Dynamics of nonnegative solutions of one-dimensional reaction-diffusion equations with localized initial data. Part I: A general quasiconvergence theorem and its consequences

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Abstract

Abstract. We consider the Cauchy problem

$$\begin{aligned} u_t &= u_{xx} + f(u), && x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), && x \in \mathbb{R}, \end{aligned}$$

where f is a locally Lipschitz function on $\mathbb{R}$ with $f(0) = 0$, and u_0 is a nonnegative function in $C_0(\mathbb{R})$, the space of continuous functions with limits at $\pm\infty$ equal to 0. Assuming that the solution u is bounded, we study its large-time behavior from several points of view. One of our main results is a general quasiconvergence theorem saying that all

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limit profiles of $u(\cdot,t)$ in $L^\infty_{loc}(\mathbb{R})$ are steady states. We also prove convergence results under additional conditions on u_0. In the bistable case, we characterize the solutions on the threshold between decay to zero and propagation to a positive steady state, and show that the threshold is sharp for each increasing family of initial data in $C_0(\mathbb{R})$.

Key words: Parabolic equations on $\mathbb{R}$, localized initial data, convergence, quasiconvergence, threshold solutions, generalized omega-limit set

AMS Classification: 35K15, 35B40

Contents

1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where f is a locally Lipschitz function on $\mathbb{R}$ with $f(0) = 0$ and $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In this paper, we are mainly interested in nonnegative initial data in $C_0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ converging to 0 at $x = \pm\infty$.

However, the introductory discussion is more general and we do not make that restriction yet.

We denote by $u(\cdot, t, u_0)$ the unique (classical) solution of (1.1), (1.2) and by $T(u_0) \in (0, \infty]$ its maximal existence time. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is *global*. We examine the large-time behavior of bounded solutions from several points of view.

Our first concern is the behavior of bounded solutions in a localized topology. We thus introduce the ω-limit set of such a solution u, denoted by $\omega(u)$ or by $\omega(u_0)$ if the initial datum of u is specified, as follows:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some } t_n \to \infty\}. \tag{1.3}$$

Here the convergence is in $L^\infty_{loc}(\mathbb{R})$, that is, the locally uniform convergence. By standard parabolic regularity estimates, the trajectory $\{u(\cdot, t) : t \geq 1\}$ of the bounded solution u is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that $\omega(u)$ is nonempty, compact and connected in $L^\infty_{loc}(\mathbb{R})$, and it attracts the solution in the following sense:

$$\operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot, t), \omega(u)) \to 0 \text{ as } t \to \infty \tag{1.4}$$

(this makes sense, as the space $L^\infty_{loc}(\mathbb{R})$ is metrizable, although not complete).

If equation (1.1) is considered on a bounded interval I, instead of $\mathbb{R}$, and one of common boundary conditions, say Dirichlet, Neumann, Robin, or periodic, is assumed, then it admits a Lyapunov functional. Specifically, the usual energy functional

$$\varphi \mapsto \int_I \left(\frac{\varphi_x^2(x)}{2} - F(\varphi(x)) \right) dx, \quad \text{with } F(u) = \int_0^u f(s)\, ds, \tag{1.5}$$

is well defined along any classical solution and is strictly decreasing in t if the solution is not a steady state. It is a standard consequence of the presence of a Lyapunov functional that each bounded solution u is *quasiconvergent:* $\omega(u)$ consists of steady states ($\omega(u)$ in this case is defined as in (1.3) with the convergence in $L^\infty(I)$). Actually, the problems on bounded intervals have a much more robust structure, and it can even be proved that each bounded solution is *convergent:* $\omega(u)$ consists of a single steady state [6, 23, 29].

In contrast, bounded solutions of (1.1) on $\mathbb{R}$ are not convergent in general, even if $f \equiv 0$. As shown in [8], if u_0 takes values -1 and 1 alternately on suitably spaced long intervals with sharp transitions between them, then, as $t \to \infty$, $u(\cdot, t)$ will oscillate between -1 and 1, thus creating a continuum $\omega(u)$ with $\pm 1 \in \omega(u)$. It is not difficult to show, using the Liouville theorem for the linear heat equation, that the ω-limit set of this solution consists of spatially constant steady states, so the solution is quasiconvergent. If $f \not\equiv 0$, then bounded solutions of (1.1) are not even quasiconvergent in general. This can be illustrated by a construction of [13] in which f is a balanced unstable nonlinearity, and u_0 is identical to -1 or 1 on large intervals and has infinitely many transitions (kinks) between these values (for related studies of a slow motion of kinks on large bounded intervals see [4, 19]). More recent examples given in [26, 27] show that bounded solutions which are not quasiconvergent occur quite frequently. For example, non-quasiconvergent bounded solutions exist whenever f is bistable in an interval $[\alpha, \gamma]$: $f'(\alpha) < 0$, $f'(\gamma) < 0$, and there is $\beta \in (\alpha, \gamma)$ such that $f < 0$ in (α, β), $f > 0$ in (β, γ). In this case, one can even find non-quasiconvergent bounded solutions with (sign-changing) initial data in $C_0(\mathbb{R})$.

Energy techniques can still be applied to a class of solutions that decay at $x = \pm\infty$ with a sufficiently fast rate and are such that both the H^1-norm and the energy functional evaluated along $u(\cdot, t)$ stay bounded as $t \to \infty$. The quasiconvergence of such solutions is then proved in a standard way. Convergence is more delicate even for solutions in the energy space, as nonconstant steady states always occur in continua due to the translation invariance of (1.1). Convergence results based on energy methods can be found in [14, 15] for the one-dimensional problems and in [3, 7, 9, 16, 20] for problems in higher dimensions. We will discuss a result from [15] in more detail below in connection with our Theorem 2.4. Let us also mention a theorem of [17], where the convergence is proved for solutions whose decay at $x = \pm\infty$ need not occur with any specific rate, but is assumed to be uniform with respect to time.

In a different way, the energy functional, more precisely the whole family of functionals (1.5) with $I \subset \mathbb{R}$ bounded, is used in [21, 22]. No decay assumption is needed in this analysis and one of its results is that if the solution $u(\cdot, \cdot, u_0)$ is bounded, then $\omega(u_0)$ contains at least one steady state. As the examples mentioned above show, this result is in some sense optimal. It cannot be further improved so as to state the quasiconvergence of the solution, unless more specific initial data are considered.

In this context, two natural specific classes of u_0 come to mind: nonnegative u_0 with compact support, and nonnegative u_0 in $C_0(\mathbb{R})$. They are both of interest, mathematically as well as physically, but only the second one is an invariant class for (1.1), (1.2): $u(\cdot, t, u_0)$ stays in $C_0(\mathbb{R})$ for all $t > 0$ if $u_0 \in C_0(\mathbb{R})$.

Initial data with compact support were considered in [10], where it was proved that for any nonnegative continuous u_0 with compact support the solution $u(\cdot, \cdot, u_0)$ is convergent, if bounded. More specifically, $\omega(u_0)$ consists of a single steady state φ, and either φ is identical to a zero of f or it is symmetric (even) about some center and symmetrically decreasing to a zero of f as $|x| \to \infty$. In [11], this result was improved slightly by specifying more precisely which zeros of f are relevant in the conclusion; under additional conditions, the result was also extended in [11] to higher dimensions.

In this paper, we consider the other specific class: $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$. One of our main result is a *general quasiconvergence theorem*: for each such u_0, the solution $u(\cdot, \cdot, u_0)$ is quasiconvergent if it is bounded. See Theorem 2.1 below for the precise statement and additional information. Using this theorem, we then give several sufficient conditions for the convergence of $u(\cdot, \cdot, u_0)$, including as a special case the condition that u_0 has compact support.

We wish to emphasize that in this kind of analysis, there is a substantial difference between initial data with compact support and those in $C_0(\mathbb{R})$. The assumption of $u_0 \geq 0$ having compact support has strong immediate implications on the solution. In particular, reflection techniques can be effectively employed to yield, among other things, the monotonicity of $u(\cdot, t, u_0)$ outside any interval containing the initial support [10, 11, 30]. This is no longer the case for general $u_0 \in C_0(\mathbb{R})$. Consequently, there appear to be different possibilities for the behavior of the solutions and our analysis necessarily relies on completely different techniques. Among them, the most prominent one is the method of spatial trajectories, as described in Section 4 (see also [28]), which links properties of the solutions of the parabolic equation (1.1) to certain structures in the phase plane portrait of the ordinary differential equation $u_{xx} + f(u) = 0$.

So far, we have only discussed the large-time behavior of the solution $u(\cdot, \cdot, u_0)$ in bounded spatial intervals, which is captured by its ω-limit set with respect to the locally uniform convergence. To go further and to account for the spatial-translation invariance of (1.1), we introduce a generalized notion of the limit set, namely the Ω-limit set of a bounded solution $u(\cdot, \cdot, u_0)$:

$$\Omega(u_0) := \{\varphi : u(\cdot + x_n, t_n) \to \varphi \text{ for some } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \qquad (1.6)$$

The convergence here is again in $L^\infty_{loc}(\mathbb{R})$. Thus, while we still consider the large-time behavior of the solution on bounded intervals, the intervals can be shifted around arbitrarily as $t \to \infty$. Obviously, $\omega(u_0) \subset \Omega(u_0)$, but the opposite inclusion is not true in general. Take, for example, a *traveling front*, that is, a solution of the form $u(x,t) = \phi(x - ct)$, where $c > 0$ is a constant and $\phi(x)$ is an increasing function with finite limits $\alpha = \phi(-\infty)$, $\gamma = \phi(\infty)$. The ω-limit set of such a solution is equal to $\{\alpha\}$, whereas its Ω-limit set contains the constants α, γ as well as all the translations $\phi(\cdot - \xi)$, $\xi \in \mathbb{R}$, of ϕ. As in the case of $\omega(u_0)$, by standard parabolic regularity estimates the set $\{u(x + \cdot, t) : t \geq 1,\ x \in \mathbb{R}\}$ is relatively compact in $L^\infty_{loc}(\mathbb{R})$ and hence $\Omega(u_0)$ is nonempty, compact and connected in $L^\infty_{loc}(\mathbb{R})$.

Within the technical framework of this paper, we examine the top of the graph of $u(\cdot, t, u_0)$ for large times, which is described in terms of a set $\mathcal{T}_\Omega(u_0)$, the top of $\Omega(u_0)$. Roughly speaking, $\mathcal{T}_\Omega(u_0)$ is the set of all elements of $\Omega(u_0)$ greater than or equal to $\gamma_{max}(u_0)$, where $\gamma_{max}(u_0)$ is a certain zero of f contained in $\Omega(u_0)$ (see Section 2 for the precise definition). Our results show that for large t either *the top of* $u(\cdot, t, u_0)$ *is flat or it has the shape of a symmetrically decreasing steady state of* (1.1)*, possibly moving with time* (see Theorem 2.4 below).

In some important cases, one can show that $\gamma_{max}(u_0) = 0$. This holds, for example, if the solution $u(\cdot, t, u_0)$ stays bounded in an integral norm, or if it is on the threshold between decay and propagation for bistable nonlinearities. With $\gamma_{max}(u_0) = 0$, one has $\mathcal{T}_\Omega(u_0) = \Omega(u_0)$, and the previous result leads to a complete description of the behavior of the solution $u(\cdot, t, u_0)$: it is *convergent in* $L^\infty(\mathbb{R})$ *(not just in* $L^\infty_{loc}(\mathbb{R})$*)* (see Theorem 2.5 in the next section). If f is of a bistable type, this conclusion allows us to describe the "threshold" between decay to 0 and propagation to a positive constant steady state. We prove that for each strictly ordered family of nonnegative initial data in $C_0(\mathbb{R})$ *the threshold is sharp* in the sense that there is only one initial datum in the family for which neither decay nor propagation of the corresponding solution occurs. Also we prove that *the corresponding threshold solution converges to a ground state,* as $t \to \infty$, uniformly on $\mathbb{R}$. These results extend theorems of [10, 30] for initial data with compact support and [24] for symmetric and symmetrically decreasing initial data in $C_0(\mathbb{R})$.

The description of $\mathcal{T}_\Omega(u_0)$ is also a starting point for the analysis we will carry out in a sequel to this paper, where we will consider problem (1.1), (1.2) with $f \in C^1$ and $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$. Imposing certain explicit and generic conditions on f, we will prove the convergence of the solution $u(\cdot, \cdot, u_0)$ in

L^∞_{loc} and give a global description of its graph in terms of traveling fronts and propagating terraces (for the definition and existence of a propagating terrace see [12]).

The rest of the paper is organized as follows. We formulate all our main results in the next section. Their proofs are given in Sections 4 and 6. Sections 3 and 5 contain preliminary results on the steady states of (1.1), the zero number of solutions of linear equations, properties of the limit sets $\omega(u_0)$ and $\Omega(u_0)$, and convergence results based on normal hyperbolicity.

In the remainder of the paper, we assume the following

$$\textbf{standing hypotheses: } f \text{ is locally Lipschitz on } \mathbb{R} \text{ and } f(0) = 0. \tag{1.7}$$

2 Main results

To formulate our main results, we need to introduce some notation and terminology.

If γ is a zero of f, a *ground state of* (1.1) *at level* γ refers to a steady state φ of (1.1) such that $\varphi > \gamma$ and $\varphi(x) \to \gamma$ as $|x| \to \infty$. If $\gamma_1 < \gamma_2$ are zeros of f, a *standing wave of* (1.1) *with limits* γ_1, γ_2 refers to a steady state φ of (1.1) such that $\gamma_1 < \varphi < \gamma_2$, and φ is strictly monotone with limits $\varphi(\pm\infty) \in \{\gamma_1, \gamma_2\}$. In terms of the trajectories of the first order system associated with the equation $u_{xx} + f(u) = 0$, a ground state corresponds to a homoclinic solution and a standing wave corresponds to a heteroclinic solution.

