On the multiplicity of nonnegative solutions with a nontrivial nodal set for elliptic equations on symmetric domains

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Abstract. We consider the Dirichlet problem for a class of fully nonlinear elliptic equations on a bounded domain Ω. We assume that Ω is symmetric about a hyperplane H and convex in the direction perpendicular to H. Each nonnegative solution of such a problem is symmetric about H and, if strictly positive, it is also decreasing in the direction orthogonal to H on each side of H. The latter is of course not true if the solution has a nontrivial nodal set. In this paper we prove that for a class of domains, including for example all domains which are convex (in all directions), there can be at most one nonnegative solution with a nontrivial nodal set. For general domains, there are at most finitely many such solutions.

1 Introduction

In this paper we continue our study of nonnegative solutions of nonlinear elliptic problems of the form

$$F(x, u, Du, D^2u) = 0, \quad x \in \Omega, \tag{1.1}$$

$$u = 0, \quad x \in \partial\Omega. \tag{1.2}$$

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Here Ω is a bounded domain in $\mathbb{R}^N$ which is convex in one direction and reflectionally symmetric about a hyperplane orthogonal to that direction. Without loss of generality, we assume that the direction is $e_1 := (1, 0, \dots, 0)$ (that is, Ω is convex in x_1) and the symmetry hyperplane is given by

$$H_0 = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 = 0\}.$$

The nonlinearity F is assumed to be sufficiently regular, elliptic, and invariant under the reflection in hyperplanes parallel to H_0 (see hypotheses (F1)–(F3) below).

It is well-known that each strictly positive solution u of (1.1), (1.2) is even in x_1:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega), \tag{1.3}$$

and decreasing with increasing $|x_1|$:

$$u_{x_1}(x_1, x') < 0 \quad ((x_1, x') \in \Omega,\ x_1 > 0). \tag{1.4}$$

For semilinear equations on smooth domains, this result was proved by Gidas, Ni, and Nirenberg [12]; the extension to the fully nonlinear equations on general symmetric domains is due to Berestycki and Nirenberg [2] (related results and a broader perspective can be found in the surveys [1, 16, 19, 20], monographs [8, 11, 26], or more recent paper [6], among others). It is also well-known that this result is not valid in general for nonnegative solutions; consider, for example, the solution $u(x) = 1+\cos x$ of the equation $u''+u-1 = 0$ on the interval $\Omega = (-3\pi, 3\pi)$. However, as proved in [22], all nonnegative solutions still enjoy the symmetry property (1.3). Of course, (1.4) necessarily fails if the solution has a nontrivial nodal set in Ω. As also shown in [22], the nodal set of each nonnegative solution u has interesting symmetry properties itself. In particular, each nodal domain of u is convex in x_1 and symmetric about a hyperplane parallel to H_0 (a nodal domain refers to a connected component of $\{x \in \Omega : u(x) \neq 0\}$). Examples of nonnegative solutions with a nontrivial nodal set can be found in [22, 25]. There are also numerous results on the nonexistence of such solutions under various additional conditions on F and Ω, see for example [4, 7, 8, 9, 11, 14, 21, 23].

In this paper, we are concerned with the multiplicity of solutions which have a nontrivial nodal set, in case such solutions do exist. To get a first insight, consider the one-dimensional problem

$$\begin{aligned} u_{xx} + f(u) = 0, \quad x \in (-\ell, \ell), \\ u(-\ell) = u(\ell) = 0, \end{aligned}$$

where $\ell > 0$ and f is a locally Lipschitz function. It is not difficult to see, by inspecting the phase-plane diagram for example, that if u is a nonnegative solution of this problem with interior zeros, then necessarily $u_x(-\ell) = u_x(\ell) = 0$. Thus u is a solution of the Cauchy problem $u(-\ell) = u_x(\ell) = 0$ for the second order ODE and as such it is uniquely determined.

