Asymptotic behavior of threshold and sub-threshold solutions of a semilinear heat equation

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Abstract. We study asymptotic behavior of global positive solutions of the Cauchy problem for the semilinear parabolic equation $u_t = \Delta u + u^p$ in $\mathbb{R}^N$, where $p > 1 + 2/N$, $p(N-2) \le N+2$. The initial data are of the form $u(x,0) = \alpha\phi(x)$, where ϕ is a fixed function with suitable decay at $|x| = \infty$ and $\alpha > 0$ is a parameter. There exists a threshold parameter α^* such that the solution exists globally if and only if $\alpha \le \alpha^*$. Our main results describe the asymptotic behavior of the solutions for $\alpha \in (0, \alpha^*]$ and in particular exhibit the difference between the behavior of sub-threshold solutions ($\alpha < \alpha^*$) and the threshold solution ($\alpha = \alpha^*$).

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1 Introduction

Let $u_0 \in L^\infty(\mathbb{R}^N)$ be nonnegative, $p > 1$, and $u(x,t) = u(x,t;u_0)$ be the (unique) solution of the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + u^p, && x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

satisfying $u(\cdot,t) \in L^\infty(\mathbb{R}^N)$ for $t > 0$. Let $T_{\max}(u_0) \le \infty$ denote the maximal existence time of this solution. It is well known (see [20] for example) that $T_{\max}(u_0) < \infty$ whenever $u_0 \not\equiv 0$ and $p \le p_F$, where $p_F := 1+2/N$ is the Fujita exponent. Since we are interested in global positive solutions, throughout the paper we will assume $p > p_F$.

Taking $u_0 = \alpha\phi$, where $\phi \in L^\infty(\mathbb{R}^N) \setminus \{0\}$ is a fixed nonnegative function and $\alpha \in \mathbb{R}^+$ is a parameter, we denote the solution of (1.1) by $u_\alpha(x,t)$ or $u_\alpha(x,t;\phi)$ and set

$$\alpha^* = \alpha^*(\phi) := \sup\{\alpha > 0 : T_{\max}(\alpha\phi) = \infty\}.$$

If $\liminf_{|x|\to\infty} \phi(x)|x|^{2/(p-1)} = \infty$ then $\alpha^* = 0$, hence all solutions u_α, $\alpha > 0$, blow up in finite time, see [20, Theorem 17.12]. On the other hand, if ϕ satisfies the growth assumption

$$\limsup_{|x|\to\infty} \phi(x)|x|^{2/(p-1)} < \infty \tag{1.2}$$

then $\alpha^* \in (0,\infty)$. Indeed, the inequality $\alpha^* > 0$ follows from a result of [11] (see also [20, Theorem 20.6]) stating in particular that if $\alpha > 0$ is small enough, then $u_\alpha(\cdot,t)$ is global and decays to 0 as $t \to \infty$. To show that $\alpha^* < \infty$ one uses a standard Kaplan-type blow-up estimate [20, Theorem 17.1]. Thus in this case the solution u_{α^*} lies on the threshold between global existence and blow-up. We therefore call the solution u_{α^*} the *threshold solution* and the solutions u_α, $\alpha \in (0,\alpha^*)$, *sub-threshold solutions*.

It is well known (see [6, 20]) that the threshold solution can blow up in finite time if $p > p_S$, where

$$p_S := \begin{cases} \dfrac{N+2}{N-2}, & \text{if } N \ge 3, \\ \infty, & \text{if } N \in \{1,2\}, \end{cases}$$

is the Sobolev exponent. For example, u_{α^*} blows up is $\phi \not\equiv 0$ is a nonnegative, smooth, radial, and radially nonincreasing function with compact support (see [13, 14] or [20, Theorem 28.7]). On the other hand, if $p \leq p_S$, it is likely that the threshold solutions are global. Several sufficient conditions guaranteeing the global existence of the threshold solution can be found in [20] and references therein. In this paper we consider the case where the threshold solution exists globally (in particular we assume $p \leq p_S$). Our main goal is to examine the asymptotic behavior of the threshold and sub-threshold solutions and reveal the differences in these behaviors.

The difference between the time decay of the threshold and sub-threshold solutions was observed a long time ago by Kavian [8] and Kawanago [9] in the subcritical case $p < p_S$ provided ϕ has an exponential decay. Several other related results can be found in [19, 20] and references therein. In the recent work [19], the second author of this paper examined the threshold and sub-threshold solutions assuming ϕ was a radial function. His results concerning $p \leq p_S$ are as follows. (Here and in what follows, $\|\cdot\|_\infty$ always denotes the norm in $L^\infty(\mathbb{R}^N)$.)

