taking infimum on the right hand side of this inequality over points $\tilde{x}$ verifying $d(\tilde{x}) \in [c_1 r, r]$, we obtain that for any $t \geq s + \delta_0$ and $r \in (0, \frac{1}{2}\min(r_0, \sqrt{\delta_0})]$ the inequality

$$u(x, t) \geq C r^{-\theta} d(x)^{\theta} \inf_{c_1 r \leq d(\tilde{x}) \leq r} u(\tilde{x}, t - 2r^2) \tag{B.2.5}$$

holds whenever $x \in \Omega$ is such that $d(x) \leq c_1 r$. The constant C is independent of u and s.

Suppose now that the conclusion of Lemma B.8 does not hold. Then there exist sequences of functions u_n, $(a_{ij}^{(n)})_{i,j=1}^{N} \in \mathcal{B}_{\alpha_0}$, $a_i^{(n)}, b_i^{(n)}, c_0^{(n)} \in \mathcal{B}$, $i = 1, \ldots, N$, real numbers s_n, $t_n \geq s_n + \delta_0$, and points $x_n \in \Omega$ such that u_n is a positive solution of (1.1) on $\Omega \times (s_n, \infty)$, with $\mathcal{A}$ replaced by $\mathcal{A}_n$, and the following inequality holds

$$\frac{u_n(x_n, t_n)}{\|u_n(t_n)\|_{L^2(\Omega)}} \leq \frac{1}{n} d(x_n)^{\theta} \tag{B.2.6}$$

for all $n \in \mathbb{N}$. For $\tau > -\delta_0$ set

$$\begin{aligned} \tilde{u}_n(\tau) &:= u_n(t_n + \tau)/\|u_n(t_n)\|_{L^2(\Omega)}, \\ \tilde{\mathcal{A}}_n(\tau) &:= \mathcal{A}_n(t_n + \tau). \end{aligned}$$

Note that $\tilde{u}_n$ is a positive solution of

$$\begin{aligned} u_\tau + \tilde{\mathcal{A}}_n(\tau) u = 0 \quad &\text{in} \quad \Omega \times (-\delta_0, \infty), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (-\delta_0, \infty), \end{aligned}$$

with $\|\tilde{u}_n(0)\|_{L^2(\Omega)} = 1$ and

$$\tilde{u}_n(x_n, 0) \leq \frac{1}{n} d(x_n)^{\theta} \qquad \text{for} \quad n \in \mathbb{N}. \tag{B.2.7}$$

Repeatedly using Corollary B.7, estimate (B.2.2) and Theorem 2.4, we find a constant $C > 1$ such that

$$\frac{1}{C} \leq \|\tilde{u}_n\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[-\frac{\delta_0}{2},\frac{\delta_0}{2}])} \leq C \tag{B.2.8}$$

for all $n \in \mathbb{N}$. Due to our assumptions on the coefficients of $\tilde{\mathcal{A}}_n$ and by the compact imbedding $C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times [-\frac{\delta_0}{2}, \frac{\delta_0}{2}]) \hookrightarrow\hookrightarrow C(\bar{\Omega} \times [-\frac{\delta_0}{2}, \frac{\delta_0}{2}])$, we see that, after passing to subsequences, $x_n \to x_0 \in \bar{\Omega}$, $\tilde{u}_n \to \tilde{u}$ in $C(\bar{\Omega} \times [-\frac{\delta_0}{2}, \frac{\delta_0}{2}])$ as $n \to \infty$, and the coefficients of $\tilde{\mathcal{A}}_n$ converge almost everywhere in $\Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2})$ to the coefficients of some uniformly elliptic operator $\tilde{\mathcal{A}}$. By Lemma A.1, $\tilde{u}$ is a solution of

$$\begin{aligned} u_\tau + \tilde{\mathcal{A}}(\tau)u = 0 \quad &\text{in} \quad \Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2}), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2}). \end{aligned}$$

Moreover, it is obvious that $\|\tilde{u}(0)\|_{L^2(\Omega)} = 1$ and $\tilde{u} \geq 0$ on $\Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2})$. From the usual Harnack inequality it follows that $\tilde{u} > 0$ in $\Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2})$. Now, since $\tilde{u}(\cdot, 0)$ is positive in Ω and $\tilde{u}_n \to \tilde{u}$ as $n \to \infty$, (B.2.7) forces $x_0 = \lim_{n\to\infty} x_n \in \partial\Omega$. Apply now inequality (B.2.5) to $u = \tilde{u}_n$, $x = x_n$, $t = 0$, $r = \frac{1}{2}\min(r_0, \frac{1}{2}\sqrt{\delta_0}) =: \tilde{r}$, use (B.2.7) and send n to infinity to conclude

$$\inf_{c_1\tilde{r}\leq d(\tilde{x})\leq\tilde{r}} \tilde{u}(\tilde{x}, -2\tilde{r}^2) = 0.$$