Next we define a specific set of zeros of f. These are the only zeros of f that are relevant for the description of the solutions of equation (1.1). As in (1.5),

$$F(u) = \int_0^u f(s)\, ds.$$

Set

$$\tilde{\Gamma} := \Big\{ \gamma \geq 0 : f(\gamma) = 0 \text{ and } F(\gamma) \geq F(v) \text{ for all } v \in [0, \gamma] \Big\}. \tag{2.1}$$

Thus $\tilde{\Gamma}$ is the set of all critical points of F in $[0, \infty)$ which are "left-global" maximizers of F. Trivially, $0 \in \tilde{\Gamma}$. It is easy to verify that the set $\tilde{\Gamma}$ is closed.

Here is our first main result–a quasiconvergence theorem.

Theorem 2.1. *Assume that* $u_0 \in C_0(\mathbb{R})$*,* $u_0 \geq 0$*, and the solution* $u(\cdot, \cdot, u_0)$ *is bounded. Then each* $\varphi \in \omega(u_0)$ *is a steady state of* (1.1) *satisfying one of the following conditions:*

(a) $\varphi \equiv \gamma$ *for some* $\gamma \in \tilde{\Gamma}$,

(b) φ *is a ground state at some level* $\gamma \in \tilde{\Gamma}$,

(c) φ *is a standing wave with limits* $\gamma_1, \gamma_2 \in \tilde{\Gamma}$.

Moreover, any two ground states in $\omega(u_0)$ *are translations of one another and for any two elements* $\varphi, \psi \in \omega(u_0)$ *one has*

$$F(\varphi(-\infty)) = F(\varphi(\infty)) = F(\psi(-\infty)) = F(\psi(\infty)). \tag{2.2}$$

Theorem 2.1 is a general quasiconvergence result for nonnegative initial data $u_0 \in C_0(\mathbb{R})$. Moreover, it gives an information as to which steady states can occur as limit profiles of $u(\cdot, \cdot, u_0)$. In particular, all periodic nonconstant steady states are excluded. We do not know if the possibility (c) occurs for some f and $u_0 \in C_0(\mathbb{R})$ with $u_0 \geq 0$ (it does occur if the condition $u_0 \geq 0$ is dropped, see [26]). It view of (2.2), (c) cannot occur in a typical situation when all the left-global global maximizers of F are strict, that is,

$$F(\gamma_1) \neq F(\gamma_2), \text{ whenever } \gamma_1, \gamma_2 \in \tilde{\Gamma} \text{ and } \gamma_1 \neq \gamma_2. \tag{2.3}$$

In this case, we have the following result, which follows directly from Theorem 2.1 and the uniqueness, up to translations, of the ground state at any given level γ (see Lemma 3.4 below).

Corollary 2.2. *Assume that* (2.3) *holds. Let* u_0 *be as in Theorem 2.1. Then there is* $\gamma \in \tilde{\Gamma}$ *such that either* $\omega(u_0) = \{\gamma\}$ *or there is a ground state* φ *at level* γ *such that*

$$\omega(u_0) \subset \{\gamma\} \cup \{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}.$$

Theorem 2.1 is also convenient to use, as a starting point, in the proof of convergence of solutions under additional hypotheses on u_0 or f, as we will see in our next result. We use the following notation. Given a function ψ on $\mathbb{R}$ and a point $\xi \in \mathbb{R}$, we set

$$V_\xi \psi(x) := \psi(2\xi - x) - \psi(x) \quad (x \in \mathbb{R}). \tag{2.4}$$

Note that $V_\xi \psi$ is odd around $x = \xi$. In one of our hypotheses, we will require that for a sufficiently large set of points $\xi \in \mathbb{R}$, the following condition is satisfied:

(CN) There is an unbounded interval $I = I(\xi)$ such that $V_\xi u_0 \not\equiv 0$ in I, and either $V_\xi u_0 \geq 0$ in I or $V_\xi u_0 \leq 0$ in I.

Theorem 2.3. *Let u_0 be as in Theorem 2.1. Assume that there are constants $a < b$ such that*

$$V_a u_0 \geq 0 \text{ in } (-\infty, a] \text{ and } V_b u_0 \geq 0 \text{ in } [b, \infty). \tag{2.5}$$

Then there is $\gamma \in \tilde{\Gamma}$ such that either $\omega(u_0) = \{\gamma\}$ or

$$\omega(u_0) = \{\varphi(\cdot + \xi) : \xi \in J\}, \tag{2.6}$$

where φ is a ground state at level γ and J is a compact interval.

If, in addition, condition (CN) *is satisfied for each ξ in a dense subset of (a, b), then the above conclusion holds with* (2.6) *replaced by $\omega(u_0) = \{\varphi\}$.*

Relation (2.6) means that $\omega(u_0)$ is contained in the translation orbit of a ground state φ. The second part of the theorem is a convergence result which extends a convergence theorem proved in [10, 11] for initial data with compact support. Indeed, it is clear that both (2.5) and the additional hypothesis of Theorem 2.3 are satisfied if u_0 has compact support. More generally, these conditions are satisfied by all nonnegative functions $u_0 \in C_0(\mathbb{R})$ of the form $u_0 = \phi_0 + \phi_1$, where ϕ_0, ϕ_1 are continuous, ϕ_0 is even and nonincreasing on $(0, \infty)$ (possibly $\phi_0 \equiv 0$), and ϕ_1 is nonnegative and has compact support.

To mention a different example, assume that $\phi_0 \in C_0(\mathbb{R})$ is even and decreasing on $(0, \infty)$, so that $V_b \phi_0 > 0$ in (b, ∞) for each $b > 0$. Then the function $u_0 = \phi_0 + \phi_1$ satisfies (2.5), provided $\phi_1 \geq 0$ and ϕ_1 is small enough outside an interval $(-b, b)$ so that

$$\phi_1(x) \leq V_b \phi_0(x) \quad (x > b), \qquad \phi_1(x) \leq V_{-b} \phi_0(x) \quad (x < -b).$$

Such a function u_0 also satisfies (CN) for each $\xi \neq 0$ if there are positive constants c and θ such that

$$\phi_0(x) e^{\theta |x|} \to c, \ \ \phi_1(x) e^{\theta |x|} \to 0, \quad \text{as } |x| \to \infty.$$

Obviously, the conclusions of Theorem 2.3 still hold if the hypotheses are satisfied with u_0 replaced by $u(\cdot, t_0, u_0)$ for some $t_0 > 0$.

We next examine the Ω-limit set, $\Omega(u_0)$, of the solution $u(\cdot, \cdot, u_0)$, assuming as usual that $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$. In order to define the "top" of $\Omega(u_0)$, we first set

$$\gamma_{max}(u_0) := \max K, \ \text{ where } K := \{\gamma \in \tilde{\Gamma} : \gamma \leq \varphi \text{ for some } \varphi \in \Omega(u_0)\}. \tag{2.7}$$

To justify the existence of the maximum, assuming that the solution $u(\cdot,\cdot,u_0)$ is bounded and nonnegative, we note that K is nonempty and compact. Indeed, K clearly contains 0 and is bounded. The closedness of K follows easily from the fact that $\tilde{\Gamma}$ is closed and $\Omega(u_0)$ is compact in $L^\infty_{loc}(\mathbb{R})$. We now define the *top of* $\Omega(u_0)$ by

$$\mathcal{T}_\Omega(u_0) := \{\varphi \in \Omega(u_0) : \varphi \geq \gamma_{max}(u_0)\}. \tag{2.8}$$

By (2.7), we have

$$\gamma_{max}(u_0) \in \tilde{\Gamma}, \quad \mathcal{T}_\Omega(u_0) \neq \emptyset.$$

The following theorem shows that $\mathcal{T}_\Omega(u_0)$ consists of steady states of (1.1).

Theorem 2.4. *Let u_0 be as in Theorem 2.1. Then either $\mathcal{T}_\Omega(u_0) = \{\gamma_{max}(u_0)\}$ or there is a ground state at level $\gamma_{max}(u_0)$ such that*

$$\mathcal{T}_\Omega(u_0) = \{\gamma_{max}(u_0)\} \cup \{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}. \tag{2.9}$$

We next devote some space to a discussion of the case $\gamma_{max}(u_0) = 0$, which is precisely the case when $\mathcal{T}_\Omega(u_0) = \Omega(u_0)$. Trivially, this holds if there are no positive elements in $\tilde{\Gamma}$. More interestingly, $\gamma_{max}(u_0) = 0$ occurs as one of two alternatives in case f is of the unbalanced bistable case:

(BS) $f \in C^1(\mathbb{R})$, $f'(0) < 0$, and for some $\gamma_1 > \beta^* > 0$ one has

$$f(\gamma_1) = 0, \quad f(u) > 0 \ \ (u \in [\beta^*, \gamma_1)), \ \ F(\beta^*) = 0, \ \ F(u) < 0 \ \ (u \in (0, \beta^*)). \tag{2.10}$$

Note that this condition implies that γ_1 is the minimal positive element of $\tilde{\Gamma}$. We shall presently see that if (BS) holds, then for any $u_0 \in C_0(\mathbb{R})$ with $0 \leq u_0 \leq \gamma_1$, either

$$\lim_{t\to\infty} u(\cdot, t, u_0) = \gamma_1 \text{ in } L^\infty_{loc}(\mathbb{R}), \tag{2.11}$$

or $\gamma_{max}(u_0) = 0$. Indeed, it is well known (see [10, 11, 17, 18]) that there are positive constants δ and ℓ with the following property:

(2.11) holds, provided for some $t_0 > 0$ one has

$$u(\cdot, t_0, u_0) > \gamma_1 - \delta \text{ on a closed interval of length } \ell. \tag{2.12}$$

The opposite of this, namely that $u(\cdot,t,u_0)$ is not greater than $\gamma_1-\delta$ on any closed interval of length ℓ for any t, clearly implies that $\Omega(u_0)$ does not contain any element ψ with $\psi\geq\gamma_1$, hence $\gamma_{max}(u_0)=0$.

If $\gamma_{max}(u_0)=0$, then (2.9) can be stated as

$$\Omega(u_0)=\{0\}\cup\{\varphi(\cdot+\xi):\xi\in\mathbb{R}\}, \tag{2.13}$$

where φ is a ground state at level 0. In this case, Theorem 2.4 tells us, in essence, that either $\Omega(u_0)=\{0\}$ or $u(\cdot,t,u_0)$ is globally approximated by the sum of suitably spaced shifts of the ground state φ. This is related to a result of Feireisl, who proved such a conclusion under the assumption that $\|u(\cdot,t,u_0)\|_{L^2(\mathbb{R})}$ stays between two positive constants [15]. Observe, that the boundedness of $\|u(\cdot,t,u_0)\|_{L^2(\mathbb{R})}$ is another sufficient condition for $\gamma_{max}(u_0)=0$, thus our theorem extends this result of Feireisl. The boundedness in $L^2(\mathbb{R})$ is needed in [15] for variational techniques (concentration compactness) to apply. In contrast, our result is proved by a completely different method and we do not need a bound in an integral norm. On the other hand, our techniques are strictly one-dimensional and, unlike [15], we have no extension of our theorem to higher dimensions.

The question whether the sum in the approximation of $u(\cdot,t,u_0)$ may actually contain several shifts of a ground state if $u_0\geq 0$ was left open in [15]. In the next result we resolve this issue. We show that not only is $u(\cdot,t,u_0)$ globally approximated by just one ground state, it actually converges to a ground state uniformly on $\mathbb{R}$. As in [15], we assume that $f'(0)<0$. This condition is essential for some of our techniques, notably the ones discussed in Section 5.

Theorem 2.5. *Assume that $f\in C^1$ and $f'(0)<0$. Let u_0 be as in Theorem 2.1. If $\gamma_{max}(u_0)=0$, then, as $t\to\infty$, $u(\cdot,t,u_0)\to\varphi$ in $L^\infty(\mathbb{R})$, where $\varphi\equiv 0$ or φ is a ground state of* (1.1) *at level 0.*

We next examine threshold solutions for bistable nonlinearities and ordered families of initial data in $C_0(\mathbb{R})$. Assume that f satisfies condition (BS) and let ℓ and δ be as in (2.12). Consider a family ψ_λ, $\lambda\in[0,1]$, of functions in $C_0(\mathbb{R})$ with the following properties:

(a1) $0\leq\psi_\lambda\leq\gamma_1$ for all $\lambda\in[0,1]$, $\psi_0\equiv 0$, and $\psi_1>\gamma_1-\delta$ on a closed interval of length ℓ.

(a2) The function $\lambda \to \psi_\lambda : [0,1] \to C_0(\mathbb{R})$ is continuous and monotone increasing in the sense that if $\lambda < \nu$, then $\psi_\lambda \leq \psi_\nu$ with the strict inequality on a nonempty (open) set.

Theorem 2.6. *Assume* (BS) *and let* ψ_λ, $\lambda \in [0,1]$, *be a family of functions in* $C_0(\mathbb{R})$ *satisfying* (a1) *and* (a2) *(with constants* ℓ *and* δ *as in* (2.12)*). Then there is* $\lambda^* \in (0,1)$ *with the following properties:*

(t1) *If* $u_0 = \psi_\lambda$ *with* $\lambda \in (0, \lambda^*)$, *then* $\lim_{t\to\infty} u(\cdot, t, u_0) = 0$ *in* $L^\infty(\mathbb{R})$.