Theorem 1.1. *Assume $p \in (p_F, p_S]$ and let ϕ be continuous and radially symmetric,*

$$\lim_{|x|\to\infty} \phi(x)|x|^{2/(p-1)} = 0. \tag{1.3}$$

(i) Let $p < p_S$. Then u_{α^*} is global and there exists a positive constant c such that

$$c^{-1} \leq \|u_{\alpha^*}(\cdot,t)\|_\infty t^{1/(p-1)} \leq c, \qquad (t > 1). \tag{1.4}$$

If $0 < \alpha < \alpha^*$ then

$$\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^{1/(p-1)} = 0. \tag{1.5}$$

(ii) Let $p = p_S$. Then the solution u_{α^*} is global and

$$\limsup_{t\to\infty} \|u_{\alpha^*}(\cdot,t)\|_\infty t^{1/(p-1)} = \infty. \tag{1.6}$$

If $\alpha \in (0, \alpha^*)$ and $\|u_\alpha(\cdot,t)\|_\infty \leq Ct^{-1/(p-1)}$ for all $t > 1$ then (1.5) *is true.*

The first part of this theorem clearly indicates the difference between the decay of the threshold and sub-threshold solutions in the subcritical case provided the initial data are radially symmetric. On the other hand, the second part of the theorem is not satisfactory because, first, it is not clear whether in (1.6) the superior limit is actually the limit or not and, second, an extra boundedness assumption on the sub-threshold solutions is needed to guarantee (1.5). The main purpose of this paper is to extend Theorem 1.1(i) to non-radial solutions and, to some extent, remove the drawbacks of Theorem 1.1(ii).

In order to formulate our main results, we need additional notation and hypotheses. Set

$$\mathcal{E} := \{w \in L^{p+1}(\mathbb{R}^N) : \nabla w \in L^2(\mathbb{R}^N)\}.$$

If ϕ is radially symmetric: $\phi(x) = \Phi(|x|)$ for some $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, we say that ϕ has only finitely many local minima if the function Φ has only finitely many local minima. The next theorem is an analogue to Theorem 1.1(i) in the critical case.

Theorem 1.2. *Let $p = p_S$. Assume that $\phi \in \mathcal{E}$ is nonnegative, nontrivial, continuous, radially symmetric, has only finitely many local minima, and satisfies* (1.3). *Then the threshold solution $u_{\alpha^*}(\cdot, t)$ is global and*

$$\lim_{t\to\infty} \|u_{\alpha^*}(\cdot, t)\|_\infty t^{1/(p-1)} = \infty. \tag{1.7}$$

If $\alpha \in (0, \alpha^*)$, then

$$\lim_{t\to\infty} \|u_{\alpha}(\cdot, t)\|_\infty t^{1/(p-1)} = 0. \tag{1.8}$$

We remark that the requirement $\phi \in \mathcal{E}$ is met if, for example, $\phi \in C^1$, $\lim_{|x|\to\infty} \phi(x)|x|^{2\gamma} = 0$, and $\lim_{|x|\to\infty} |\nabla\phi|(x)|x|^{2\gamma+1} = 0$ for some $\gamma > 1/(p-1)$.

If the hypotheses of Theorem 1.2 are satisfied and (1.3) is strengthened to $\lim_{|x|\to\infty} \phi(x)|x|^{2\gamma} = 0$ for some $\gamma > 1/(p-1)$, then (1.8) can be complemented with the following information. For any $\alpha \in (0, \alpha^*)$, u_α behaves like the solution of the linear heat equation: there exists $C > 0$ such that

$$e^{t\Delta}(\alpha\phi) \le u_\alpha(\cdot, t) \le C e^{t\Delta}(\alpha\phi) \quad (t > 1),$$

where $e^{t\Delta}(\alpha\phi)$ denotes the solution of the linear heat equation with initial data $\alpha\phi$ (see [4, 19] or [20, Remark 28.11(ii)]).

For the general class of initial data considered here, it is probably not possible to give a more accurate statement than (1.7) on the asymptotic behavior of the threshold solution. Indeed, if $N = 3$ and $\phi(x)$ behaves like

$|x|^{-\gamma}$ for $|x|$ large, $\gamma > 1/(p-1)$, then formal matched asymptotics expansions [10] suggest that for t large, $\|u_{\alpha^*}(\cdot,t)\|_\infty$ behaves like $t^{(\gamma-1)/2}$ or $t^{1/2}$ provided $\gamma \in (1/2, 2)$ or $\gamma > 2$, respectively. (Notice that $1/(p-1) = 1/4$ in this case and that there exist stationary threshold solutions with spatial decay like $|x|^{-1}$.)

In order to formulate a generalization of Theorem 1.1(i) to the non-radial case, we will need the following assumption.

(LT) There are no bounded positive entire solutions of

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

While (LT) is likely to be true in the whole subcritical range $1 < p < p_S$, so far the proof is only available for $p < p_{BV}$ with

$$p_{BV} = p_{BV}(N) := \begin{cases} \dfrac{N(N+2)}{(N-1)^2}, & \text{if } N \geq 2, \\ \infty, & \text{if } N = 1, \end{cases}$$

see [1, 20]. It is also known that (LT) is true if the class of solutions considered is restricted to the radial ones (and $1 < p < p_S$), see [17]. The following universal a priori bound was derived from (LT) in [18] (see also [20]). If u is a global positive solution of

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.9}$$

then, assuming either that u is radial (and $1 < p < p_S$) or that (LT) holds, we have $|u(x,t)| \leq Ct^{-1/(p-1)}$ for all $x \in \mathbb{R}^N$ and $t > 0$, where $C > 0$ is a constant depending only on N and p.