From our choice of $\tilde{r}$ and from continuity of $\tilde{u}$ it follows that there is a point $(\tilde{x}, \tilde{t}) \in \Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2})$ such that $\tilde{u}(\tilde{x}, \tilde{t}) = 0$. This, however, is a contradiction to $\tilde{u}$ being positive in $\Omega \times (-\frac{\delta_0}{2}, \frac{\delta_0}{2})$. This finishes the proof of Lemma B.8. □

Remark B.9. So far we have discussed properties of positive solutions of (1.1). In view of our discussion in Section 2, analogous properties can be proved for the adjoint problem (1.5). In particular, statements of Theorem B.1, Theorem B.2, and Theorem B.3 remain valid when applied to nonnegative solutions of (1.5).

Finally, let us prove an estimate that is used in the proof of (5.3). We will use $U^*(\cdot, \cdot)$ to denote the evolution operator defined in Section 2.

Lemma B.10. *Let $\delta_0 > 0$. Then there is a constant $C = C(\delta_0) \in (0,1)$ such that if $(a_{ij})_{i,j=1}^N \in \mathcal{B}_{\alpha_0}$, $a_i, b_i, c_0 \in \mathcal{B}$, $i = 1, \ldots, N$ and ψ is a positive entire solution of* (1.5) *then*

$$C(\delta_0) \leq \frac{\frac{\psi(x,t)}{\|\psi(t)\|_{L^2(\Omega)}}}{(U^*(t, t+\delta_0)1)(x)} \leq \frac{1}{C(\delta_0)} \quad ((x,t) \in \Omega \times \mathbb{R}).$$

Proof. Consider the solution $\tilde{v}$ of (1.5) with $v(\cdot, t+\delta_0) \equiv 1$, that is, for any $\tau \leq t + \delta_0$ we have $\tilde{v}(x,\tau) = (U^*(\tau, t+\delta_0)1)(x)$. Using the same limiting arguments as in the proof of Lemma B.8, one proves that for each $\delta_0 > 0$ and $r > 0$, r sufficiently small, there exists a constant $C = C(r, \delta_0) > 0$ such that

$$\inf_{\substack{x \in \overline{\Omega^{c_1 r}} \\ t-\delta_0 \leq \tau \leq t+\frac{\delta_0}{2}}} \tilde{v}(x,\tau) \geq C(r, \delta_0). \tag{B.2.9}$$

Again, C is independent of $t \in \mathbb{R}$. Moreover, by (B.2.2), Proposition 2.2 (with $p = q = \infty$) and Corollary B.7, the fraction $\|\psi(t)\|_{L^2(\Omega)}/\|\psi(s)\|_{L^2(\Omega)}$ is bounded below and above by positive constants independent of $t, s \in \mathbb{R}$, and ψ, whenever $|t - s| \leq \frac{r_0}{2}$ and ψ is a positive entire solution of (1.5). This and an application of estimate (B.2.4) from Lemma B.8 shows that (B.2.9) holds with $\tilde{v}$ replaced by the function $\tilde{\psi}(\tau) := \psi(\tau)/\|\psi(t+\delta_0)\|_{L^2(\Omega)}$ ($C(r, \delta_0)$ may have to be made smaller). Using these facts, $L^2 - L^\infty$ estimates from Proposition 2.2 and Theorem B.3, first with $u = \tilde{\psi}$ and $v = \tilde{v}$ then with $u = \tilde{v}$ and $v = \tilde{\psi}$, we obtain that, for each x in a fixed neighborhood of the boundary $\partial\Omega$, the quotient

$$\frac{\frac{\psi(x,t)}{\|\psi(t)\|_{L^2(\Omega)}}}{\tilde{v}(x,t)} \tag{B.2.10}$$

is uniformly bounded below and above by positive constants, which are independent of ψ, t. Finally, applying estimate (B.2.4) to ψ and the $L^\infty - L^\infty$ estimate from Proposition 2.2 to $\tilde{v}$, we find a positive lower bound for the quotient (B.2.10) depending only on r and δ_0, whenever $d(x, \partial\Omega) \geq r$ ($0 < r \leq \frac{r_0}{2}$). An analogous upper bound is proved by applying Corollary B.7 to ψ and employing inequality (B.2.9). This completes the proof. $\square$

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