(t2) *If* $u_0 = \psi_\lambda$ *with* $\lambda \in (\lambda^*, 1]$, *then* $\lim_{t\to\infty} u(\cdot, t, u_0) = \gamma_1$ *in* $L^\infty_{loc}(\mathbb{R})$.

(t3) *If* $u_0 = \psi_{\lambda^*}$, *then* $\lim_{t\to\infty} u(\cdot, t, u_0) \to \phi$ *in* $L^\infty(\mathbb{R})$, *where* ϕ *is a ground state of* (1.1) *at level 0.*

As remarked in the introduction, this theorem extends results of [10, 30], where the functions ψ_λ are assumed to have compact supports, and [24], where they are assumed to be even and symmetrically decreasing.

Remark 2.7. All our theorems concern a bounded solution. Thus, without affecting the validity of the theorems, we can always modify the nonlinearity outside a bounded interval containing the range of the solution in question. This allows us to make assumptions on the behavior of f near infinity, without losing any generality. We shall make such assumptions to simplify some preliminaries. Also, although our theorems concern nonnegative solutions only, it will be convenient to have f defined on the whole real line.

3 Preliminaries I

This section consists of three parts. First, we discuss the steady states of (1.1) and their orbits in the phase plane. Then we summarize the basic properties of the zero-number functional. In the last part, we recall the invariance and other properties of the ω and Ω-limit sets.

3.1 Steady states and their trajectories in the phase plane

In this subsection we examine the steady states of (1.1). We assume that f satisfies the standing hypotheses (see (1.7)), but, with the exception of Lemmas 3.5 and 3.6, the condition $f(0) = 0$ is not needed and can be dropped.

For convenience, in the whole subsection we assume in addition that f satisfies the following identity near $\pm\infty$:

(C1) There is $\kappa > 0$, such that for $|u| > \kappa$ one has $f(u) = u/2$.

As noted in Remark 2.7, this is at no cost to the generality of our results.

As in (1.5), $F(u) = \int_0^u f(s)\,ds$. Condition (C1) implies that for some constants k^-, k^+ one has

$$F(u) = u^2 + k^- \quad (u < -\kappa), \qquad F(u) = u^2 + k^+ \quad (u > \kappa). \tag{3.1}$$

The steady states of (1.1) are solutions of the equation

$$u_{xx} + f(u) = 0, \quad x \in \mathbb{R}. \tag{3.2}$$

The first-order system associated with (3.2),

$$u_x = v, \quad v_x = -f(u), \tag{3.3}$$

is a Hamiltonian system with respect to the energy

$$H(u, v) := v^2/2 + F(u).$$

Thus each trajectory of (3.3) is contained in a level set of H. Note that the level sets of H are symmetric about the u-axis and (3.1) implies that they are all bounded. We next list, without detailed proofs, further simple and well-known consequences of the Hamiltonian structure.

System (3.3) has only four types of orbits: equilibria–all of them on the u-axis, nonstationary periodic orbits (or, closed orbits), homoclinic orbits, and heteroclinic orbits. This follows easily from the symmetry of the level sets of H and the fact that in the half-plane $v > 0$ the u component of the solutions is increasing, whereas in $v < 0$ it is decreasing. Thus, trajectories with more than one intersection with the v-axis are periodic orbits. Any nonstationary orbit with exactly one intersection with the v-axis is a homoclinic orbit. Indeed, the corresponding solution u of (3.2) has a unique critical point a and is symmetric about a. Consequently, the limits $u(-\infty)$, $u(\infty)$ exist and are equal, and from (3.3) one obtains that, as $x \to \pm\infty$, $(u(x), u_x(x)) \to (\gamma, 0)$, where γ is a zero of f. Hence, $\{(u(x), u_x(x)) : x \in \mathbb{R}\}$ is a homoclinic orbit and the solution u itself is a ground state at level γ, in the sense of our definition in Section 2, if $u > \gamma$. Similar considerations show that if an orbit

$\{(u(x), u_x(x)) : x \in \mathbb{R}\}$ does not intersect the v-axis at all, so $|u_x| > 0$, then $(u(x), u_x(x)) \to (\pm\gamma, 0)$ as $x \to \pm\infty$, for some distinct zeros γ^-, γ^+ of f. Hence, $\{(u(x), u_x(x)) : x \in \mathbb{R}\}$ is a heteroclinic orbit and the solution u is a standing wave of (1.1).

We view orbits as subsets of $\mathbb{R}^2$, although our descriptive terminology, like periodic solutions, reflects properties of the corresponding solutions of (3.3). This should cause no confusion. Of course, our classification of orbits uses the Lipschitz continuity of f (for uniqueness) and assumption (C1) (for boundedness of all orbits).

Each nonstationary periodic orbit $\mathcal{O}$ is symmetric about the u axis and for some $p < q$ one has

$$\begin{aligned}
&\mathcal{O} \cap \{(u, 0) : u \in \mathbb{R}\} = \{(p, 0), (q, 0)\}, \\
&\mathcal{O} \cap \{(u, v) : v > 0\} = \{(u, \sqrt{2(c - F(u))}) : u \in (p, q)\},
\end{aligned}$$

where $c = F(p) = F(q)$. Viewing the periodic orbit $\mathcal{O}$ as a Jordan curve, we denote by $\mathrm{Int}(\mathcal{O})$ and $\mathrm{Ext}(\mathcal{O})$ the two connected components of $\mathbb{R}^2 \setminus \mathcal{O}$, $\mathrm{Int}(\mathcal{O})$ being the bounded one. Then

$$\begin{aligned}
&\{(a, 0) : p < a < q\} \subset \mathrm{Int}(\mathcal{O}), \\
&\{(a, 0) : a < p \text{ or } a > q\} \subset \mathrm{Ext}(\mathcal{O}),
\end{aligned} \tag{3.4}$$

and, for any $(a_0, b_0) \in \mathcal{O}$ with $b_0 > 0$,

$$\begin{aligned}
&\{(a_0, b) : |b| < b_0\} \subset \mathrm{Int}(\mathcal{O}), \\
&\{(a_0, b) : |b| > b_0\} \subset \mathrm{Ext}(\mathcal{O}).
\end{aligned} \tag{3.5}$$

We next give a description of the phase plane portrait of (3.3) with all the periodic orbits removed. This is one of the key ingredients of the proofs of Theorems 2.1 and 2.4. We use the following notation:

$$\begin{aligned}
\mathcal{E} &:= \{(a, 0) : f(a) = 0\} \quad \text{(the set of all equilibria of (3.3))}, \\
\mathcal{P}_0 &:= \{(a, b) \in \mathbb{R}^2 : (a, b) \text{ lies on a nonstationary periodic orbit of (3.3)}\}, \\
\mathcal{P} &:= \mathcal{P}_0 \cup \mathcal{E}.
\end{aligned}$$

Lemma 3.1. *The following two statements are valid:*

(i) *Let Σ be a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$. Then Σ is a compact set contained in a level set of the Hamiltonian H and one has*

$$\Sigma = \{(u, v) \in \mathbb{R}^2 : u \in J,\ v = \pm\sqrt{2(c - F(u))}\}, \tag{3.6}$$

where c is the value of H on Σ and $J = [p, q]$ for some $p, q \in \mathbb{R}$ with $p \leq q$. Moreover, if $(u, 0) \in \Sigma$ and $p < u < q$, then $(u, 0)$ is an equilibrium. The point $(p, 0)$ is an equilibrium or it lies on a homoclinic orbit; the same is true of the point $(q, 0)$.

(ii) *Each connected component of the set $\mathbb{R}^2 \setminus \mathcal{P}$ consists of a single orbit of (3.3), either a homoclinic orbit or a heteroclinic orbit.*

In preparation for the proof of this lemma, we prove the following approximation result.

Lemma 3.2. *Given any $(a_0, b_0) \in \mathbb{R}^2$, there are monotone sequences $\{b_n^-\}$, $\{b_n^+\}$ such that*

$$b_n^- \nearrow b_0, \ \ b_n^+ \searrow b_0, \ \ \text{and } (a_0, b_n^\pm) \in \mathcal{P}_0 \quad (n = 1, 2, \dots). \tag{3.7}$$

If $b_0 = 0$ and $f(a_0) \neq 0$ (that is, $(a_0, 0)$ is not an equilibrium of (3.3)), then there are also monotone sequences $\{a_n^-\}$, $\{a_n^+\}$ such that

$$a_n^- \nearrow a_0, \ \ a_n^+ \searrow a_0, \ \ \text{and } (a_n^\pm, 0) \in \mathcal{P}_0 \quad (n = 1, 2, \dots). \tag{3.8}$$

Proof. Before starting, recall that (3.1) implies that the level sets of H are all bounded, hence they are compact, by the continuity of H.

For the first statement, it is sufficient to prove that for any $\epsilon \neq 0$, there is b between b_0 and $b_0 + \epsilon$ such that the point (a_0, b) lies on a periodic orbit. The latter is guaranteed if the level set of H containing (a_0, b) contains no equilibrium of (3.3) or, in other words, there is no $s \in \mathbb{R}$ such that

$$f(s) = 0 \text{ and } F(s) = \frac{b^2}{2} + F(a_0). \tag{3.9}$$

Clearly, therefore, it is sufficient to choose b between b_0 and $b_0 + \epsilon$ such that $b^2/2 + F(a_0)$ is not a critical value of F and such a choice is possible by Sard's theorem.

The proof of the second statement is analogous, only this time we need to show that for each $\epsilon \neq 0$ it is possible to choose a between a_0 and $a_0 + \epsilon$ such that $F(a)$ is not a critical value of F. Since $F'(a_0) = f(a_0) \neq 0$ by assumption, Sard's theorem again shows that such a choice is possible. $\square$

Proof of Lemma 3.1. We start with the proof of (i). First we note that according to the classification of all orbits of (3.3) from the beginning of this subsection, Σ consists of equilibria, homoclinic orbits, and heteroclinic orbits. Therefore, being a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$, with each orbit, Σ must contain its reflection about the u axis (which is an orbit of the same type). Thus Σ is symmetric about the u-axis.

Let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ denote the projection onto the the first component and set $J := \Pi(\Sigma)$. This is an interval or a single point, as Σ is connected. We show that Π is one-to-one on $\Sigma \cap \{(a, b) : b \geq 0\}$. Indeed, if not then Σ contains two points (a_0, b_0), $(a_0, \tilde{b}_0)$ with $0 \leq b_0 < \tilde{b}_0$. By Lemma 3.2, there is $\hat{b} \in (b_0, \tilde{b}_0)$ such that $(a_0, \hat{b}) \in \mathcal{P}_0$. Let $\mathcal{O}$ be the periodic orbit containing $(a_0, \hat{b})$. Then (3.5) implies that the points (a_0, b_0), $(a_0, \tilde{b}_0)$ lie in different connected components of $\mathbb{R}^2 \setminus \mathcal{O}$, thus they cannot both be contained in the connected set $\Sigma \subset \mathbb{R}^2 \setminus \mathcal{P}_0$, a contradiction.

To proceed, pick any $a_0 \in J$ and let b_0 be the (unique) nonnegative number such that $(a_0, b_0) \in \Sigma$. Corresponding to (a_0, b_0), we find a sequence $\{b_n^+\}$ as in Lemma 3.2. Let $\mathcal{O}_n$ be the periodic orbit containing (a_0, b_n^+). By (3.5), $(a_0, b_0) \in \mathrm{Int}(\mathcal{O}_n)$, hence $\mathrm{Int}(\mathcal{O}_n)$ contains the whole (connected) set Σ. In particular, Σ is bounded.

We next claim that for any $(a_1, b_1) \in \Sigma$ one has

$$\mathrm{dist}((a_1, b_1), \mathcal{O}_n) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Assume this is not true: for some $(a_1, b_1) \in \Sigma$ there is $\epsilon > 0$ such that $\mathrm{dist}((a_1, b_1), \mathcal{O}_n) \geq \epsilon$ for all n. Since both Σ and $\mathcal{O}_n$ are symmetric about the u-axis, we may assume that $b_1 \geq 0$. As $(a_1, b_1) \in \Sigma \subset \mathrm{Int}(\mathcal{O}_n)$, there is $\tilde{b}_n > b_1$ such that $(a_1, \tilde{b}_n) \in \mathcal{O}_n$ (refer to (3.4), (3.5)). By the assumption on (a_1, b_1), we have $\tilde{b}_n \geq b_1 + \epsilon$ for all n. Applying Lemma 3.2 to the point (a_1, b_1), we find $\hat{b}_1 \in (b_1, b_1 + \epsilon)$ such that $(a_1, \hat{b}_1)$ lies on some periodic orbit $\hat{\mathcal{O}}$. By (3.5), $(a_1, b_1) \in \mathrm{Int}(\hat{\mathcal{O}})$ hence also $\Sigma \subset \mathrm{Int}(\hat{\mathcal{O}})$, whereas $(a_1, \tilde{b}_n) \in \mathrm{Ext}(\hat{\mathcal{O}})$. On the other hand, from

$$(a_0, b_n^+) \to (a_0, b_0) \in \Sigma \subset \mathrm{Int}(\hat{\mathcal{O}})$$

we infer that $(a_0, b_n^+) \in \mathrm{Int}(\hat{\mathcal{O}})$ for large enough n. We have thus arrived at the conclusion that $\mathcal{O}_n$ contains a point $(a_0, b_n^+) \in \mathrm{Int}(\hat{\mathcal{O}})$, as well as a point $(a_1, \tilde{b}_n) \in \mathrm{Ext}(\hat{\mathcal{O}})$. Consequently the two distinct periodic orbits $\hat{\mathcal{O}}$ and $\mathcal{O}_n$ intersect, which is a contradiction. Our claim is proved.