Theorem 1.3. *Let $p_F < p < p_S$. Assume that $\phi \in C(\mathbb{R}^N)$ is nonnegative, $\phi \not\equiv 0$, and* (1.3) *holds. Assume also that* (LT) *holds or that ϕ is a radial function. Then the following statements hold true:*

(i) u_{α^*} is global and there is a positive constant c such that

$$c^{-1} \leq \|u_{\alpha^*}(\cdot,t)\|_\infty t^{1/(p-1)} \leq c \quad (t > 0). \tag{1.10}$$

(ii) If $\alpha \in (0, \alpha^*)$, then $\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^{1/(p-1)} = 0$.

Observe that $\phi \in \mathcal{E}$ is not assumed in this theorem. Statement (i) can be made more precise using similarity variables. If $u = u(x,t)$ is a solution of (1.1) and

$$v(y,s) := e^{\beta s} u(e^{s/2} y, e^s - 1), \qquad y \in \mathbb{R}^N,\ s \geq 0, \tag{1.11}$$

then v solves the equation

$$v_s = \Delta v + \frac{y}{2} \cdot \nabla v + \beta v + v^p, \tag{1.12}$$

where

$$\beta := \frac{1}{p-1}.$$

Denote by v_α the solution of (1.12) corresponding to u_α. In Lemma 4.4 we will show that, as $s \to \infty$, $v_{\alpha^*}(\cdot, s)$ converges in $L^\infty(\mathbb{R}^N)$ to a (bounded) positive steady state of (1.12).

Similarly as in the case of Theorem 1.2, statement (ii) of Theorem 1.3 guarantees the linear behavior of sub-threshold solutions provided

$$\lim_{|x|\to\infty} \phi(x)|x|^{2\gamma} = 0 \quad \text{for some } \gamma > 1/(p-1).$$

The remainder of the paper is organized as follows. The proofs of our main theorems, Theorems 1.2 and 1.3, are given in Sections 3 and 4, respectively. At several places in the paper, when dealing with radial solutions, we use properties of the zero number functional, or intersection comparison arguments. We recall these properties in Section 2. Also we collect in that section basic properties of radial stationary solutions of (1.12). In the appendix we prove a symmetry theorem on positive ancient solutions of (1.12) which extends a theorem of [15] concerning steady states. This extension is crucial for the proof of Theorem 1.3.

Let us close this introductory part by making a few notational conventions. Sometimes, when there is no danger of confusion, we suppress the spatial argument and write $u(t)$ for $u(\cdot, t)$. We denote by $\|\cdot\|_q$ the norm in $L^q(\mathbb{R}^N)$ and by B_ρ the open ball in $\mathbb{R}^N$ of radius ρ centered at the origin.

2 Preliminaries: Radial solutions and zero number

Radially symmetric solutions of (1.12) have been well understood, thanks to contribution of several people. In the following proposition due to [7, 23, 22, 3] we summarize the basic results in the subcritical and critical cases.

Proposition 2.1. *Let $p > 1$, $\lambda \geq 0$ and let $w_\lambda = w_\lambda(\rho)$ be the solution of the problem*

$$w'' + \frac{N-1}{\rho} w' + \frac{\rho}{2} w' + \beta w + |w|^{p-1} w = 0 \quad \textit{for } \rho > 0, \quad w(0) = \lambda, \quad w'(0) = 0.$$

Then w_λ is defined for all $\rho > 0$ and there exists finite $\lim_{\rho \to \infty} w_\lambda(\rho)\rho^{2/(p-1)} =: A(\lambda)$. Given $\lambda > 0$, set $\rho_\lambda := \sup\{\rho > 0 : w_\lambda > 0$ on $[0, \rho)\}$. Then the following is true:

(i) If $p_F < p < p_S$ then there exists $\lambda_0 \in (0, \infty)$ with the following property: $\rho_\lambda < \infty$ if and only if $\lambda > \lambda_0$. In addition, $A(\lambda) > 0$ for $\lambda \in (0, \lambda_0)$ and $A(\lambda_0) = 0$.

(ii) If $p = p_S$ then $\rho_\lambda = \infty$ and $A(\lambda) > 0$ for all $\lambda > 0$.

Let us now consider the solution $u_\alpha(x, t) = u(x, t; \alpha\phi)$ assuming that ϕ is radially symmetric. By the uniqueness for the Cauchy problem (1.1), u_α is also radially symmetric in x and the corresponding solution v_α of (1.12) is radially symmetric in y. In this case, when no confusion is likely, we often view ϕ and $u_\alpha(\cdot, t)$ as functions of $r = |x| \in [0, \infty)$. Similarly, the radial functions $v_\alpha(\cdot, s)$ and w_λ will be considered as functions of $y \in \mathbb{R}^N$ or $\rho = |y| \in [0, \infty)$, depending on the context.