One naturally wonders whether a similar uniqueness result may hold in higher space dimension. It turns out that it does, under some geometric conditions on Ω, for example, if Ω is convex (in all directions). And again, the uniqueness for the Cauchy problem plays a role here. This may look curious at the first glance, given that we are making no smoothness assumptions on Ω, so let us explain. Assume for now that Ω is convex. Let $u \not\equiv 0$ be a solution of (1.1), (1.2), which vanishes somewhere in Ω. We will prove that $\partial\Omega$ necessarily has a smooth portion S on which u vanishes together with ∇u. There are two reasons why this happens. The first one is that $\nabla u = 0$ on the nodal set of u in Ω (because $u \geq 0$). This and a reflectional symmetry property of u imply that ∇u vanishes on a portion $\tilde{S}$ of $\partial\Omega$ near its "rightmost" part. The second reason is that the function u_{x_1}, whose nodal set includes the nodal set of u in Ω, solves a linear equation. Thus, excluding a set of Hausdorff dimension not greater than $N-2$, the nodal set of u_{x_1} is smooth (of class $C^{1,1}$ in our setting). Using this and the symmetries of u, we find a smooth portion S of $\tilde{S}$. This way we show that any two solutions with a nontrivial nodal set in Ω vanish on a smooth portion of $\partial\Omega$ together with their gradients. The uniqueness for the Cauchy problem for elliptic equations then implies that any two such solutions coincide on a nonempty open subset. Consequently, by unique continuation, they coincide everywhere in Ω, hence the uniqueness.

The above arguments give the uniqueness if Ω is convex or if other geometric conditions are imposed. Without any additional conditions on Ω, we can prove that the number of solutions with a nontrivial nodal set is finite and can be estimated above by a quantity derived in an explicit way from geometric properties of Ω.

The finite multiplicity result is of some importance in studies of the parabolic problem

$$u_t = F(x, u, Du, D^2u), \quad x \in \Omega, \quad t > 0, \tag{1.5}$$

$$u = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.6}$$

for which solutions of (1.1), (1.2) are equilibria. Equilibria with a nontrivial nodal set play a distinguished role in the global dynamics of (1.5), (1.6), in particular when it comes to the asymptotic symmetry properties of positive solutions (more details on this will appear in [10]).

The multiplicity results have also some symmetry consequences for the solutions of (1.1), (1.2) themselves. For example, if both Ω and F are invariant under a continuous group of rotations, then each solution with a nontrivial nodal set must be symmetric with respect to that group (otherwise its group orbit yields infinitely many such solutions). In the case of convex domains, the same applies even to discrete symmetry groups.

The rest of the paper is organized as follows. In the next section we give precise statements of our main results. The proofs are given in Section 3.

2 Statement of the main results

We first introduce some notation and state our hypotheses. Recall that our standing assumption is that Ω is a bounded domain in $\mathbb{R}^N$, which is convex in e_1, the direction of the x_1-axis, and symmetric about the hyperplane H_0. Let $\mathcal{S}$ denote the space of $N \times N$ symmetric (real) matrices. We consider a function $F : (x, u, p, q) \mapsto F(x, u, p, q)$ defined on $\bar{\Omega} \times \mathcal{B}$, where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}$, which is invariant under the transformation Q defined by

$$\begin{aligned} &Q(u, p, q) = (u, -p_1, p_2, \dots, p_N, \bar{q}), \\ &\bar{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i,\ j \text{ equals } 1, \\ \ \ q_{ij} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1}$$

We assume that $F : \bar{\Omega} \times \mathcal{B} \to \mathbb{R}$ satisfies the following conditions

(F1) (Regularity) F is continuous in all variables and it is differentiable with bounded derivatives with respect to u, p, q. In particular, F is Lipschitz in (u, p, q): there is $\beta_0 > 0$ such that

$$\begin{aligned} |F(x, u, p, q) - F(x, \tilde{u}, \tilde{p}, \tilde{q})| &\le \beta_0 |(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})| \\ &((x, u, p, q), (x, \tilde{u}, \tilde{p}, \tilde{q}) \in \bar{\Omega} \times \mathcal{B}). \end{aligned} \tag{2.2}$$

Moreover, we assume that the derivatives $F_{q_{ij}}$, $i, j = 1, \dots, N$, are Lipschitz (in all variables) on $\Omega \times \mathcal{B}$.

(F2) (Ellipticity) There is a constant $\alpha_0 > 0$ such that

$$F_{q_{ij}}(x,u,p,q)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad ((x,u,p,q) \in \Omega \times \mathcal{B},\ \xi \in \mathbb{R}^N). \qquad (2.3)$$

Here and below we use the summation convention (summation over repeated indices). In the above formula, the left-hand side represents the sum over $i, j = 1, \dots, N$.

(F3) (Symmetry) F is independent of x_1 and for any $(x,u,p,q) \in \Omega \times \mathcal{B}$ one has

$$F(x, Q(u,p,q)) = F(x,u,p,q) \quad (= F((0,x'),u,p,q)).$$

We consider classical solutions u of (1.1), (1.2). By this we mean functions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$(u(x), Du(x), D^2u(x)) \in \mathcal{B} \quad (x \in \Omega)$$

and (1.1), (1.2) are satisfied everywhere. We shall require the following stronger regularity of the solutions:

(U) For $i, j = 1, \dots, N$, the derivatives $u_{x_i x_j}$ are locally Lipschitz continuous on Ω.