From (3.10) and the fact that H is continuous and constant on $\mathcal{O}_n$, we conclude that H is constant on Σ, that is, Σ is contained in a level set of H. We shall presently see that Σ is closed. Indeed, using the continuity of solutions of (3.3) with respect to initial data, it is easy to verify that $\bar{\Sigma}$ cannot contain any nonstationary periodic orbit. Hence $\bar{\Sigma}$ is a connected subset of $\mathbb{R}^2 \setminus \mathcal{P}_0$. As Σ is a connected component, necessarily $\Sigma = \bar{\Sigma}$. As observed above, Σ is also bounded, hence it is compact. Therefore, the interval $J = \Pi(\Sigma)$ is compact: $J = [p, q]$ for some $p, q \in \mathbb{R}$ (possibly $p = q$).

To prove (3.6), let c be the constant value of H on Σ. For each $u \in J$ there is a unique $v \geq 0$ such that $(u, v) \in \Sigma$ and for such v one has $H(u, v) = v^2/2 + F(u) = c$. This and the symmetry of Σ give (3.6).

By the previous argument, we also have $F \leq c$ on J. Thus if $(u, 0) \in \Sigma$, then $F(u) = c$ and u is a maximum point of F in $[p, q]$. If $p < u < q$, this implies $f(u) = F'(u) = 0$, hence $(u, 0)$ is an equilibrium, as stated in the theorem. Consider now the point $(p, 0)$. If it is not an equilibrium, then it lies on a homoclinic orbit, for it clearly cannot lie on a heteroclinic orbit and there are no periodic orbits in Σ. A similar remark applies to $(q, 0)$. The proof of statement (i) is now complete.

Statement (ii) follows from (i). Indeed, (i) implies that each connected component of the set $\Sigma \setminus \mathcal{E}$ is a single orbit, either a homoclinic orbit or a heteroclinic orbit. Now, if Σ_0 is a connected component of $\mathbb{R}^2 \setminus \mathcal{P}$, let Σ be the connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$ containing Σ_0. Then, Σ_0 is also a connected component of $\Sigma \setminus \mathcal{E}$, which proves (ii). $\square$

The following property of Σ, which we established in (3.10) will also be useful below.

Lemma 3.3. *Let Σ be a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$. There exists a sequence $\mathcal{O}_n$, $n = 1, 2, \dots,$ of periodic orbits such that $\Sigma \subset \mathrm{Int}(\mathcal{O}_n)$ ($n = 1, 2, \dots$) and for any $(a_1, b_1) \in \Sigma$ one has*

$$\mathrm{dist}((a_1, b_1), \mathcal{O}_n) \to 0 \text{ as } n \to \infty. \tag{3.11}$$

It will also be useful to recall the uniqueness of ground states:

Lemma 3.4. *Given $\gamma \in f^{-1}(0)$, there is at most one ground state of* (1.1) *at level γ, up to translations.*

See [2, Section 6] for a simple proof.

If Σ is a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$, we set

$$N_\Sigma := \{\gamma \in \mathbb{R} : f(\gamma) = 0 \text{ and } (\gamma, 0) \in \Sigma\} = \Pi(\Sigma \cap \mathcal{E}) \tag{3.12}$$

($\Pi : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection onto the the first component). It is obvious from Lemma 3.1 that N_Σ is a nonempty compact set and $F\big|_{N_\Sigma}$ is constantly equal to $H(\Sigma)$, the value of the Hamiltonian on Σ. Moreover, the constant value of $F\big|_{N_\Sigma}$ is the maximum of F on the interval $\Pi(J)$.

Consider now the set $\tilde{\Gamma}$ defined in (2.1). If $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$, then the previous remarks imply that all nonnegative elements of N_Σ are contained in $\tilde{\Gamma}$. For our analysis, a key difference between this case and the opposite case, $N_\Sigma \cap \tilde{\Gamma} = \emptyset$, is expressed in the following lemma (here, the assumption $f(0) = 0$ is needed).

Lemma 3.5. *Let Σ be a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$ and let $\mathcal{O}_n$, $n = 1, 2, \dots,$ be a sequence of periodic orbits as in Lemma 3.3. Assume that $N_\Sigma \cap (0, \infty) \neq \emptyset$. Then the following statements hold.*

- (i) *If $N_\Sigma \cap \tilde{\Gamma} = \emptyset$, then $\Sigma \subset \{(u, v) : u > 0\}$ and $\mathcal{O}_n \subset \{(u, v) : u > 0\}$ for all sufficiently large n.*
- (ii) *If $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$, then $N_\Sigma \cap [0, \infty) \subset \tilde{\Gamma}$ and for each n the periodic orbit $\mathcal{O}_n$ intersects the v-axis at two points different from the origin.*

Proof. Take any $\gamma^* \in N_\Sigma \cap (0, \infty)$. Thus $(\gamma^*, 0) \in \Sigma \cap \mathcal{E}$ and $\gamma^* > 0$.

Assume that $N_\Sigma \cap \tilde{\Gamma} = \emptyset$. Then $\gamma^* \notin \tilde{\Gamma}$, hence there is a constant a satisfying

$$0 < a < \gamma^* \text{ and } F(a) > F(\gamma^*). \tag{3.13}$$

We claim that if n is sufficiently large, then $\mathcal{O}_n$ does not intersect the vertical line $\{(a, v) : v \in \mathbb{R}\}$. To prove this, assume that $(a, v) \in \mathcal{O}_n$ for some v. For the constant value of the Hamiltonian H on $\mathcal{O}_n$ we then have

$$H(\mathcal{O}_n) = H(a, v) = \frac{v^2}{2} + F(a) \geq F(a) > F(\gamma^*) = H(\gamma^*, 0).$$

This is not possible for large n by (3.11). Hence our claim is true and, as $\mathrm{Int}(\mathcal{O}_n) \supset \Sigma$, statement (i) is proved.

Next assume that $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$. As remarked above, this implies $N_\Sigma \cap [0, \infty) \subset \tilde{\Gamma}$ and in particular $\gamma^* \in \tilde{\Gamma}$. We prove statement (ii) by contraction. Assume that for some n the periodic orbit $\mathcal{O}_n$ does not intersect the v-axis. From the fact that $\mathrm{Int}(\mathcal{O}_n)$ contains $\Sigma \ni (\gamma^*, 0)$, we deduce that, first, $\mathcal{O}_n$ contains a point (γ^*, b) with $b > 0$ and, second, $\mathcal{O}_n$ intersect the u-axis at some point $(a, 0)$ with $0 \leq a < \gamma^*$. Actually, a has to be positive since $(0, 0)$ is an equilibrium ($f(0) = 0$). We thus have

$$\frac{b^2}{2} + F(\gamma^*) = H(\gamma^*, b) = H(a, 0) = F(a).$$

This gives $F(\gamma^*) < F(a)$, which contradicts the fact that $\gamma^* \in \tilde{\Gamma}$. This contradiction shows that $\mathcal{O}_n$ intersects the v-axis; necessarily the intersections are away from the origin, as $(0, 0)$ is an equilibrium, hence by symmetry about the u-axis, there are two of them. $\square$

We conclude this section with the following observation (here, too, condition $f(0) = 0$ is needed):

Lemma 3.6. *With Σ as in Lemma 3.5, assume that $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$. If $\{(\varphi(x), \varphi_x(x) : x \in \mathbb{R}\}$ is a homoclinic orbit of* (3.3) *contained in $\Sigma \cap \{(u, v) : u \geq 0\}$ and $\gamma = \varphi(\infty)$ $(= \varphi(-\infty))$, then $\gamma \in \tilde{\Gamma}$ and $\varphi > \gamma$, that is, φ is a ground state of* (1.1) *at level γ. Moreover, up to translations, there is at most one ground state φ such that $\{(\varphi(x), \varphi_x(x) : x \in \mathbb{R}\} \subset \Sigma$.*

Proof. Obviously, $(\gamma, 0)$ is the limit equilibrium of the homoclinic orbit, hence $(\gamma, 0) \in \Sigma$. Since $\varphi \geq 0$ by assumption, we have $\gamma \geq 0$. Therefore, by Lemma 3.5(ii), $\gamma \in \tilde{\Gamma}$.

Now, $\varphi < \gamma$ or $\varphi > \gamma$, so we just need to rule out the former. Assume it holds. Then φ achieves its minimal value $m \geq 0$ at some $x_0 \in \mathbb{R}$. Using a translation, we may assume that $x_0 = 0$: $\varphi(0) = m$ and $\varphi'(0) = 0$. Also, by (3.3), $f(m) = -\varphi''(0) \leq 0$. In fact, the strict inequality has to hold, as $(m, 0)$ is not an equilibrium. By the same reason, $m \neq 0$ (the assumption $f(0) = 0$ is in effect), hence $m > 0$. Since the Hamiltonian H is constant on Σ, we obtain

$$F(m) = H(m, 0) = H(\gamma, 0) = F(\gamma).$$

But then the relations $m > 0$ and $F'(m) = f(m) < 0$ imply the existence of some $\tilde{m} \in (0, m)$ with $F(\tilde{m}) > F(m) = F(\gamma)$, which is impossible due to $\gamma \in \tilde{\Gamma}$. This contradiction rules out the case $\varphi < \gamma$.

The last statement follows from Lemma 3.1. Indeed, if φ is a ground state with maximum at $x = 0$ such that $\{(\varphi(x), \varphi_x(x) : x \in \mathbb{R}\} \subset \Sigma$, then the point $(\varphi(0), 0)$ necessarily equals $(q, 0)$, with q as in Lemma 3.1. By uniqueness for the Cauchy problem for the second order ODE, two different ground states with maximum at 0 cannot have the same value at 0. $\square$

3.2 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + c(x,t)v, \quad x \in \mathbb{R},\ t \in (s,T), \tag{3.14}$$

where $-\infty \le s < T \le \infty$ and c is a bounded measurable function. Note that if u, $\tilde{u}$ are global solutions of (1.1), then their difference $v = u - \tilde{u}$ is a solution of (3.14), where we can take

$$c(x,t) = \int_0^1 f'(\tilde{u}(x,t) + s(\tilde{u}(x,t) - u(x,t)))\, ds$$

if $u(x,t) \ne \tilde{u}(x,t)$ (which is well defined as f is locally Lipschitz) and $c(x,t) = 0$ otherwise.

For an interval $I = (a,b)$, with $-\infty \le a < b \le \infty$, we denote by $z_I(v(\cdot,t))$ the number, possibly infinite, of all zeros $x \in I$ of the function $x \to v(x,t)$. If $I = \mathbb{R}$, we usually omit the subscript $\mathbb{R}$:

$$z(v(\cdot,t)) := z_{\mathbb{R}}(v(\cdot,t)).$$

The following intersection-comparison principle holds (see [1, 5]).

Lemma 3.7. *Let v be a nontrivial solution of* (3.14) *and $I = (a,b)$, where $-\infty \le a < b \le \infty$. Assume that the following conditions are satisfied:*

(c1) *if $b < \infty$, then $v(b,t) \ne 0$ for all $t \in (s,T)$,*

(c2) *if $a > -\infty$, then $v(a,t) \ne 0$ for all $t \in (s,T)$.*

Then the following statements hold true:

(i) *For each $t \in (s,T)$, all zeros of $v(\cdot,t)$ are isolated. In particular, if $a > -\infty$ and $b < \infty$, then $z_I(v(\cdot,t)) < \infty$ for all $t \in (s,T)$.*

(ii) *$t \mapsto z_I(v(\cdot,t))$ is a monotone nonincreasing function on (s,T) with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s,T)$, the function $v(\cdot,t_0)$ has a multiple zero in I and $z_I(v(\cdot,t_0)) < \infty$, then for any $t_1, t_2 \in (s,T)$ with $t_1 < t_0 < t_2$ one has*

$$z_I(v(\cdot,t_1)) > z_I(v(\cdot,t_2)). \tag{3.15}$$

If (3.15) holds, we say that $z_I(v(\cdot,t))$ drops in the interval (t_1,t_2).

Remark 3.8. It is clear that if the assumptions of Lemma 3.7 are satisfied and for some $s_0 \in (s,T)$ one has $z_I(v(\cdot,s_0)) < \infty$, then $z_I(v(\cdot,t))$ can drop at most finitely many times in (s_0,T) and if it is constant on (s_0,T), then $v(\cdot,t)$ has only simple zeros in I for each $t \in (s_0,T)$.

Corollary 3.9. *Assume that v is continuous on $\mathbb{R}\times[0,\infty)$ and it is a solution of (3.14) on $\mathbb{R}\times(0,\infty)$. Assume further that $I=(a,b)$ and the following conditions are satisfied:*

- (ci) *Either $b<\infty$ and $|v(b,t)|>0$ for all $t\geq 0$, or $b=\infty$ and there is $\tilde{b}\in(a,\infty)$ such that $v(\tilde{b},0)\neq 0$ and $v(\tilde{b},0)v(x,0)\geq 0$ for each $x\in[\tilde{b},\infty)$.*
- (cii) *Either $a>-\infty$ and $|v(a,t)|>0$ for all $t\geq 0$, or $a=-\infty$ and there is $\tilde{a}\in(-\infty,b)$ such that $v(\tilde{a},0)\neq 0$ and $v(\tilde{a},0)v(x,0)\geq 0$ for each $x\in(-\infty,\tilde{a}]$.*

Then there is $t_0>0$ such that for $t\geq t_0$ the function $v(\cdot,t)$ has only finitely many zeros in I and all of them are simple.