The following intersection comparison principle is used at several places below. Assume v is a (classical) radially symmetric solution of (1.12) on some time interval $(0, S)$, $0 < S \leq \infty$, and let $w = w_\lambda$ for some $\lambda > 0$. Given $0 < \rho_0 \leq \infty$, we define the *zero number* $z_{[0,\rho_0)}(v(\cdot, s) - w)$ of $v(\cdot, s) - w$ to be the number of zeros of the function $v_\alpha(\cdot, s) - w_\lambda$ in $[0, \rho_0)$. The following proposition follows from [2] and the fact that the difference $v(\cdot, s) - w$ is a radial solution of a linear parabolic equation on $\mathbb{R}^N \times (0, S)$. The term $y \cdot \nabla v$, not included in [2], is harmless for the application of [2], as shown in [20, Remark 52.29(i)].

Proposition 2.2. *Under the above assumptions and notation, assume also that either $\rho_0 < \infty$ and $v(\rho_0, s) - w(\rho_0) \neq 0$ for all $s \in (0, S)$ or $\rho_0 = \infty$ and there is $s_0 \in (0, S)$ such that $z_{[0,\rho_0)}(v(\cdot, s) - w) < \infty$ for all $s \in (0, s_0)$. Then the following statements hold true:*

(i) $z_{[0,\rho_0)}(v(\cdot, s) - w) < \infty \quad (s \in (0, S))$;

(ii) $s \mapsto z_{[0,\rho_0)}(v(\cdot,s)-w)$ *is a monotone nonincreasing function;*

(iii) *if for some* $s_1 \in (0,S)$ *the function* $v(\cdot,s_1)-w$ *has a multiple zero in* $[0,\rho_0)$*, then* $s \mapsto z_{[0,\rho_0)}(v(\cdot,s)-w)$ *is discontinuous at* $s=s_1$ *(hence, by (ii), it drops at* $s=s_1$*).*

Note that, by (i) and (ii), given any $\tilde{s} \in (0,S)$ the dropping as in (iii) can occur for only finitely many values of $s_1 \in [\tilde{s},S)$. It does occur whenever $v(0,s_1)-w(0)=0$ because the relation $v_\rho(0,s_1)-w_\rho(0)=0$ is automatically satisfied by the radially symmetry.

3 Radial solutions in the critical case

Set

$$E(w) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(x)|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |w(x)|^{p+1}\,dx, \qquad w \in \mathcal{E}. \tag{3.1}$$

Recall from [20, Example 51.28] (see the case $\lambda=0$ considered there) that (1.1) is well posed in $\mathcal{E}$ if $p \le p_S$ and, given $u_0 \in \mathcal{E}$, the energy function $E_{u_0}(t) := E(u(\cdot,t))$ satisfies $E_{u_0} \in C^1((0,T_{\max}(u_0))) \cap C([0,T_{\max}(u_0)))$. In addition, the regularity properties of u proved in [20, Example 51.28] guarantee that the following energy identity is true for $0 \le t_1 < t_2 < T_{\max}(u_0)$:

$$E_{u_0}(t_2) - E_{u_0}(t_1) = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} u_t^2(x,t)\,dx\,dt. \tag{3.2}$$

Throughout this section we assume that $p=p_S$ and $\phi \in L^\infty(\mathbb{R}^N)\setminus\{0\}$ is a fixed radial nonnegative function (which we mostly view as a function of $r=|x|$ in this section). It is well known that the threshold solution u_{α^*} exists globally in this case. In fact, assume to the contrary that $T_{\max}(\alpha^*\phi) < \infty$. Then [6, Theorem 5.1] guarantees that u_{α^*} blows up completely at $T_{\max}(\alpha^*\phi)$. This means in particular that choosing $t_1 > T_{\max}(\alpha^*\phi)$ and a sequence $\alpha_k \nearrow \alpha^*$ we obtain $u_{\alpha_k}(x,t_1) \nearrow \infty$ as $k \to \infty$ for all $x \in \mathbb{R}^N$. Consequently,

$$\int_{\mathbb{R}^N} u_{\alpha_k}(x,t_1)e^{-|x|^2}\,dx > (2N)^{1/(p-1)}\pi^{N/2}$$

for k large enough so that $T_{\max}(\alpha_k\phi) < \infty$ by [20, Theorem 17.1(ii)], which yields a contradiction.

We start with the following lemma (which is true for any $p \le p_S$).

Lemma 3.1. *Let $\phi \in \mathcal{E}$ and $0 < \alpha \le \alpha^*$. Then $E(u_\alpha(\cdot,t)) \ge 0$ for all $t \ge 0$. Consequently,*

$$\int_0^\infty \int_{\mathbb{R}^N} (\partial_t u_\alpha(x,t))^2 \, dx \, dt < \infty.$$

Proof. Assume to the contrary that $E(u_\alpha(\cdot,t_0)) < 0$ for some $t_0 \ge 0$. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function satisfying $\varphi(r) = 1$ for $r \le 1$ and $\varphi(r) = 0$ for $r \ge 2$, and let $\varphi_k(r) := \varphi(r-k)$ for $k = 0,1,2,\dots$. Set $u_{0,k}(x) := u_\alpha(x,t_0)\varphi_k(|x|)$, $x \in \mathbb{R}^N$. Then $u_{0,k} \in H^1(\mathbb{R}^N)$ and $E(u_{0,k}) < 0$ for k sufficiently large, hence the solution $u^{(k)}$ of (1.1) with initial data $u_{0,k}$ blows up in finite time by [20, Theorem 17.6]. Since $u_\alpha(\cdot,t+t_0) \ge u^{(k)}(\cdot,t) \ge 0$ by the comparison principle, we obtain a contradiction with the global existence of u_α. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By (1.3), we have $\alpha^* \in (0,\infty)$ (see (1.2) and the paragraph following it). Fix $\alpha \in (0,\alpha^*]$. By [16], there exists $t_d \ge 0$ such that $u_\alpha(\cdot,t)$ is radially decreasing for all $t \ge t_d$. Without loss of generality we may assume $t_d = 0$.