We remark that one can often establish the validity of (U) for each classical solution if additional assumptions are made on F and Ω. A sufficient condition is that F is differentiable (in all variables) on $\Omega \times \mathcal{B}$ and all its first order derivatives are locally Hölder continuous (see [13, Lemma 17.16]).

The assumptions we are making on the nonlinearity are the same as in [22], except that in (F1) we require, in addition to the Lipschitz continuity of F in (u,p,q), that its derivatives in (u,p,q) be defined everywhere (and not just almost everywhere). The reason is that we use the chain rule, which does not always hold for Lipschitz functions, unless additional structural conditions are assumed (for semilinear equations, for example, one can use different arguments and the Lipschitz continuity is sufficient, see [24, Section 5]). We need the Lipschitz continuity of the derivatives $F_{q_{ij}}$ and condition (U) in applications of the weak unique continuation theorem and the related uniqueness for the Cauchy problem for linearizations (1.1) (see Lemma 3.3 below).

Another difference, as compared to [22], is that in condition (U) the solutions are required to be locally Lipschitz, rather than Lipschitz. This is easier to verify (see the reference to [13, Lemma 17.16] above) and the results and proofs of [22] remain valid with no modifications under the weaker condition. In fact, local Lipschitz continuity is obviously sufficient for the weak unique continuation and in all applications of the uniqueness for the Cauchy problem in [22] the underlying hypersurface is contained in Ω, so again the local Lipschitz continuity is sufficient. In the present paper, we need to consider some hypersurfaces in $\partial\Omega$, thus it does make a difference that the solutions are not assumed Lipschitz continuous up to the boundary. Fortunately, we will be able to use symmetry to obtain the regularity up to the boundary whenever needed.

Let

$$\ell := \max\{x_1 : (x_1, x') \in \partial\Omega, \text{ for some } x' \in \mathbb{R}^{N-1}\},$$

so that $\{(x_1, x') \in \partial\Omega : x_1 = \ell\}$ is the "right-most" part of the boundary. For $\lambda \in [0, \ell)$, denote

$$\Sigma_\lambda := \{(x_1, x') \in \Omega : x_1 > \lambda\}. \tag{2.4}$$

Let E_{nod} be the set of all nonnegative solutions u of (1.1), (1.2), which satisfy (U) and for which $u^{-1}(0) \cap \Omega \neq \emptyset$. Note in particular that the function $u \equiv 0$, if it is a solution (that is, if $F(\cdot, 0, 0, 0) \equiv 0$), is included in E_{nod}.

Our main theorem can now be stated as follows.

Theorem 2.1. *Assume that* (F1)–(F3) *hold. Then the set E_{nod} is finite. If the set Σ^λ is connected for each $\lambda > 0$, then E_{nod} has at most one element.*

Note that Σ^λ is connected for each $\lambda > 0$ if Ω is convex (in all directions) or, more generally, if it is convex in all directions perpendicular to e_1.

In the next theorem, we make the multiplicity statement a little more precise, giving an estimate on the number of the solutions. Let $\mathcal{K}$ be the collection of all sets $D \subset \Omega$ such that D is a connected component of Σ_λ for some $\lambda > 0$. Given $\delta > 0$, let $m(\mathcal{K}, \delta)$ be the maximal integer m such that one can find m mutually disjoint sets from $\mathcal{K}$ each having measure at least δ. Note that $m(\mathcal{K}, \delta)$ is well defined and

$$m(\mathcal{K}, \delta) \leq \frac{|\Omega|}{2\delta}, \tag{2.5}$$

where $|\Omega|$ stands for the measure of Ω.

Theorem 2.2. *Assume that* (F1)–(F3) *hold. There exists* $\delta > 0$ *determined by* $N = \dim \Omega$ *and the constants* β_0, α_0 *from* (F1), (F2) *such that the set* E_{nod} *has at most* $m(\mathcal{K}, \delta)$ *elements.*

The proof of Theorem 2.2 is given in the next section. Theorem 2.1 follows from Theorem 2.2. Indeed, the only argument that perhaps needs to be made is that if Σ^λ is connected for each $\lambda > 0$, then $\mathcal{K} = \{\Sigma^\lambda : \lambda \in (0, \ell)\}$ and it is simply ordered by inclusion. Hence $m(\mathcal{K}, \delta) \leq 1$ for each $\delta > 0$.