Proof. In view of Remark 3.8, we obtain the desired conclusion once we prove that $z_I(v(\cdot,t_1))<\infty$ for some t_1. If a and b are both finite, this follows immediately from Lemma 3.7(i).

Consider the case $b=\infty$. Assume for definiteness that $v(\tilde{b},0)>0$, so that assumption (ci) gives $v(\cdot,0)\geq 0$ in $[\tilde{b},\infty)$ (the case $v(\tilde{b},0)<0$ can be treated similarly). By continuity, if $t_1>0$ is sufficiently small we have $v(\tilde{b},t)>0$ for all $t\in(0,t_1]$. By the comparison principle, for all $t\in(0,t_1]$ we have $v(\cdot,t)>0$ in $[\tilde{b},\infty)$, hence all zeros of $v(\cdot,t)$ are contained in $(a,\tilde{b})$. By similar arguments, if $a=-\infty$, then making t_1 smaller, if necessary, we achieve that for $t\in(0,t_1)$ all zeros of $v(\cdot,t)$ are contained in $(\tilde{a},\tilde{b})$. Thus in all cases, Lemma 3.7(i) gives the desired finiteness. $\square$

Remark 3.10. Corollary 3.9 in particular applies to the difference $u-\psi$, where u is a global solution of (1.1) with initial datum $u_0\in C_0(\mathbb{R})$, $u_0\geq 0$, and ψ is a periodic steady state of (1.1). More precisely, let $I=(a,b)$ be a connected component of the set $\{x\in\mathbb{R}:\psi(x)>0\}$. Conditions (ci), (cii) are readily verified for $v:=u-\psi$ if $I=\mathbb{R}$ (i.e., $\varphi>0$). If I is bounded, conditions (ci), (cii) hold for the function $v(x,t)=u(x,t+\delta)-\psi$ for any $\delta>0$.

Finally, we state a persistence property of multiple zeros in solutions of (3.14). The following lemma is a reformulation of [10, Lemma 2.6].

Lemma 3.11. *Assume that v is a nontrivial solution of* (3.14) *such that for some $s_0 \in (s,T)$ the function $v(\cdot,s_0)$ has a multiple zero at some x_0: $v(x_0,s_0) = v_x(x_0,s_0) = 0$. Assume further that for some $\delta > 0$, v_n is a sequence in $C^1([x_0-\delta,x_0+\delta]\times[s_0-\delta,s_0+\delta])$ which converges in this space to v. Then for all sufficiently large n the function $v_n(\cdot,t)$ has a multiple zero in $(x_0-\delta,x_0+\delta)$ for some $t \in (s_0-\delta,s_0+\delta)$.*

3.3 Invariance and connectedness of the limit sets

Recall that an *entire solution* of (1.1) refers to a solution defined for each $t \in \mathbb{R}$ (as above, the term "global solution" is reserved for a solution defined for all $t \geq 0$).

If u is a bounded solution of (1.1), then standard parabolic regularity estimates imply that the derivatives u_t, u_x, u_{xx} are bounded on $\mathbb{R}\times[1,\infty)$ and they are globally α-Hölder there for each $\alpha \in (0,1)$. Is is also a standard observation that for each $\varphi \in \omega(u)$ there is an entire solution U of (1.1) such that $U(\cdot,0)=\varphi$ and $U(\cdot,t) \in \omega(u)$ for each $t \in \mathbb{R}$. Let us recall precisely how U can be found. Let $t_n \to \infty$ be a sequence of times such that $u(\cdot,t_n) \to \varphi$ (in $L^\infty_{loc}(\mathbb{R})$). Consider the sequence of functions $(x,t) \mapsto u(x,t_n+t)$, $(x,t) \in \mathbb{R}\times(-t_n,\infty)$, $n = 1,2,\dots$. For a subsequence of this sequence one has

$$D^{2,1}u(\cdot,t_n+\cdot) \to D^{2,1}U, \tag{3.16}$$

uniformly on each compact set in $\mathbb{R}^2$, where $U(x,t)$ is an entire solution with the indicated properties and the symbol $D^{2,1}u$ stands for (u,u_x,u_{xx},u_t).

The same invariance statement holds for $\Omega(u)$: if $\varphi \in \Omega(u)$, then there is an entire solution U of (1.1) such that $U(\cdot,0)=\varphi$ and $U(\cdot,t) \in \Omega(u)$ for each $t \in \mathbb{R}$. One finds U in an analogous way, only this time one has

$$D^{2,1}u(x_n+\cdot,t_n+\cdot) \to D^{2,1}U, \tag{3.17}$$

uniformly on each compact set in $\mathbb{R}^2$, where $\{(x_n,t_n)\}$ is a suitable sequence in $\mathbb{R}\times(0,\infty)$ such that $t_n \to \infty$ and $u(x_n+\cdot,t_n) \to \varphi$.

Remark 3.12. In addition to the usual flow invariance, $\Omega(u_0)$ also has the translation invariance property. More precisely, the definition of $\Omega(u_0)$ implies that if $\varphi \in \Omega(u_0)$, then $\Omega(u_0)$ contains the $L^\infty_{loc}(\mathbb{R})$-closure of the translation group orbit of φ, $\{\varphi(\cdot+\xi) : \xi \in \mathbb{R}\}$. In particular, if the limit $\gamma := \varphi(\infty)$ exists, then $\gamma \in \Omega(u_0)$. The same is true of the limit at $-\infty$.

It is also well-known and easy to prove using the compactness of

$$\{u(x+\cdot,t,u_0): x \in \mathbb{R},\ t \geq 1\}$$

that $\omega(u_0)$ and $\Omega(u_0)$ are connected in metric spaces such as $L^\infty_{loc}(\mathbb{R})$, $C^1_{loc}(\mathbb{R})$. Consequently, the sets

$$\{(\varphi(x), \varphi_x(x)) : \varphi \in \omega(u_0),\ x \in \mathbb{R}\}, \quad \{(\varphi(x), \varphi_x(x)) : \varphi \in \Omega(u_0),\ x \in \mathbb{R}\}$$

are connected in $\mathbb{R}^2$.

4 Proofs of Theorems 2.1, 2.3, and 2.4

In the whole section, we assume that $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and the solution $u(\cdot,\cdot,u_0)$ is bounded. Also, in addition to the standing hypotheses on f (f is locally Lipschitz and $f(0)=0$), we assume, without any loss of generality (cp. Remark 2.7), that f satisfies condition (C1) from Section 3.1.

In the proofs of our results, we often consider multiple zeros of the difference of two functions φ, ψ. Existence of such zeros means in particular that

$$\{(\varphi(x), \varphi_x(x)) : x \in \mathbb{R}\} \cap \{(\psi(x), \psi_x(x)) : x \in \mathbb{R}\} \neq \emptyset.$$

It will therefore be convenient to define the *spatial trajectory (or orbit)* of a C^1 function φ on $\mathbb{R}$ by

$$\tau(\varphi) := \{(\varphi(x), \varphi_x(x)) : x \in \mathbb{R}\}. \tag{4.1}$$

We are especially interested in spatial trajectories of solutions of (1.1). Note that if φ is a steady state of (1.1), then $\tau(\varphi)$ is the usual orbit of the solution (φ, φ_x) of the system (3.3).

For the proofs of Theorems 2.1, 2.3, and 2.4 the following property of spatial trajectories is crucial.

Lemma 4.1. *Assume that $\varphi \in \omega(u_0)$ or $\varphi \in \mathcal{T}_\Omega(u_0)$, ψ is a periodic steady state of* (1.1)*, and at least one of the functions φ, ψ is not identical to any zero of f. Then $\tau(\varphi) \cap \tau(\psi) = \emptyset$.*

In the proof of this lemma, we use the following technical result (see Section 2 for the definition of $\mathcal{T}_\Omega(u_0)$ and $\gamma_{max}(u_0)$).

Lemma 4.2. *Let ψ be a periodic steady state of* (1.1)*. Then for any bounded interval I the following statement holds*

(S) *The function $u(\cdot, t, u_0) - \psi$ has only simple zeros in I for all sufficiently large t.*

Moreover, either (S) holds with $I = \mathbb{R}$ or $\psi < \gamma_{max}(u_0)$.

We first complete the proof of Lemma 4.1, then give the proof of Lemma 4.2.

Proof of Lemma 4.1. First of all we prove that $\Omega(u_0)$ does not contain any nonconstant periodic steady state and, consequently, $\omega(u_0)$ and $\mathcal{T}_\Omega(u_0)$ do not contain any nonconstant periodic steady state. For a contradiction, assume $\phi \in \Omega(u_0)$ is a nonconstant periodic steady state. Then $\phi \geq 0$, as $u_0 \geq 0$, and the relation $f(0) = 0$ implies $\phi > 0$. Considering $\mathcal{O} := \tau(\phi)$ as a closed orbit of (3.3), we find an equilibrium $(\nu, 0)$ of (3.3) in $\mathrm{Int}(\mathcal{O})$. Then $f(\nu) = 0$, $\nu > 0$, and $\phi - \nu$ has infinitely many zeros, all of them simple. Since $\phi \in \Omega(u_0)$, there is a sequence $\{(x_n, t_n)\}$ in $\mathbb{R} \times (0, \infty)$ such that $t_n \to \infty$ and $u(x_n + \cdot, t_n) \to \phi$ in $C^1_{loc}(\mathbb{R})$. This implies that $z(u(\cdot, t_n, u_0) - \nu) \to \infty$ as $n \to \infty$. On the other hand, Corollary 3.9 and Remark 3.10 imply that $z(u(\cdot, t, u_0) - \nu)$ is finite and independent of t for t large enough, which is a contradiction.

Let now φ and ψ satisfy the assumptions of Lemma 4.1. Assume that $\tau(\varphi) \cap \tau(\psi) \neq \emptyset$. This means that there is a shift $\tilde{\psi} := \psi(\cdot - y)$ of ψ such that the function $\varphi - \tilde{\psi}$ has a multiple zero, say x_0. We show that this leads to a contradiction with Lemma 4.2.

As noted in Section 3.3, there is an entire solution of (1.1) such that $U(\cdot, 0) = \varphi$, and for some sequence $\{(x_n, t_n)\}$ in $\mathbb{R} \times (0, \infty)$ one has $t_n \to \infty$ and

$$D^{2,1} u(x_n + \cdot, t_n + \cdot) \to D^{2,1} U \text{ in } L^\infty_{loc}(\mathbb{R}^2). \tag{4.2}$$

In case $\varphi \in \omega(u_0)$, we can take $x_n = 0$ for all n. Since $\varphi - \tilde{\psi} \not\equiv 0$ (by what we proved in the first paragraph of this proof), the function $v := U - \tilde{\psi}$ is a nontrivial entire solution of a linear equation (3.14). Therefore, an application of Lemma 3.11 shows that for each large n there is $\tilde{t}_n \in (t_n - 1, t_n + 1)$ such that the function $u(x_n + \cdot, \tilde{t}_n) - \tilde{\psi}$ has a multiple zero in $(x_0 - 1, x_0 + 1)$.

If $\varphi \in \omega(u)$, we take $x_n = 0$, $n = 1, 2, \ldots$, and the previous conclusion gives us immediately a sought-after contradiction with Lemma 4.2 (applied to $\tilde{\psi}$ in place of ψ): just take $I = (x_0 - 1, x_0 + 1)$ in statement (S).

In the case $\varphi \in \mathcal{T}_\Omega(u_0)$, we have $\varphi \geq \gamma_{max}(u_0)$, hence the assumption $\tau(\varphi) \cap \tau(\psi) \neq \emptyset$ rules out the possibility $\psi < \gamma_{max}(u_0)$. Therefore, statement (S) holds for $I = \mathbb{R}$ (with $\tilde{\psi}$ in place of ψ). If ψ is constant, then $u(x_n + \cdot, \tilde{t}_n) - \tilde{\psi}$ having a multiple zero means that $u(\cdot, \tilde{t}_n) - \tilde{\psi}$ has a multiple zero and we have a contradiction. If ψ is not constant, write x_n in the form $x_n = k_n\rho + \zeta_n$, where $\rho > 0$ is the minimal period of ψ, $k_n \in \mathbb{Z}$, and $\zeta_n \in [0, \rho)$. Passing to subsequences, we may assume that $\zeta_n \to \zeta_0 \in [0, \rho]$. Then (4.2) implies that

$$u(k_n\rho + \zeta_0 + \cdot, t_n + \cdot) \to U \text{ in } C^1_{loc}(\mathbb{R}^2).$$

Therefore, using Lemma 3.11, similarly as above, we see that for all large n the function $u(k_n\rho + \zeta_0 + \cdot, \bar{t}_n) - \tilde{\psi}$ has a multiple zero for some $\bar{t}_n \in (t_n - 1, t_n + 1)$. But then $u(\cdot, \bar{t}_n) - \tilde{\psi}(\cdot - k_n\rho - \zeta_0) = u(\cdot, \bar{t}_n) - \tilde{\psi}(\cdot - \zeta_0)$ has a multiple zero and this is a contradiction to statement (S).

$\square$

Proof of Lemma 4.2. Statement (S) is obviously true with $I = \mathbb{R}$ if $\tilde{\psi} \equiv 0$, since $u(\cdot, t, u_0) > 0$ (we are assuming that $u_0 \geq 0$ and $u_0 \not\equiv 0$). Likewise, if $\tilde{\psi} > 0$, then $u(\cdot, t) - \tilde{\psi}$ has only simple zeros in $\mathbb{R}$ by virtue of Corollary 3.9 (cp. Remark 3.10), so again (S) holds with $I = \mathbb{R}$.