Let v_α be the rescaled solution defined by (1.11) with u replaced by u_α. Then v_α satisfies (1.12) and $v_\alpha(\cdot,s)$ is radially decreasing for any $s \ge 0$, in particular $\|v(\cdot,s)\|_\infty = v_\alpha(0,s)$. In what follows we will consider $v_\alpha(\cdot,s)$ as a function of $\rho = |y| \in [0,\infty)$. For any $\lambda > 0$ let $z_\lambda(s) := z_{[0,\infty)}(v_\alpha(\cdot,s) - w_\lambda)$ denote the zero number of the function $v_\alpha(\cdot,s) - w_\lambda$ in $[0,\infty)$.

We next prove that there exists $\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^\beta$ or, equivalently, there exists $\lim_{s\to\infty} v_\alpha(0,s)$. Indeed, if not then there exists

$$\lambda \in (\liminf_{s\to\infty} v_\alpha(0,s), \limsup_{s\to\infty} v_\alpha(0,s)).$$

Fix such λ and recall that $A(\lambda) := \lim_{\rho\to\infty} w_\lambda(\rho)\rho^{2\beta} > 0$ (see Proposition 2.1). Also fix constants $S \in (0,\infty)$ and $a \in (0, A(\lambda))$ and choose $C_S > 0$ such that $v_\alpha(\rho,s) \le C_S$ for all $\rho \ge 0$ and $s \in [0,S]$. Proceeding as in the proof of [19, Theorem 4.1], one can find $R_2 > R_1 > 0$ (depending on a, S, λ) such that $v_\alpha(\rho,s) \le a(\rho - R_1)^{-2\beta} < w_\lambda(\rho)$ for all $\rho \ge R_2$ and $s \in [0,S]$. Consequently, $z_\lambda(s) = z_{[0,R_2)}(v_\alpha(\cdot,s) - w_\lambda)$ for $s \in [0,S]$, hence, by Proposition 2.2, $z_\lambda(s)$ is finite for $s > 0$, nonincreasing in s and drops whenever $v_\alpha(0,s) = \lambda$. But by the choice of λ the latter occurs for infinitely many s and we have a contradiction.

Once the existence of a limit $\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^\beta$ has been proved, (1.7) follows from Theorem 1.1(ii) and (1.8) will follow from that theorem if we prove that $\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^\beta \neq \infty$.

Assume to the contrary that $0 < \alpha < \alpha^*$ and

$$\lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty t^\beta = \infty. \tag{3.3}$$

Then there exists $t_0 > 0$ such that $u_\alpha(0,t)t^\beta > 8^\beta$ for all $t \geq t_0$. Fix $\tau \geq t_0$ and set $\eta(t) := 2^{-p} u_\alpha(0,t)^{1-p}$. The differential inequality $\partial_t u_\alpha(0,t) \leq u_\alpha(0,t)^p$ guarantees that

$$u_\alpha(0,t+\tilde{t}) \leq 2u_\alpha(0,t) \quad \text{for all } \tilde{t} \in [0,\eta(t)]. \tag{3.4}$$

In addition, we infer from [21, Lemma 5.2] (used with $\delta := \tau$ and $M(t) := u_\alpha(0,\tau+t)$) that there exist constants $\theta,\mu \in (0,1)$ (independent of τ) with the following property: the measure of the set

$$H(\tau) := \{t \in (\tau, 2\tau - \tau/8) : (\exists T \in [t+\theta\eta(t), t+\eta(t)])\ u_\alpha(0,T) \geq \mu u_\alpha(0,t)\}$$

is greater than or equal to $\tau/8$.

Set $\tau_k := 2^k t_0$, $k = 0,1,2,\ldots$, and $H := \bigcup_k H(\tau_k)$. We claim that there exist $t_k \in H$, $k = 1,2,\ldots$, such that $t_k \to \infty$ and

$$\int_{\mathbb{R}^N} (\partial_t u_\alpha(x,t_k))^2\,dx < \frac{1}{t_k \log t_k}. \tag{3.5}$$

In fact, if such sequence t_k did not exist then there would exist k_0 such that $\int_{\mathbb{R}^N} (\partial_t u_\alpha(\cdot,t))^2\,dx \geq \frac{1}{t\log t}$ for all $t \in H(\tau_k)$ and all $k \geq k_0$, hence

$$\int_{\tau_{k_0}}^\infty \int_{\mathbb{R}^N} (\partial_t u_\alpha)^2 \geq \sum_{k=k_0}^\infty \int_{H(\tau_k)} \int_{\mathbb{R}^N} (\partial_t u_\alpha)^2 \geq \sum_{k=k_0}^\infty \frac{1}{16} \frac{1}{\log(2\tau_k)} = \infty$$

which contradicts Lemma 3.1.