We remark that we currently have no example of a problem (1.1), (1.2) with more than one solution in E_{nod}.

We finish this section with remarks concerning the global Lipschitz continuity assumptions on F, see (F1) (similar remarks apply to the ellipticity condition (F2), which we assume to hold globally). If it is known that all classical solutions are in $C^2(\bar{\Omega})$, as is often the case when F and Ω are sufficiently regular, then the uniqueness result in Theorem 2.1 in unaffected if the global Lipschitz continuity is relaxed to the local Lipschitz continuity. Indeed, in this case Theorem 2.1 gives the uniqueness for the solutions $u \in E_{nod}$ such that the range of (u, Du, D^2u) is contained in a given bounded set (just modify F outside that bounded set so it becomes globally Lipschitz). This, of course, implies the uniqueness globally. It is not so clear whether the finite multiplicity result remains valid as stated under the relaxed Lipschitz continuity assumptions. One can prove that the number of all $u \in E_{nod}$ such that the range of (u, Du, D^2u) is contained in a given bounded set is finite. However, as the constant δ in Theorem 2.2 depends on β_0, the theorem does not yield a uniform estimate on this number, unless $\sup_{\delta>0} m(\mathcal{K}, \delta) < \infty$.

3 Proof of Theorem 2.2

In the whole section we assume that (F1)–(F3) are satisfied. For any $\lambda \in [0, \ell)$, we set

$$\begin{aligned} H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Gamma_\lambda &:= H_\lambda \cap \Omega. \end{aligned} \tag{3.1}$$

Let P_λ stand for the reflection in the hyperplane H_λ. Note that since Ω is convex in x_1 and symmetric in the hyperplane H_0, $P_\lambda(\Sigma_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$. For a function u on $\bar{\Omega}$, we define $V_\lambda u$ by

$$V_\lambda u(x) := u(P_\lambda x) - u(x) = u(2\lambda - x_1, x') - u(x) \quad (x = (x_1, x') \in \bar{\Sigma}_\lambda).$$

We recall the following two results of [22].

Lemma 3.1. *Let u be a solution of* (1.1), (1.2) *and let $x^0 = (x_1^0, \dots, x_N^0) \in \Omega$ be such that $u(x^0) > 0 = u_{x_1}(x^0)$. Let G be the nodal domain of u containing x^0 and let $\lambda = x_1^0$. Then G is convex in x_1 and symmetric about the hyperplane H_λ, $V_\lambda u \equiv 0$ in G, and $u_{x_1} \neq 0$ everywhere in $G \setminus H_\lambda$.*

This result follows directly from Theorem 2.2 of [22]. This theorem states in particular that for each nodal domain G, there is λ such that the above conclusions hold. We only need to add to this that the assumption $u(x^0) > 0 = u_{x_1}(x^0)$ forces $x^0 \in H_\lambda$, which means that $\lambda = x_1^0$.

Lemma 3.2. *There exists $\delta_0 > 0$ determined by N and the constants β_0, α_0 from* (F1), (F2) *such that the following statement is valid. For each $u \in E_{nod} \setminus \{0\}$ there is $\lambda_1 \in (0, \ell)$ and a connected component D of Σ_{λ_1} such that $|D| \geq \delta_0$ and*

$$V_{\lambda_1} u \equiv 0 \text{ in } D. \tag{3.2}$$

For any solution satisfying (U), the existence of $\lambda_1 \in [0, \ell)$ such that such that (3.2) holds for some connected component D of Σ_{λ_1} is stated in [22, Theorem 2.2]; λ_1 is strictly positive if $u \in E_{nod} \setminus \{0\}$ (see [22, Remark 2.3]). The fact that $|D| \geq \delta_0$ for any such component is stated in formula (4.21) of [22]. Here δ_0 is a positive constant determined by N, β_0, and α_0 as follows (cp. Lemma 4.3 and Proposition 3.1 in [22]). Consider a linear equation of the form

$$a_{ij}(x) v_{x_i x_j} + b_i(x) v_{x_i} + c(x) v = 0, \quad x \in G, \tag{3.3}$$

where G is an open set in $\mathbb{R}^N$ and

(L1) a_{ij}, b_i, c are measurable functions on G satisfying

$$\begin{aligned} |a_{ij}(x)|,\ |b_i(x)|,\ |c(x)| &\leq \beta_0 \quad (i, j = 1, \dots, N, x \in G), \\ a_{ij}(x) \xi_i \xi_j &\geq \alpha_0 |\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in G). \end{aligned}$$