In the remainder of the proof we consider the case when $\tilde{\psi}$ changes sign. Let (a, b) be a connected component of the set $\{x : \tilde{\psi}(x) > 0\}$. Then all the other connected components of this set have the form (a_k, b_k), $k \in \mathbb{Z}$, where $a_k := a + k\varrho$ and $b_k := b + k\varrho$, ϱ being the minimal period of $\tilde{\psi}$. For each fixed k, Corollary 3.9 and Remark 3.10 imply that there is t_k such that all zeros of $u(\cdot, t, u_0) - \tilde{\psi}$ in (a_k, b_k) are simple. Consequently, for each finite n there is $\bar{t}_n$ such that if $t \geq \bar{t}_n$, that all zeros of $u(\cdot, t, u_0) - \tilde{\psi}$ in

$$\bigcup_{k=-n,\dots n} (a_k, b_k)$$

are simple. Clearly, $u(\cdot, t, u_0) - \tilde{\psi}$ has no zeros in the complement of the union $\bigcup_{k\in\mathbb{Z}}(a_k, b_k)$, for $u(\cdot, t, u_0) > 0 > \tilde{\psi}$ there. This shows that statement (S) holds for any bounded interval I.

To prove the last statement of Lemma 4.2, fix any $t_0 > 0$. Since $u_0 \in C_0(\mathbb{R})$, the functions $u(\cdot, t_0, u_0)$, $u_x(\cdot, t_0, u_0)$ are both contained in $C_0(\mathbb{R})$. By the periodicity of $\tilde{\psi}$, there are $\epsilon > 0$, $\delta > 0$ such that $|\tilde{\psi}_x| > \delta$ in the ε-neighborhoods of the zero points a_k, b_k of $\tilde{\psi}$. Consequently, there is n such that if $|k| \geq n$, then $u(\cdot, t_0, u_0) - \tilde{\psi}$ has exactly two zeros, both simple, in (a_k, b_k). By the monotonicity of the zero number, there are now two possibilities. Either

$$z_{(a_k,b_k)}(u(\cdot,t,u_0) - \tilde{\psi}) = 2 \quad (t \geq t_0,\ |k| \geq n), \tag{4.3}$$

or else there is k with $|k| \geq n$ such that $z_{(a_k,b_k)}(u(\cdot,t,u_0) - \tilde{\psi})$ drops in (t_0, ∞). If (4.3) holds, then, as noted in Remark 3.8, the two zeros in (a_k, b_k) have to be simple for all $t \geq t_0$. Combining this with what we proved above for bounded intervals, we conclude that statement (S) holds with $I = \mathbb{R}$. In this case we are done.

Consider the other possibility: there is k with $|k| \geq n$ such that the zero number $z_{(a_k,b_k)}(u(\cdot,t,u_0) - \tilde{\psi})$ drops in (t_0, ∞). Necessarily, then, there is $t_1 > t_0$ such that $u(\cdot,t_1,u_0) - \tilde{\psi} \geq 0$ in $[a_k, b_k]$. We show that this implies $\gamma_{max}(u_0) > \tilde{\psi}$.

We have $u(\cdot,t_1,u_0) \geq \psi^*$ on $\mathbb{R}$, where ψ^* is the continuous function defined by

$$\psi^*(x) = \begin{cases} \tilde{\psi}(x), & \text{if } x \in [a_k, b_k], \\ 0, & \text{if } x \in \mathbb{R} \setminus [a_k, b_k]. \end{cases}$$

By the comparison principle, for all $t > t_1$

$$u(\cdot,t,u_0) > u(\cdot,t - t_1,\psi^*). \tag{4.4}$$

Let us now examine the solution $u(\cdot,\cdot,\psi^*)$. Obviously, it is nonnegative, and since $u(\cdot,\cdot,u_0)$ is bounded, $u(\cdot,\cdot,\psi^*)$ is bounded as well. It is well-known and easy to prove using the comparison principle that, being the maximum of two distinct steady states of (1.1), ψ^* is a strict subsolution of (1.1): $u(t,\cdot,\psi^*) > \psi^*$ for all $t > 0$. This also implies that $u(\cdot,t,\psi^*)$ is increasing in t (indeed, for any $s > 0$ the function $u(\cdot,t+s,\psi^*) - u(\cdot,t,\psi^*)$ is positive, since it solves a linear parabolic equation and has a positive initial value). Hence, as $t \to \infty$, $u(\cdot,t,\psi^*)$ converges to a steady state $\phi > \psi^*$. In fact, this steady state must be constant, say $\phi \equiv \gamma^*$. Indeed, otherwise there is $\xi \approx 0$ such that $\tilde{\phi} := \phi(\cdot - \xi) > \psi^*$ and $\tilde{\phi}(x_0) < \phi(x_0)$ for some $x_0 \in \mathbb{R}$. But then $\tilde{\phi} > u(\cdot,t,\psi^*)$ for all t and, in particular, $\tilde{\phi}(x_0) > u(x_0,t,\psi^*)$ makes $u(x_0,t,\psi^*) \to \phi(x_0)$ impossible. Clearly, $\gamma^* > \psi^*$ implies that $\gamma^* > \tilde{\psi}$, by the periodicity of $\tilde{\psi}$, and γ^* is the minimal constant steady state greater than $\tilde{\psi}$. We shall prove in a moment that γ^* must be contained in $\tilde{\Gamma}$. Using (4.4) and the relations $\Omega(u_0) \supset \omega(u_0) \neq \emptyset$, we see that there is $\phi \in \Omega(u_0)$ with $\phi \geq \gamma^*$.

Therefore, by the definition of $\gamma_{max}(u_0)$, one has $\gamma_{max}(u_0) \geq \gamma^*$, which gives the desired conclusion $\gamma_{max}(u_0) > \tilde{\psi}$.

To prove that $\gamma^* \in \tilde{\Gamma}$, we denote by μ the maximum of $\tilde{\psi}$. Since, γ^* is the minimal constant steady state greater than $\tilde{\psi}$, we have

$$|f(s)| > 0 \quad (s \in [\mu, \gamma^*)). \tag{4.5}$$

Moreover, if y is a maximum point of $\tilde{\psi}$, then $f(\mu) = f(\tilde{\psi}(y)) = -\tilde{\psi}_{xx}(y) \geq 0$. Thus relation (4.5) is valid without the absolute value. Consequently,

$$F(\gamma^*) > F(s) \quad (s \in [\mu, \gamma^*)). \tag{4.6}$$

Now, since the trajectory $\tau(\tilde{\psi})$ intersects the v-axis ($\tilde{\psi}$ changes sign) and

$$\frac{\tilde{\psi}_x^2}{2} + F(\tilde{\psi}) \equiv const = F(\mu),$$

we have

$$F(\mu) \geq F(r) \quad (r \in [0, \mu)).$$

This and (4.6) imply that $\gamma^* \in \tilde{\Gamma}$.

□

One more lemma is needed for the proof of our theorems.

Lemma 4.3. *Let u be a solution of* (1.1) *on $\mathbb{R} \times (t_1, t_2)$ for some $t_1 < t_2$ and let ψ be a steady state of* (1.1). *If $\tau(u(\cdot, t_0)) \subset \tau(\psi)$ for some $t_0 \in (t_1, t_2)$, then $u \equiv \psi(\cdot + c)$ for some constant c.*

Proof. The statement is trivial if ψ is a constant steady state, thus we assume that $\psi_x \not\equiv 0$.

The relation $\tau(u(\cdot, t_0)) \subset \tau(\psi)$ means that for each $x \in \mathbb{R}$ there is $\zeta(x)$ such that

$$u(x, t_0) = \psi(\zeta(x)), \tag{4.7}$$
$$u_x(x, t_0) = \psi_x(\zeta(x)). \tag{4.8}$$

Clearly, $\zeta(x)$ is unique in case $\tau(\psi)$ is a homoclinic or heteroclinic orbit of (3.3). In case $\tau(\psi)$ is a (nonstationary) periodic orbit, we determine $\zeta(x)$ uniquely by postulating that it be contained in $[y_0, y_0 + \varrho)$, where y_0 is a fixed constant and ϱ is the minimal period of ψ. We choose y_0 such that y_0

and $y_0 + \varrho/2$ are the only two critical points, one maximum the other one minimum, of ψ in $[y_0, y_0+\varrho)$ (here we are using the symmetry of $\tau(\psi)$ around the u-axis, cp. Section 3.1).

Fix any x_0 with $u_x(x_0,t_0) \neq 0$ and set $\zeta_0 = \zeta(x_0)$. By (4.8), $\psi_x(\zeta_0) \neq 0$; in particular, if ψ is periodic and y_0, ϱ are as above, then $\zeta_0 \in (y_0, y_0+\varrho)$. The implicit function theorem therefore implies that ζ is of class C^1 in a neighborhood of x_0. Differentiating (4.7) and comparing the result to (4.8), we obtain that $\zeta'(x) \equiv 1$ for $x \approx x_0$. Hence there is a constant c such that for $x \approx x_0$ we have $\zeta(x) = x+c$. This means that the function $u(x,t_0)-\psi(x+c)$ is identically equal to zero on an open interval. By Lemma 3.7, this is possible only if $u - \psi(\cdot + c) \equiv 0$. □

The proofs of Theorems 2.1 and 2.4 have a common part comprising the following proposition.

Proposition 4.4. *Let $\varphi \in \omega(u_0) \cup \mathcal{T}_\Omega(u_0)$. Then φ is a steady state of* (1.1) *and one of the statements (a)-(c) of Theorem 2.1 holds.*

Proof. First we prove that φ is a steady state. We assume that φ is not identical to a zero of f, otherwise there is nothing to prove.

As in Section 3.3, we find an entire solution U of (1.1) such that $U(\cdot,0) = \varphi$ and

$$\begin{aligned} &U(\cdot,t) \in \omega(u_0) \quad (t \in \mathbb{R}), \quad \text{if } \varphi \in \omega(u_0), \\ &U(\cdot,t) \in \Omega(u_0) \quad (t \in \mathbb{R}), \quad \text{if } \varphi \in \mathcal{T}_\Omega(u_0). \end{aligned}$$

In the latter case, the relations $U(\cdot,0) = \varphi \geq \gamma_{max}(u_0)$ and the comparison principle also give

$$U(\cdot,t) \in \mathcal{T}_\Omega(u_0) \quad (t \geq 0).$$

Of course, $U(\cdot,t)$ is not identical to a zero of f for any $t \in \mathbb{R}$. Set

$$K := \{(U(x,t), U_x(x,t)) : x \in \mathbb{R},\, t \geq 0\} = \bigcup_{t\geq 0} \tau(U(\cdot,t)). \tag{4.9}$$

Then K is a connected set in $\mathbb{R}^2$. By Lemma 4.1, $K \cap \tau(\psi) = \emptyset$ if ψ is any periodic (or constant) steady state of (1.1). In the notation of Lemma 3.1, this translates to $K \cap \mathcal{P} = \emptyset$. Hence K is contained in a connected component of $\mathbb{R}^2 \setminus \mathcal{P}$. Therefore, by Lemma 3.1(ii), there is a homoclinic or heteroclinic solution $(\tilde\varphi, \tilde\varphi_x)$ of (3.3) such that $K \subset \tau(\tilde\varphi)$. By Lemma 4.3, $U \equiv \tilde\varphi(\cdot + c)$ for some constant c. In particular $\varphi \equiv \tilde\varphi(\cdot + c)$.

Thus we have proved that φ is a steady state, as desired. Moreover, we have shown that if φ is not constant, then $\tau(\varphi)$ is a heteroclinic or homoclinic orbit of (3.3).

To prove that one of the statements (a)-(c) of Theorem 2.1 holds, we consider the connected component Σ of $\mathbb{R}^2 \setminus \mathcal{P}_0$ such that $\tau(\varphi) \subset \Sigma$ (recall that $\mathcal{P}_0 := \mathcal{P} \cup \mathcal{E}$, as in Lemma 3.1). Note that if

$$\gamma := \varphi(-\infty) \text{ or } \gamma := \varphi(\infty), \tag{4.10}$$

then $\gamma \geq 0$, as $u_0 \geq 0$, and $(\gamma, 0)$ is an equilibrium of (3.3). Also, $(\gamma, 0)$ being the limit of $(\varphi(x), \varphi_x(x))$ as $x \to \infty$ or $x \to -\infty$, one has $(\gamma, 0) \in \Sigma$, as Σ is compact by Lemma 3.1.

We now claim that the following implication is valid:

$$\text{if } \gamma \text{ is as in (4.10), then } \gamma \in \tilde{\Gamma}. \tag{4.11}$$

Suppose for a while this is true. If φ is a constant steady state or a standing wave, then (4.11) shows that statement (a) or statement (c) of Theorem 2.1 holds. If $\tau(\varphi)$ is a homoclinic orbit, Lemma 3.6 implies that φ is a ground state at level γ. Note that the assumption $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$ of Lemma 3.6 is satisfied due to (4.11) (we recall that N_Σ was defined in (3.12)). From (4.11) we further conclude that statement (b) of Theorem 2.1 holds. Thus the proof of Proposition 4.4 will be complete once we prove (4.11).