Set $M_k := u_\alpha(0,t_k)$. Due to the definition of H and (3.4), there exist $T_k \in [t_k + \theta\eta(t_k), t_k + \eta(t_k)]$ such that

$$2M_k \geq u_\alpha(0,T_k) \geq \mu M_k. \tag{3.6}$$

We claim that given $q \geq 2$,

$$\|\partial_t u_\alpha(\cdot,T_k)\|_q = o\Big(M_k^{(p-1)(\frac{1}{2}+\frac{N}{2}(\frac{1}{2}-\frac{1}{q}))}\Big) \quad \text{as } k \to \infty. \tag{3.7}$$

To prove this, we employ the following equation satisfied by $z := \partial_t u_\alpha$

$$z_t - \Delta z = p u_\alpha^{p-1} z. \tag{3.8}$$

Observe that (3.3) and (3.5) guarantee $\|\partial_t u_\alpha(\cdot, t_k)\|_2 = o(M_k^{(p-1)/2})$. Using this and obvious estimates in the variation of constants formula for equation (3.8), we obtain for $t \in [t_k, t_k + \eta(t_k)]$

$$\begin{aligned} \|z(t)\|_2 &\leq \|z(t_k)\|_2 + \int_{t_k}^t p(2M_k)^{p-1} \|z(s)\|_2 \, ds \\ &\leq o(M_k^{(p-1)/2}) + CM_k^{p-1}(t-t_k)^{1/2} \Big(\int_{t_k}^t \|z(s)\|_2^2 \, ds \Big)^{1/2} \\ &= o(M_k^{(p-1)/2}), \end{aligned} \tag{3.9}$$

where $z(t) = z(\cdot, t)$ and C is a constant (we have used (3.4), $t - t_k \leq \eta(t_k) = 2^{-p} M_k^{1-p}$, and $\int_{t_k}^\infty \|z(s)\|_2^2 \, ds \to 0$ as $k \to \infty$).

Next choose an integer $m \geq 1$ and a sequence $2 = q_0 \leq q_1 \leq \cdots \leq q_m = q$ such that $\frac{N}{2}(\frac{1}{q_j} - \frac{1}{q_{j+1}}) < 1$ for $j = 0, 1, \ldots, m-1$. Set $t_{k,j} := t_k + (T_k - t_k) j/m$, $j = 0, 1, \ldots, m$. We will prove that for $j = 0, 1, \ldots, m$

$$\sup_{t \in [t_{k,j}, T_k]} \|z(t)\|_{q_j} = o\Big(M_k^{(p-1)(\frac{1}{2} + \frac{N}{2}(\frac{1}{2} - \frac{1}{q_j}))} \Big) \quad \text{as } k \to \infty. \tag{3.10}$$

In particular, claim (3.7) will follow. Estimate (3.10) for $j = 0$ follows from (3.9). Next assume that (3.10) is true for some $j < m$. Using $L^p - L^q$ estimates in the variation of constants formula for equation (3.8) we obtain for $t \in [t_{k,j+1}, T_k]$,

$$\begin{aligned} \|z(t)\|_{q_{j+1}} &\leq (t - t_{k,j})^{-\frac{N}{2}(\frac{1}{q_j} - \frac{1}{q_{j+1}})} \|z(t_{k,j})\|_{q_j} \\ &\qquad + p(2M_k)^{p-1} \int_{t_{k,j}}^t (t-s)^{-\frac{N}{2}(\frac{1}{q_j} - \frac{1}{q_{j+1}})} \|z(s)\|_{q_j} \, ds \\ &= o\Big(M_k^{(p-1)(\frac{1}{2} + \frac{N}{2}(\frac{1}{2} - \frac{1}{q_{j+1}}))} \Big). \end{aligned}$$

Here we have also used (3.4) and the existence of constants $C_1, C_2 > 0$ such that $C_1 M_k^{1-p} \leq t - t_{k,j} \leq C_2 M_k^{1-p}$ (these estimates follow from $2^{-p} M_k^{1-p} = \eta(t_k) \geq T_k - t_k \geq t - t_{k,j} \geq (T_k - t_k)/m \geq \theta \eta(t_k)/m$). We have thus proved (3.7).

Set $\lambda_k := M_k^{-(p-1)/2}$ and $U_k(y,s) := \frac{1}{M_k} u_\alpha(\lambda_k y, T_k + \lambda_k^2 s)$. Then $W_k(y) := U_k(y,0)$ satisfies the elliptic equation

$$-\Delta W_k = W_k^p - F_k \qquad \text{in } \mathbb{R}^N, \tag{3.11}$$

where $F_k(y) := -\partial_s U_k(y,0)$. In addition, by (3.6), $\max W_k = W_k(0) \in [\mu, 2]$. Observe that

$$\|F_k\|_q = M_k^{\frac{p-1}{2}\frac{N}{q}-p} \|\partial_t u_\alpha(\cdot, T_k)\|_q = o(1), \qquad q \geq 2,$$

due to (3.7) and $p = p_S$. Therefore, using standard compactness arguments and passing to a subsequence if necessary, one easily shows that W_k converges to V_c in $C^{1+\kappa}_{loc}(\mathbb{R}^N)$ for any $\kappa < 1$ and some $c \in [\mu, 2]$, where V_c is the unique positive radial solution of the equation $-\Delta V = V^p$ in $\mathbb{R}^N$ satisfying $V(0) = c$.