We say that the maximum principle holds for (3.3) if for any $v \in C(\bar{G})$ which is a solution of (3.3) on G, the relation $v \geq 0$ on ∂G implies $v \geq 0$ in $\bar{G}$. Here and below, by a solution of the linear problem (3.3) we mean a strong solution, that is, a function $v \in W^{2,N}_{loc}(G)$ such that (3.3) is satisfied almost everywhere in G. It is well known (see [2, 3]) that there is $\delta_0 = \delta_0(N, \beta_0, \alpha_0) > 0$ such that the maximum principle holds for any equation

(3.3) whose coefficients satisfy (L1), provided $|G| < \delta_0$. Lemma 3.2 is valid with this choice of δ_0. Note that for any solution u of (1.1), the function $v = V_\lambda u$ solves a linear equation (3.3) on $G = \Sigma_\lambda$ and $v \geq 0$ on ∂G. This explains why the constant δ determined by the linear equation is relevant for (1.1).

Below we shall deal with different linear equations linked to (1.1). Consider any solution $u \in E_{nod}$. By (U), the function $v = u_{x_1}$ is $C^{1,1}_{loc}(\Omega)$; hence it is in $W^{2,\infty}_{loc}(\Omega)$. Differentiating (1.1) with respect to x_1, we see that v solves equation (3.3) with

$$a_{ij}(x) = F_{q_{ij}}, \quad b_i(x) = F_{p_i} \quad (i, j = 1, \ldots, N), \quad c(x) = F_u, \tag{3.4}$$

where the derivatives of F are evaluated at $(x, u(x), Du(x), D^2u(x))$. Note that the use of the chain rule is justified by (F1), and $F_{x_1} \equiv 0$ by (F3).

We shall also consider a linear equation (3.3) satisfied by the difference $v = u - \tilde{u}$ of two solutions u, $\tilde{u}$ of (1.1). In this case the coefficients of (3.3) are given by

$$\begin{aligned} a_{ij}(x) &= \int_0^1 F_{q_{ij}}(x, z^\theta(x), Dz^\theta(x), D^2 z^\theta(x))\, d\theta, \\ b_i(x) &= \int_0^1 F_{p_i}(x, z^\theta(x), Dz^\theta(x), D^2 z^\theta(x))\, d\theta, \\ c(x) &= \int_0^1 F_u(x, z^\theta(x), Dz^\theta(x), D^2 z^\theta(x))\, d\theta, \end{aligned} \tag{3.5}$$

where $z^\theta = (1-\theta)\tilde{u} + \theta u$.

Observe that if both u and $\tilde{u}$ satisfy (U), then the functions in (3.4) and (3.5) satisfy (L1) on $G = \Omega$. Moreover, the following condition is satisfied, provided the second derivatives of u and $\tilde{u}$ are Lipschitz on G (by (U) this is the case if $\bar{G} \subset \Omega$)

(L2) The functions a_{ij}, $i, j = 1, \ldots, N$ are Lipschitz on G.

We shall use the following results concerning the linear equation (3.3).

Lemma 3.3. *Let G be a domain in $\mathbb{R}^N$. Assume that* (L1), (L2) *hold and let v be a solution of* (3.3).

(i) If $v \equiv 0$ in a nonempty open subset of G, then $v \equiv 0$ in G.

(ii) *Let S be a $C^{1,1}$ hypersurface of $\mathbb{R}^N$ such that for some nonempty open $U \subset \mathbb{R}^N$ one has $U \cap \partial G = U \cap S$. If v and ∇v extend continuously to $G \cup S$ and $v = |\nabla v| = 0$ on S, then $v \equiv 0$ in G.*

Statement (i) is a well known (weak) unique continuation theorem, (see [15, Theorem 17.2.6], for example; the statement also follows from Lemma 3.4 below). Statement (ii), the uniqueness for the Cauchy problem for elliptic equations, follows from (i). Indeed, the assumption implies that there is a point $y \in S$ and a small ball B centered at y such that $B \setminus S$ has two connected components B_1, B_2 with $B_1 \subset G$ (possibly also $B_2 \subset G$). Keeping v untouched in B_1 and defining or redefining it to be identically zero on B_2, also suitably extending the coefficients of (3.3) if needed, one obtains a solution on B vanishing of B_2. By (i), this solution has to be identical to 0 on $B_1 \subset G$ as well, hence, again by (i), v has to be identical to 0 on G (see [18, p. 60 and Section VI.40] for more details; note that since the leading coefficients are Lipschitz continuous, one can rewrite the equation in the divergence form and deal with weak solutions).