We prove (4.11) by contradiction. Assume γ is as in (4.10) and $\gamma \notin \tilde{\Gamma}$. Then, by Lemma 3.5(ii), $N_\Sigma \cap \tilde{\Gamma} = \emptyset$. Consequently, by Lemma 3.5(i) there is a nonconstant periodic orbit $\mathcal{O}$ of (3.3) such that $\mathcal{O} \subset \{(u, v) : u > 0\}$ and $\Sigma \subset \operatorname{Int}(\mathcal{O})$. In particular, $(\gamma, 0) \in \operatorname{Int}(\mathcal{O})$. Now, $\mathcal{O} = \tau(\psi)$ for some positive periodic solution ψ of (3.2). From $(\gamma, 0) \in \operatorname{Int}(\mathcal{O})$ it follows that $\psi - \gamma$ has infinitely many zeros. Of course, all these zeros are simple as ψ, γ both solve the second-order ODE (3.2). Since $\varphi \in \omega(u_0) \cup \mathcal{T}_\Omega(u_0) \subset \Omega(u_0)$, we have $\gamma \in \Omega(u_0)$ (see Remark 3.12). Therefore, there are real sequences $\{t_n\}$ and $\{x_n\}$ such that $t_n \to \infty$, and $u(\cdot + x_n, t_n, u_0) \to \gamma$ in $L^\infty_{loc}(\mathbb{R})$. Using the periodicity of ψ and the fact that $\psi - \gamma$ has infinitely many simple zeros, one shows easily that $z(u(\cdot, t_n, u_0) - \psi) \to \infty$ as $n \to \infty$. This contradicts Corollary 3.9 (see also Remark 3.10). This contradiction proves (4.11) and completes the proof of Proposition 4.4. $\square$

Proof of Theorem 2.1. In view of Proposition 4.4, it remains to prove the last two statements of Theorem 2.1.

Let Σ be as in the proof of Proposition 4.4, namely, it is a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$ containing $\tau(\varphi)$ for some $\varphi \in \omega(u_0)$. Then Σ actually contains the whole set

$$M := \bigcup_{\varphi \in \omega(u_0)} \tau(\varphi) = \{(\varphi(x), \varphi_x(x)) : \varphi \in \omega(u_0),\ x \in \mathbb{R}\},$$

for this set is connected in $\mathbb{R}^2$ (see Section 3.3) and, by Lemma 4.1, $M \subset \mathbb{R}^2 \setminus \mathcal{P}_0$.

By Lemma 3.1(i), the Hamiltonian H takes a constant value $H(\Sigma)$ on Σ, hence if γ is as in (4.10) for any $\varphi \in \omega(u_0)$, then $F(\gamma) = H(\gamma, 0) = H(\Sigma)$. This proves the last statement of Theorem 2.1. Further, as in the proof of Proposition 4.4, (4.11) implies that $N_\Sigma \cap \tilde{\Gamma} \neq \emptyset$. Hence, by Lemma 3.6, up to translations, there is at most one ground state in $\omega(u_0)$. The proof of Theorem 2.1 is now complete. $\square$

Proof of Theorem 2.4. By Proposition 4.4, for each $\varphi \in \mathcal{T}_\Omega(u_0)$ one of the statements (a)-(c) of Theorem 2.1 holds. We shall presently see that (c) is impossible. Indeed, if it holds, then the larger of the two constants γ_1, γ_2, say $\hat{\gamma}$, is an element of $\tilde{\Gamma}$ and a strict upper bound on φ. Also $\hat{\gamma} \in \Omega(u_0)$ (see Remark 3.12). Hence $\varphi < \hat{\gamma} \leq \gamma_{max}(u_0)$, which implies that φ is not contained in $\mathcal{T}_\Omega(u_0)$ - a contradiction.

Thus for each $\varphi \in \mathcal{T}_\Omega(u_0)$ one of the statements (a) or (b) must hold. We now show that in either case one has $\gamma = \gamma_{max}(u_0)$ ($\gamma \in \tilde{\Gamma}$ is the constant in (a), (b)). Indeed, $\varphi \in \mathcal{T}_\Omega(u_0)$ implies that $\varphi \geq \gamma_{max}(u_0)$, therefore $\gamma \geq \gamma_{max}(u_0)$. On the other hand, we also have $\varphi \geq \gamma$, hence $\gamma \leq \gamma_{max}(u_0)$ by the maximality of $\gamma_{max}(u_0)$.

Having proved that for each $\varphi \in \mathcal{T}_\Omega(u_0)$ one of the statements (a) or (b) holds with $\gamma = \gamma_{max}(u_0)$, we can conclude easily. If there is any ground state in $\mathcal{T}_\Omega(u_0)$, then the uniqueness of the ground state at level $\gamma_{max}(u_0)$ (see Lemma 3.4) and Remark 3.12 imply that (2.9) holds. If there is no ground state in $\mathcal{T}_\Omega(u_0)$, then statement (a) with $\gamma = \gamma_{max}(u_0)$ has to hold for each $\varphi \in \mathcal{T}_\Omega(u_0)$, which gives $\mathcal{T}_\Omega(u_0) = \{\gamma_{max}(u_0)\}$. Theorem 2.4 is proved. $\square$

Proof of Theorem 2.3. Assume that (2.5) holds for some $a < b$. We apply a standard reflection principle. The function $V_a u$ solves a linear parabolic equation in the set $(-\infty, a) \times (0, \infty)$ and is nonnegative on its parabolic boundary. Hence by the comparison principle, $V_a u(\cdot, t) \geq 0$ in $(-\infty, a)$ for all $t > 0$. Similarly, $V_b u(\cdot, t) \geq 0$ in (b, ∞) for all $t > 0$. These relations imply in particular that

$$u_x(a,t) = -\frac{1}{2}(V_a u)_x(a,t) \geq 0 \text{ and } u_x(b,t) = -\frac{1}{2}(V_b u)_x(b,t) \leq 0 \quad (t > 0). \tag{4.12}$$

Consequently,

$$\varphi_x(a) \geq 0 \text{ and } \varphi_x(b) \leq 0 \quad (\varphi \in \omega(u_0)). \tag{4.13}$$

This is clearly not satisfied by any standing wave, thus for each $\varphi \in \omega(u_0)$ one of the statements (a)-(b) of Theorem 2.1 holds. Also, (4.13) implies that if $\varphi \in \omega(u_0)$ is a ground state, then its maximum point is contained in $[a, b]$. By Theorem 2.1, all ground states in $\omega(u_0)$ are translations of one another. Hence, by (4.13), if $\omega(u_0)$ contains a ground state φ, then all the ground states in $\omega(u_0)$ are contained in the set $\{\varphi(\cdot + \xi) : \xi \in J\}$, where J is a compact interval. In this case, the connectedness of $\omega(u_0)$ and Theorem 2.1 imply that the whole of $\omega(u_0)$ is contained in $\{\varphi(\cdot + \xi) : \xi \in J\}$, therefore, making J smaller if necessary, $\omega(u_0) = \{\varphi(\cdot + \xi) : \xi \in J\}$.

Assume now that $\omega(u_0)$ contains no ground state. Then $\omega(u_0)$ consists of constants, all elements of $\tilde{\Gamma}$. We need to rule out the possibility that $\omega(u_0)$ contains two constants $\gamma_1 < \gamma_2$. Assume it does. Then, since $\omega(u_0)$ is connected and consists of constant steady states, $\gamma := (\gamma_1 + \gamma_2)/2$ is also a constant steady state (contained in $\omega(u_0)$). From $\gamma_1 \in \omega(u_0)$ we infer that for some $t_0 > 0$ one has $u(\cdot, t_0) < \gamma$ on $[a, b]$. Using this, (4.12), and the comparison principle, we conclude that $u(\cdot, t) < \gamma$ on (a, b) for all $t > t_0$, contradicting the assumption that $\gamma_2 \in \omega(u_0)$. This contradiction shows that $\omega(u_0) = \{\gamma\}$ for some $\gamma \in \tilde{\Gamma}$.

Now, to prove the last statement of Theorem 2.3, assume that condition (CN) is satisfied for each $\xi \in S$, where S is a dense subset of (a, b). If $\omega(u_0) = \{\gamma\}$, for some $\gamma \in \tilde{\Gamma}$, there is nothing else to be proved. Thus we assume that (2.6) holds for some ground state φ.

We already know that the ground states in $\omega(u_0)$ have their maximum points in $[a, b]$. Now take any $\xi \in S$. Condition (CN) allows us to apply Corollary 3.9 to the solutions $u(\cdot, \cdot, u_0)$ and $u(2\xi - \cdot, \cdot, u_0)$, which shows that for all sufficiently large $t > 0$ the function $V_\xi u(\cdot, t)$ has only simple zeros. In particular, since $V_\xi u(\xi, t) = 0$, we have $-2u_x(\xi, t) = (V_\xi u)_x(\xi, t) \neq 0$ for all large t. If $u_x(\xi, t) > 0$ for all large t, then all the ground states in $\omega(u_0)$ have their maximum points in $[\xi, b]$. Otherwise, if $u_x(\xi, t) < 0$ for all large t, they have their maximum points in $[a, \xi]$. We can now continue this division process, proving eventually that all the ground states in $\omega(u_0)$ have their maximum point located at one fixed point in $[a, b]$. In view of (2.6) this means that $\omega(u_0) = \{\tilde{\varphi}\}$, where $\tilde{\varphi}$ is a translation of φ. The theorem is proved. $\square$

5 Preliminaries II: normal hyperbolicity and convergence

In the whole section we assume that $f \in C^1(\mathbb{R})$, $f(0) = 0$, and $f'(0) < 0$.

We recall two results, both closely related to the fact that if φ is a ground state at level 0, then its translation orbit $M = \{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}$ is normally hyperbolic. This is to say that the dimension of the kernel of the linearized operator $\partial_x^2 + f'(\varphi(x))$ (in $H^2(\mathbb{R})$, say) is 1, the same as the dimension of M.

The first result is the convergence for localized solutions:

Theorem 5.1. *Let $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$. Assume that the solution $u(\cdot, \cdot, u_0)$ is bounded and localized:*

$$\lim_{|x| \to \infty} u(x, t, u_0) = 0, \text{ uniformly in } t \geq 0. \tag{5.1}$$

Then $\lim_{t \to \infty} u(\cdot, t, u_0) = \varphi$, where $\varphi \equiv 0$ or it is a ground state of (1.1) *at level 0, and the convergence is in $L^\infty(\mathbb{R})$.*

For a proof of this theorem see [17], where more general one-dimensional problems are considered (and condition $u_0 \geq 0$ is not needed) or [20], where this convergence result is proved in any dimension (see also [3] and references therein for different proofs under additional assumptions). An interested reader can also consult [20] for a broader discussion of connections between normal hyperbolicity and convergence, including some abstract results and relevant references.

The second result concerns the existence of a convergent solution with a 3-step initial condition. We consider a function g defined by

$$g(x) = \begin{cases} \beta & (x \in (-\infty, -q)), \\ \vartheta & (x \in [-q, 0]), \\ 0 & (x \in (0, \infty]), \end{cases} \tag{5.2}$$

with suitable constants $0 < \beta < \vartheta$ and $q > 0$.

Lemma 5.2. *Assume that condition* (BS) *from Section 2 holds and let* γ_1 *be as in* (2.10)*. There is* $\beta_0 > 0$ *with the following property. For each* $\beta \in (0, \beta_0]$ *there exist* $\vartheta \in (\beta, \gamma_1)$ *and* $q > 0$ *such that if* g *is as in* (5.2)*, then* $\lim_{t\to\infty} u(\cdot, t, g) = \varphi$*, where* φ *is a ground state of* (1.1) *at level 0 and the convergence is in* $L^\infty(\mathbb{R})$*.*

This is proved in [27, Lemma 3.5]. Although the standing hypotheses in [27] are a little bit more restrictive than (BS), the additional conditions are not used in the proof of this result.

The significance of Lemma 5.2 is in that it provides an example of a solution which is neither localized nor symmetric and which converges to a ground state uniformly on $\mathbb{R}$. Below, we will employ this solution in a comparison argument ruling out the possibility that other solutions near the translation orbit M drift indefinitely along M, without converging to any element of M.

6 Proofs of Theorems 2.5 and 2.6

In the whole section we assume that $f \in C^1(\mathbb{R})$ and $f(0) = 0 > f'(0)$.

The following result is used in the proofs of both the theorems.

Lemma 6.1. *Assume that the hypotheses of Theorem 2.5 are satisfied:* $u_0 \in C_0(\mathbb{R})$*,* $u_0 \geq 0$*, the solution* $u(\cdot, \cdot, u_0)$ *is bounded, and* $\gamma_{max}(u_0) = 0$*. Then the solution* $u(\cdot, \cdot, u_0)$ *is localized (that is,* (5.1) *holds).*

We now give the proofs of the theorems, then prove the lemma.

Proof of Theorem 2.5. The conclusion follows from Lemma 6.1 upon application of Theorem 5.1. □

Proof of Theorem 2.6. Assume that the hypotheses of Theorem 2.6 are satisfied and let ℓ and δ be as in those hypotheses. To simplify the notation, let $u^\lambda := u(\cdot, \cdot, \psi_\lambda)$. Set

$$\begin{aligned} K_0 &:= \{\lambda \in [0, 1] : \lim_{t\to\infty} u^\lambda(\cdot, t) = 0 \text{ in } L^\infty(\mathbb{R})\}, \\ K_1 &:= \{\lambda \in [0, \gamma] : \text{there is } t_0 > 0 \text{ such that } u^\lambda(\cdot, t_0) > \gamma_1 - \delta \\ &\qquad\qquad \text{on a closed interval of length } \ell\ \}. \end{aligned}$$

These sets are both open in $[0,1]$: the openness of K_1 is an easy consequence of the continuity with respect to the initial conditions and the openness of K_0 follows from the asymptotic stability of the trivial steady state (given by the assumption that $f'(0) < 0$). Hypothesis (a1) of Theorem 2.6 implies that $0 \in K_0$ and $1 \in K_1$. Also, by the comparison principle, the sets K_0, K_1 are intervals: there are $0 < \lambda_* \leq \lambda^* < 1$ such that

$$K_0 = [0, \lambda_*), \quad K_1 = (\lambda^*, 1]. \tag{6.1}$$

Consider now any $\lambda \in [\lambda_*, \lambda^*] = [0,1] \setminus (K_0 \cup K_1)$ and set $u_0 := \psi_\lambda$. Since $\lambda \notin K_1$, $u(\cdot, t, u_0) = u^\lambda(\cdot, t)$ is not greater than $\gamma_1 - \delta$ on any closed interval of length ℓ for any t. As already remarked in Section 2, this means that $\gamma_{max}(u_0) = 0$. Therefore, Theorem 2.5 applies and we have

$$\lim_{t\to\infty} u^\lambda(\cdot, t) = \varphi \text{ in } L^\infty(\mathbb{R}), \tag{6.2}$$

where either $\varphi \equiv 0$ or it is a ground state at level 0. The former cannot hold, as $\lambda \in [0,1] \setminus K_0$, hence the latter is the case. Since this conclusion is valid for each $\lambda \in [\lambda_*, \lambda^*]$, the proof of Theorem 2.6 will be complete if we show that $\lambda_* = \lambda^*$.