Next fix $\vartheta \in (\alpha, \alpha^*)$ and rescale u_ϑ in the same way as u_α: Set $\tilde{U}_k(y,s) := \frac{1}{M_k} u_\vartheta(\lambda_k y, T_k + \lambda_k^2 s)$, where $M_k = u_\alpha(0, t_k)$ and λ_k, t_k, T_k are defined above, and $\tilde{W}_k(y) := \tilde{U}_k(y,0)$. Since $\frac{\vartheta}{\alpha} u_\alpha$ is a subsolution of (1.1) satisfying the same initial condition as u_ϑ, we have $u_\vartheta \geq \frac{\vartheta}{\alpha} u_\alpha$ and, consequently, $\tilde{W}_k \geq \frac{\vartheta}{\alpha} W_k$. Fix $\varepsilon \in (0, \vartheta/\alpha - 1)$. With $c \in [\mu, 2]$ as above, given any $R > 0$, we have

$$\tilde{W}_k \geq (1+\varepsilon) V_c \quad \text{on } B_R \text{ for } k \text{ large enough.} \tag{3.12}$$

The solution $u(\cdot,\cdot;(1+\varepsilon)V_c)$ of (1.1) with initial data $(1+\varepsilon)V_c$ blows up completely in a finite time T^* (see [6] and the proof of [20, (22.26)], for example). This means in particular that if $u^{(j)} = u^{(j)}(\cdot,\cdot;(1+\varepsilon)V_c)$ denotes the solution of (1.1) with initial data $(1+\varepsilon)V_c$ and the nonlinearity u^p replaced by $f_j(u) := \min(u^p, j)$, $j = 1, 2, \ldots$, then

$$\int_{\mathbb{R}^N} u^{(j)}(x,T) e^{-|x|^2}\, dx \to \infty \quad \text{as } j \to \infty,$$

whenever $T > T^*$. Fix $T > T^*$ and j such that

$$\int_{\mathbb{R}^N} u^{(j)}(x,T) e^{-|x|^2}\, dx > \pi^{N/2}(2N)^{1/(p-1)}.$$

Let $u_R^{(j)}$ denote the solution of (1.1) with initial data $(1+\varepsilon)V_c\chi_{B_R}$ and the nonlinearity u^p replaced by $f_j(u)$. Then $u_R^{(j)}(\cdot,T)$ converge monotonically to $u^{(j)}(\cdot,T)$ as $R \to \infty$, hence there exists $R > 0$ such that

$$\int_{\mathbb{R}^N} u_R^{(j)}(x,T) e^{-|x|^2}\, dx > \pi^{N/2}(2N)^{1/(p-1)}.$$

Now [20, Theorem 17.1] guarantees that the solution $u(\cdot,\cdot;(1+\varepsilon)V_c\chi_{B_R})$ of (1.1) blows up in finite time, hence $\tilde{U}_k$ blows up in finite time if k is large enough by (3.12) and the comparison principle. This contradicts the global existence of u_ϑ and concludes the proof. $\square$

Remark 3.2. The arguments used in the proof of Theorem 1.2 show the existence of $T_k \to \infty$ such that $\frac{1}{M_k}u_{\alpha^*}(\lambda_k y, T_k) \to V_c$ in $C^1_{loc}(\mathbb{R}^N)$, where M_k, λ_k and V_c are as in the proof of Thm 1.2. This convergence result can be compared with an analogous result for the Cauchy-Dirichlet problem in a ball, see [20, Proposition 23.11] (cf. also [12, Proposition 4.1]). The asymptotic behavior of the L^∞ norm of threshold solutions of the Cauchy-Dirichlet problem was studied in [5].

4 The subcritical case

The proof of Theorem 1.3 is carried out in several steps, comprising the following lemmas. In all of them we assume the hypotheses of Theorem 1.3 to be satisfied. As a consequence of (1.3) we have $\alpha^* \in (0,\infty)$ (see (1.2) and the paragraph following it).

Lemma 4.1. *There are constants C_0, C_1 such that*

$$\|u_{\alpha^*}(\cdot,t)\|_\infty \le C_0 t^{-\beta} \quad (t>0), \tag{4.1}$$

$$\|v_{\alpha^*}(\cdot,s)\|_\infty \le C_1 \qquad (s>0). \tag{4.2}$$

Proof. The arguments here are essentially those used in [19, Proof of Theorem 4.1]. The estimate on $u_{\alpha*}$, as given in (4.1), follows from the fact that for the global solutions u_α, $\alpha \in (0,\alpha^*)$, such an estimate holds with a constant C_0 independent of α. This is a consequence of universal estimates of [18] (which depend on the assumption (LT)). Estimate (4.2) follows from (4.1) and the fact that $\phi \in L^\infty(\mathbb{R}^N)$ (hence u_{α^*} remains bounded for $t \approx 0$). $\square$

Next we establish a uniform spatial decay rate of $v_\alpha(\cdot,s)$.