Below, $\dim_H$ stands for the usual Hausdorff dimension for subsets of $\mathbb{R}^N$ and $\mathcal{H}^{N-1}$ for the $N-1$ dimensional Hausdorff (outer) measure on $\mathbb{R}^N$. The following result is contained in [17, Theorem 2.1] (see also [5, Section 6] for more general results).

Lemma 3.4. *Let G be a domain in $\mathbb{R}^N$. Assume that* (L1), (L2) *hold and let v be a nontrivial solution of* (3.3). *Then* $\dim_H(\mathcal{Z}_1(v))) \leq N-1$ *and* $\dim_H(\mathcal{Z}_2(v)) \leq N-2$, *where*

$$\mathcal{Z}_1(v) := \{x \in G : v(x) = 0\},$$
$$\mathcal{Z}_2(v) := \{x \in G : v(x) = 0 = |\nabla v(x)|\}$$

are the nodal set and the singular nodal set of v, respectively.

We remark that the statements of Lemma 3.4 and 3.3(i) are local in the sense that the validity of the statements for each subdomain of G with closure contained G implies the validity of the statements for the whole domain G. Thus in applications of these results we do not have to worry about the coefficients a_{ij} being globally Lipschitz, as long as they are locally Lipschitz.

For $\lambda \in (0, \ell)$, let Π_λ denote the projection in the hyperplane H_λ along the x_1-axis. In terms of the reflection P_λ,

$$\Pi_\lambda := \frac{1}{2}(I + P_\lambda),$$

where I is the identity on $\mathbb{R}^N$. Note that by the x_1-convexity of Ω, one has

$$\Gamma_\lambda = \Pi_\lambda(\partial\Sigma_\lambda \setminus H_\lambda). \tag{3.6}$$

The following result is a crucial ingredient of the proof of Theorem 2.2.

Lemma 3.5. *Let $u \in E_{nod} \setminus \{0\}$ and let λ_1 and D be as in Lemma 3.2. Then there is a a dense subset M of $\Gamma_{\lambda_1} \cap \partial D$ such that for each $x \in M$ the following statement is valid. The set $(\Pi_{\lambda_1})^{-1}(x) \cap (\partial D \setminus H_{\lambda_1})$ consists of a single point y and there is a neighborhood U of y in $\mathbb{R}^N$ such that $S = U \cap \partial\Omega$ is a $C^{1,1}$ hypersurface.*

Proof. Set

$$\Upsilon := (\Pi_{\lambda_1})^{-1}(\Gamma_{\lambda_1}) \cap (\partial D \setminus H_{\lambda_1}). \tag{3.7}$$

This is a relatively open subset of $\partial\Omega$. In view of the x_1-convexity of Ω,

$$\Pi_{\lambda_1}(\Upsilon) = \Gamma_{\lambda_1} \cap \partial D \subset \Omega. \tag{3.8}$$

The x_1-convexity of Ω and the relation $\lambda_1 > 0$ further imply that

$$\tilde{\Upsilon} := P_{\lambda_1}(\Upsilon) \subset \Omega.$$

By the identity $V_{\lambda_1} u \equiv 0$ in D and the Dirichlet boundary condition, we have $u = 0$ on $\tilde{\Upsilon}$. As $u \geq 0$ in Ω, $\nabla u = 0$ on $\tilde{\Upsilon}$. Consider now the function $v := u_{x_1}$. Obviously,

$$\tilde{\Upsilon} \subset Z_1(v). \tag{3.9}$$

As noted above, v is a $C^{1,1}_{loc}$ solution of a linear equation (3.3) whose coefficients (3.4) satisfy (L1) on $G = \Omega$ and they satisfy (L2) on any domain with $\bar{G} \subset \Omega$. Of course, v is nontrivial as $u \in E_{nod} \setminus \{0\}$. By Lemma 3.4, $\dim_H(Z_2(v)) \leq N-2$. Since the Hausdorff dimension is not increased by Lipschitz maps, we also have $\dim_H(K_0) \leq N-2$, where