To that end, we first note that the limit ground state φ, as in (6.2), is independent of $\lambda \in [\lambda_*, \lambda^*]$. This follows easily from hypothesis (a2) combined with the comparison principle (which gives that the limit ground states for any two values in $[\lambda_*, \lambda^*]$ are ordered) and Lemma 3.4 (uniqueness of the ground state at level zero, up to translations). We now quote a general instability result of [25, Theorem 5.1], which applies in particular to any solution of (1.1) which is bounded, localized, and does not decay to zero as $t \to \infty$. Note that all these conditions are satisfied by u^λ for any $\lambda \in [\lambda_*, \lambda^*]$ by Lemma 6.1 and (6.2). Fix any such λ. Then, according to [25, Theorem 5.1], there is a positive constant d such that for any $\mu \in (0,1)$, $\mu \neq \lambda$, one has

$$\liminf_{t\to\infty} |u^\mu(x,t) - u^\lambda(x,t)| \geq d \quad (x \in [-1,1]). \tag{6.3}$$

In particular, u^μ, u^λ cannot both converge to the same steady state, which shows that $[\lambda_*, \lambda^*]$ cannot contain two different elements: $\lambda_* = \lambda^*$. The proof is now complete. $\square$

Proof of Lemma 6.1. As remarked in Section 2, condition $\gamma_{max}(u_0) = 0$ means that $\mathcal{T}_\Omega(u_0) = \Omega(u_0)$. Hence, by Theorem 2.4, either $\Omega(u_0) = \{0\}$ or

$$\Omega(u_0) = \{0\} \cup \{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}, \tag{6.4}$$

where φ is a ground state of (1.1) at level 0. If $\Omega(u_0) = \{0\}$, then, as one easily verifies, $u(\cdot, \cdot, u_0)$ is localized (and $u(\cdot, t, u_0) \to 0$, as $t \to \infty$, in $L^\infty(\mathbb{R})$). Hence, in this case we are done. Henceforth we therefore assume that (6.4) holds. We use the solution from Lemma 5.2 and intersection comparison arguments to show that the solution $u(\cdot, t, u_0)$ is localized in this case as well.

We have $u(\cdot, t, u_0) > 0$ for each positive t. Replacing u_0 with $u(\cdot, 1, u_0) > 0$, we may therefore assume that $u_0 > 0$. Modifying f outside the ranges of φ and $u(\cdot, \cdot, u_0)$, if necessary, we may further assume without loss of generality that f has a zero greater than $m_0 := \max \varphi$. We let γ_1 denote the minimal zero of f greater than m_0. Then condition (BS) holds. Indeed, since $\varphi_x^2/2 + F(\varphi) \equiv 0$, one has $F(u) < 0$ in $(0, m_0)$ and $F(m_0) = 0$. Moreover, $f(m_0) \neq 0$ and equation $\varphi_{xx} + f(\varphi) = 0$ gives $f(m_0) > 0$. Consequently, $F'(u) = f(u) > 0$ in $[m_0, \gamma_1)$,which verifies (BS).

Hence Lemma 5.2 applies. It yields constants $0 < \beta < \vartheta < \gamma_1$ and $q > 0$ such that $\lim_{t\to\infty} u(\cdot, t, g) = \phi$, where g is as in (5.2) and ϕ is a ground state of (1.1) at level 0 (thus ϕ is a translation of φ by uniqueness).

Since $u_0 \in C_0(\mathbb{R})$, for all sufficiently large $x_0 > 0$ one has $u_0 < \beta$ in $(-\infty, -x_0]$. For any such x_0,

$$\tilde{g} := g(\cdot + x_0) > u_0 \text{ in } (-\infty, -x_0]. \tag{6.5}$$

Of course, in $(-x_0, \infty)$ we have $u_0 > 0 \equiv \tilde{g}$. As one easily verifies (for example, by taking smooth approximations g_n of g with a sharp transition from ϑ to 0),

$$z(u(\cdot, t, \tilde{g}) - u(\cdot, t, u_0)) = 1$$

for all sufficiently small $t > 0$. By the nonincrease of the zero number,

$$z(u(\cdot, t, \tilde{g}) - u(\cdot, t, u_0)) \leq 1 \quad (t > 0). \tag{6.6}$$

Also, it is clear that as long as $z(u(\cdot, t, \tilde{g}) - u(\cdot, t, u_0))$ is equal to 1, $u(\cdot, t, \tilde{g}) - u(\cdot, t, u_0)$ cannot change its signature $(+, -)$, that is,

$$\begin{aligned} u(x, t, \tilde{g}) > u(x, t, u_0) \quad (x < \zeta(t)), \\ u(x, t, \tilde{g}) < u(x, t, u_0) \quad (x > \zeta(t)), \end{aligned} \tag{6.7}$$

where $\zeta(t)$ is the unique zero of $u(\cdot, t, \tilde{g}) - u(\cdot, t, u_0)$.

We intend to use these properties, with a suitable choice of x_0, in order to prove that

$$\lim_{x\to-\infty} u(x, t, u_0) = 0, \text{ uniformly in } t \geq 0. \tag{6.8}$$

We start by noting that, in view of the translation invariance of (1.1), one has

$$u(\cdot,t,\tilde{g}) = u(\cdot + x_0,t,g) \to \phi(\cdot + x_0) =: \tilde{\phi} \text{ in } L^\infty(\mathbb{R}). \tag{6.9}$$

Now, assume that (6.8) is not true. Then there is a sequence $\{(x_n,t_n)\}$ in $\mathbb{R}\times(0,\infty)$ such that $x_n \to -\infty$ and

$$u(x_n,t_n,u_0) \geq \epsilon_0 \quad (n=1,2,\dots), \tag{6.10}$$

for some $\epsilon_0 > 0$. Since $u_0 \in C_0(\mathbb{R})$, necessarily $t_n \to \infty$. Passing to a subsequence, we may assume that $u(\cdot,t_n,u_0) \to \psi$ in $L^\infty_{loc}(\mathbb{R})$ for some $\psi \in \omega(u_0) \subset \Omega(u_0)$. By (6.4), either $\psi \equiv 0$ or ψ is a ground state at level 0 (a shift of φ). Pick $x_0 > 0$ such that (6.5) holds and that, in case $\psi \not\equiv 0$, we also have $\psi \not\equiv \tilde{\phi}$, thus ψ and $\tilde{\phi}$ are different shifts of φ. Then

$$\psi(y_0) < \tilde{\phi}(y_0),$$

where y_0 is the maximum point of $\tilde{\phi}$. Consequently, for all sufficiently large n, we have

$$u(y_0,t_n,\tilde{g}) > u(y_0,t_n,u_0). \tag{6.11}$$

On the other hand, using (6.10) and the uniform convergence of $u(\cdot,t,\tilde{g})$ to $\tilde{\phi} \in C_0(\mathbb{R})$, we obtain that for all large n one has

$$u(x_n,t_n,\tilde{g}) < u(x_n,t_n,u_0). \tag{6.12}$$

From (6.11), (6.12) we conclude that for all large n the function $u(\cdot,t_n,\tilde{g}) - u(\cdot,t_n,u_0)$ has a zero in the interval (x_n,y_0). Thus, the equality must hold in (6.6) and then (6.7) holds for all t. This, however, contradicts (6.11), (6.12). The contradiction proves that (6.8) holds.

Applying (6.8) to the reflection $u_0(x) = u_0(-x)$, we see that (6.8) also holds with $x \to -\infty$ replaced by $x \to \infty$, hence the solution $u(\cdot,t,u_0)$ is localized. The proof is now complete. $\square$

References

[1] S. Angenent, *The zeroset of a solution of a parabolic equation*, J. reine angew. Math. **390** (1988), 79–96.

[2] H. Berestycki and P.-L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[3] J. Busca, M.-A. Jendoubi, and P. Poláčik, *Convergence to equilibrium for semilinear parabolic problems in* $\mathbb{R}^N$, Comm. Partial Differential Equations **27** (2002), 1793–1814.

[4] J. Carr and R.L. Pego, *Metastable patterns in solutions of* $u_t = \epsilon^2 u_{xx} - f(u)$, Commun. Pure Appl. Math. **42** (1989), 523–576.

[5] X.-Y. Chen, *A strong unique continuation theorem for parabolic equations*, Math. Ann. **311** (1998), 603–630.

[6] X.-Y. Chen and H. Matano, *Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations*, J. Differential Equations **78** (1989), 160–190.

[7] R. Chill and M. A. Jendoubi, *Convergence to steady states of solutions of non-autonomous heat equations in* $\mathbb{R}^N$, J. Dynam. Differential Equations **19** (2007), 777–788.

[8] P. Collet and J.-P. Eckmann, *Space-time behaviour in problems of hydrodynamic type: a case study*, Nonlinearity **5** (1992), 1265–1302.

[9] C. Cortázar, M. del Pino, and M. Elgueta, *The problem of uniqueness of the limit in a semilinear heat equation*, Comm. Partial Differential Equations **24** (1999), 2147–2172.

[10] Y. Du and H. Matano, *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*, J. Eur. Math. Soc. **12** (2010), 279–312.

[11] Y. Du and P. Poláčik, *Locally uniform convergence to an equilibrium for nonlinear parabolic equations on* $\mathbb{R}^N$, Indiana Univ. Math. J. **64** (2015), 787–824.

[12] A. Ducrot, T. Giletti, and H. Matano, *Existence and convergence to a propagating terrace*, Trans. Amer. Math. Soc., **366** (2014), 5541–5566.

[13] J.-P. Eckmann and J. Rougemont, *Coarsening by Ginzburg-Landau dynamics*, Comm. Math. Phys. **199** (1998), no. 2, 441–470.

[14] E. Fašangová, *Asymptotic analysis for a nonlinear parabolic equation on* $\mathbb{R}$, Comment. Math. Univ. Carolinae **39** (1998), 525–544.

[15] E. Feireisl, *On the long time behavior of solutions to nonlinear diffusion equations on* $\mathbb{R}^N$, NoDEA Nonlinear Differential Equations Appl. **4** (1997), 43–60.

[16] E. Feireisl and H. Petzeltová, *Convergence to a ground state as a threshold phenomenon in nonlinear parabolic equations*, Differential Integral Equations **10** (1997), 181–196.

[17] E. Feireisl and P. Poláčik, *Structure of periodic solutions and asymptotic behavior for time-periodic reaction-diffusion equations on* $\mathbb{R}$, Adv. Differential Equations **5** (2000), 583–622.

[18] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335–361.

[19] G. Fusco and J.K. Hale, *Slow-motion manifolds, dormant instability, and singular perturbations*, J. Dyn. Differ. Equations **1** (1989), 75–94.

[20] J. Földes and P. Poláčik, *Convergence to a steady state for asymptotically autonomous semilinear heat equations on* $\mathbb{R}^N$, J. Differential Equations **251** (2011), 1903–1922.

[21] T. Gallay and S. Slijepčević, *Energy flow in extended gradient partial differential equations*, J. Dynam. Differential Equations **13** (2001), 757–789.

[22] ———, *Distribution of energy and convergence to equilibria in extended dissipative systems*, J. Dynam. Differential Equations, **27** (2015), 653–682.

[23] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18** (1978), 221–227.

[24] C. B. Muratov and X. Zhong, *Threshold phenomena for symmetric decreasing solutions of reaction-diffusion equations*, NoDEA Nonlinear Differential Equations Appl. **20** (2013), 1519–1552.

[25] P. Poláčik, *Threshold solutions and sharp transitions for nonautonomous parabolic equations on* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **199** (2011), 69–97, Addendum: www.math.umn.edu/∼polacik/Publications.

[26] ———, *Examples of bounded solutions with nonstationary limit profiles for semilinear heat equations on* $\mathbb{R}$, J. Evol. Equ., **15** (2015), 281-307.

[27] ———, *Threshold behavior and non-quasiconvergent solutions with localized initial data for bistable reaction-diffusion equations*, J. Dynam. Differential Equations, to appear.

[28] P. Poláčik, *Spatial trajectories and convergence to traveling fronts for bistable reaction-diffusion equations*, Contributions to nonlinear differential equations and systems, A tribute to Djairo Guedes de Figueiredo on the occasion of his 80th Birthday, Progress in Nonlinear Differential Equations and Their Applications, A.N. Carvalho et.al. (eds), Springer 2016, 405–423.

[29] T. I. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4** (1968), 17–22.

[30] A. Zlatoš, *Sharp transition between extinction and propagation of reaction*, J. Amer. Math. Soc. **19** (2006), 251–263.