$$K_0 := \Pi_{\lambda_1}(Z_2(v)) \subset H_{\lambda_1}.$$

By the implicit function theorem, the set $\mathcal{N} := Z_1(v) \setminus Z_2(v)$ is a $C^{1,1}$ submanifold of $\mathbb{R}^N$ of dimension $N-1$. Let $K_1 \subset H_{\lambda_1}$ be the set of the critical values of the map $\Pi_{\lambda_1}\big|_{\mathcal{N}} : \mathcal{N} \to H_{\lambda_1}$ (here H_{λ_1} is viewed as a submanifold of $\mathbb{R}^N$ of dimension $N-1$). Note that $x \in H_{\lambda_1}$ is in K_1 if and only if $x = \Pi_{\lambda_1}(z)$ for some $z \in \mathcal{N}$ such that the vector $e_1 = (1, 0, \ldots, 0)$ is tangent to $\mathcal{N}$ at z.

By Sard's theorem, $\mathcal{H}^{N-1}(K_1) = 0$, hence also $\mathcal{H}^{N-1}(K) = 0$ for $K := K_0 \cup K_1$. Set

$$M := \Gamma_{\lambda_1} \cap \partial D \setminus K.$$

This is clearly a dense subset of $\Gamma_{\lambda_1} \cap \partial D$. We verify that the conclusion of Lemma 3.5 holds for this set M.

For this aim, take any $x \in M$. Note first of all that (3.8) implies that the set $(\Pi_{\lambda_1})^{-1}(x) \cap \Upsilon$ is nonempty hence also

$$(\Pi_{\lambda_1})^{-1}(x) \cap \tilde{\Upsilon} = P_{\lambda_1}\left((\Pi_{\lambda_1})^{-1}(x) \cap \Upsilon\right) \neq \emptyset.$$

Next, if this set contained two different points, then, by the x_1-convexity of Ω, it would also contain a line segment J parallel to e_1. Then J would be contained in $\mathcal{N}$ (as $\tilde{\Upsilon} \subset Z_1(v)$ and $x \notin K_0$) and e_1 would be tangent to $\mathcal{N}$ at the points of J. This is impossible as $x \notin K_1$. Thus there is z such that

$$(\Pi_{\lambda_1})^{-1}(x) \cap \tilde{\Upsilon} = \{z\}, \tag{3.10}$$

hence $(\Pi_{\lambda_1})^{-1}(x) \cap \Upsilon = \{y\}$ with $y := P_{\lambda_1}(z)$.

We next claim that for some neighborhood $\tilde{U}$ of z, one has $\tilde{\Upsilon} \cap \tilde{U} = \mathcal{N} \cap \tilde{U}$. This implies that near y the set Υ coincides with the $C^{1,1}$ manifold $P_{\lambda_1}(\mathcal{N})$, completing the proof of Lemma 3.5.

To prove the claim, we first use the fact that $e_1 \notin T_z\mathcal{N}$ to find a neighborhood U_0 of z such that the map

$$\Pi_{\lambda_1}\Big|_{U_0 \cap \mathcal{N}} \tag{3.11}$$

is a diffeomorphism of $U_0 \cap \mathcal{N}$ onto its image $\Pi_{\lambda_1}(U_0 \cap \mathcal{N})$, which is an open subset of $\Gamma_{\lambda_1} \cap \partial D$. Next observe that near z the manifold $\mathcal{N}$ coincides with $Z_1(v)$ (because $\nabla v(\xi) \neq 0$ for $\xi = z$, hence for $\xi \approx z$). Therefore, by (3.9), $\tilde{\Upsilon} \subset \mathcal{N}$ near z. Assume now that our claim is not true. Then there is a sequence $z^n \in \mathcal{N}$ converging to z such that $z^n \notin \tilde{\Upsilon}$. Since the set $\Pi_{\lambda_1}(\tilde{\Upsilon}) = \Pi_{\lambda_1}(\Upsilon)$ is a relative neighborhood of $x = \Pi_{\lambda_1}(z)$ in Γ_{λ_1}, for all large n there are points $\tilde{z}^n \in \tilde{\Upsilon}$ with $\Pi_{\lambda_1}(\tilde{z}^n) = \Pi_{\lambda_1}(z^n)$. Passing to a subsequence, we may assume $\tilde{z}^n \to \tilde{z}$ for some z with $\Pi_{\lambda_1}(\tilde{z}) = \Pi_{\lambda_1}(z)$. One verifies easily (using the closedness of $\partial\Omega$) that $\tilde{z} \in \tilde{\Upsilon}$. Hence, by (3.10), $\tilde{z} = z$. This implies that for large n, we have $z_n \in U_0$, contradicting the injectivity of the map (3.11). $\square$

Lemma 3.6. *Let $u \in E_{nod} \setminus \{0\}$ and let λ_1 and D be as in Lemma 3.2. If y is a point in $\partial D \setminus H_{\lambda_1}$ with $\Pi_{\lambda_1}(y) \in \Omega$, then there is a neighborhood $U \subset \mathbb{R}^N$ of y such that ∇u and $D^2 u$ extend to Lipschitz functions on $U \cap \bar{\Omega}$ with $\nabla u = 0$ on $\partial\Omega \cup U$.*

Proof. Define Υ by (3.7). As in the proof of Lemma 3.5, $\nabla u = 0$ on $\tilde{\Upsilon} := P_{\lambda_1}(\Upsilon) \subset \Omega$. Now the conclusion of Lemma 3.6 follows easily from the facts that u is of class $C^{2,1}_{loc}$ in Ω and $V_{\lambda_1} u \equiv 0$ in D. □

Lemma 3.7. *Let $u, \tilde{u} \in E_{nod} \setminus \{0\}$, and let λ_1, $\tilde{\lambda}_1$ and D, $\tilde{D}$ be the corresponding numbers in $(0, \ell)$ and subdomains of Ω as in Lemma 3.2 (D, $\tilde{D}$ are connected components of Σ_{λ_1}, $\Sigma_{\tilde{\lambda}_1}$, respectively). If $u \not\equiv \tilde{u}$, then $D \cap \tilde{D} = \emptyset$.*

Proof. Without loss of generality we may assume that $\tilde{\lambda}_1 \leq \lambda_1$. Then $\Sigma_{\lambda_1} \subset \Sigma_{\tilde{\lambda}_1}$, hence either $D \cap \tilde{D} = \emptyset$ or $D \subset \tilde{D}$. We show that the latter implies that $\tilde{u} \equiv u$, which proves the conclusion of Lemma 3.7.

Thus assume $D \subset \tilde{D}$. Then clearly

$$\partial D \setminus H_{\lambda_1} \subset \partial \tilde{D} \setminus H_{\tilde{\lambda}_1}.$$

By Lemma 3.5 there is a point $y \in \partial D \setminus H_{\lambda_1}$ such that $\Pi_{\lambda_1} y \in \Gamma_{\lambda_1}$ and for some neighborhood U of y, the set $S = U \cap \partial\Omega$ is a $C^{1,1}$ hypersurface. By the x_1-convexity, $\Pi_{\tilde{\lambda}_1} y \in \Gamma_{\tilde{\lambda}_1}$. Therefore, by Lemma 3.6, making U smaller if necessary, we have

$$\nabla u = \nabla \tilde{u} = 0 \text{ on S}$$

and the functions $\tilde{u}$, u are $C^{2,1}$ on $\bar{U} \cap \bar{\Omega}$. Also $u = \tilde{u} = 0$, by the Dirichlet boundary condition.

Consider now the function $v = u - \tilde{u}$, which solves of a linear equation (3.3) on Ω with coefficients 3.5. The coefficients satisfy (L1) on $G = \Omega$ and (L2) on any domain G where $\tilde{u}$, u are globally of class $C^{2,1}$. This applies to any domain G with $\bar{G} \subset \Omega$ and also, as shown above, to $G = U$. Applying, Lemma 3.3 to v, we obtain $v \equiv 0$, first in U, then in Ω. Hence $u \equiv \tilde{u}$, as desired.

□

Completion of the proof of Theorem 2.2. First we note that if $u \equiv 0$ is a solution of (1.1), (1.2), then $E_{nod} = \{0\}$. Indeed, if $u \in E_{nod}$, then $u = u - 0$ can itself be viewed as a solution of a linear equation satisfying (L1), (L2). Applying Lemma 3.3 as in the previous proof we obtain $u \equiv 0$ (alternatively, one can use a version of the strong maximum principle). In this case Theorem 2.2 holds trivially. We continue assuming that $u \equiv 0$ is not a solution, hence $E_{nod} = E_{nod} \setminus \{0\}$.

Let δ_0 be as in Lemma 3.2. By the same lemma, to each $u \in E_{nod}$, we can associate an element $D = D(u)$ of $\mathcal{K}$ with measure at least δ_0. By Lemma 3.7 the sets $D(u)$, $u \in E_{nod}$, are mutually disjoint. This implies that E_{nod} has at most $m(\mathcal{K}, \delta_0)$ elements. The proof is complete. $\square$

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