Lemma 4.2. *For each $\alpha \in (0,\alpha^*)$ one has*

$$v_\alpha(y,s) = o(|y|^{-2\beta}) \text{ as } |y| \to \infty, \quad \text{uniformly in } s \ge 0. \tag{4.3}$$

Proof. Fix any $\epsilon > 0$. We need to find R such that

$$v_\alpha(y,s) \leq \frac{\epsilon}{|y|^{2\beta}} \quad (|y| > R, s \geq 0). \tag{4.4}$$

Choose $\delta > 0$ so small that

$$\frac{\epsilon\delta^{-2\beta}}{2} > C_1, \tag{4.5}$$

where C_1 is in (4.2). One can easily verify (or see [19, Proof of Theorem 4.1]) that if R_1 is sufficiently large, then the function $w(y) = \epsilon(|y| - R_1)^{-2\beta}/2$ is a supersolution of (1.12) on $\{y : |y| > R_1 + \delta\}$. Fix such R_1, assuming also that

$$\alpha^*\phi(y) \leq \frac{\epsilon}{2(|y| - R_1)^{2\beta}} \quad (|y| > R_1 + \delta) \tag{4.6}$$

(the latter is justified by (1.3)). Then (4.6) and (4.5) guarantee that w dominates v_{α^*} (hence also v_α) on the parabolic boundary of the set $\{(y,s) : |y| > R_1 + \delta, s > 0\}$. The comparison principle therefore yields

$$v_\alpha(y,s) \leq w(y) \quad (|y| > R_1 + \delta, s > 0, \alpha \in (0, \alpha^*]).$$

Since

$$w(y) \leq \frac{\epsilon}{|y|^{2\beta}} \quad (|y| > R)$$

provided $R > R_1 + \delta$ is large enough, we see that (4.4) holds for such R. □

Recall that λ_0 is defined in Proposition 2.1. The next lemma gives a lower bound on v_{α^*}. Its proof is contained in the proof of Theorem 4.1 [19] (see the two paragraphs in [19] containing (4.9)-(4.13), note that the symmetry of v is not used there).

Lemma 4.3. *Fix $\lambda \in (0, \lambda_0)$. There exists R_0 with the following property. For each $s > 0$ there is $y \in B_{R_0}$ such that $v_{\alpha^*}(y,s) = w_\lambda(y)$. Consequently, one has*

$$\|v_{\alpha^*}(\cdot, s)\|_\infty \geq C_2 \quad (s > 0),$$

where C_2 is a positive constant.

For any $\alpha \in (0, \alpha^*]$ we now introduce the ω-limit set of v_α:

$$\omega(v_\alpha) = \{\psi : \psi = \lim_{k\to\infty} v_\alpha(\cdot, s_k) \text{ for some } s_k \to \infty\},$$

where the limit is taken in $L^\infty(\mathbb{R}^N)$. Observe that (4.2),(4.3), and parabolic estimates imply that the set $O(v_\alpha) := \{v_\alpha(\cdot, s) : s \geq 1\}$ is relatively compact in $L^\infty(\mathbb{R}^N)$. Hence $\omega(v_\alpha)$ is a nonempty compact connected subset of $L^\infty(\mathbb{R}^N)$ and it attracts v_α in the sense that

$$\lim_{s\to\infty} \operatorname{dist}_{L^\infty(\mathbb{R}^N)}(v_\alpha(\cdot, s), \omega(v_\alpha)) = 0.$$

Moreover, the compactness of $O(v_\alpha)$ and standard limiting arguments show that with each ψ, $\omega(v_\alpha)$ contains an entire solution of (1.12). Specifically, for each $\psi \in \omega(v_\alpha)$, there exists a solution $\zeta(y, s)$ of (1.12) on $\mathbb{R}^N \times \mathbb{R}$ such that $\zeta(\cdot, 0) = \psi$ and $\zeta(\cdot, s) \in \omega(v_\alpha)$ for all s. By (4.3), we also know that each such entire solution ζ also satisfies

$$\zeta(y, s) = o(|y|^{-2\beta}) \text{ as } |y| \to \infty, \text{ uniformly in } s \in \mathbb{R}. \tag{4.7}$$

In addition, ζ is nonnegative (because v_α is) and therefore Theorem 5.1 proved in the appendix applies to ζ. It implies that for each $s \in \mathbb{R}$ the function $\zeta(\cdot, s)$ is radially symmetric. Thus to understand $\omega(v_\alpha)$ we need to examine radial entire solutions. Our ultimate goal is prove the following lemma.

Lemma 4.4. *One has*

(a) $\omega(v_\alpha) = \{0\}$ *for each* $\alpha \in (0, \alpha^*)$ *and*

(b) $\omega(v_{\alpha^*}) = \{w_{\lambda_0}\}$.

These statements can be equivalently written as

(a') $\lim_{s\to\infty} v_\alpha(\cdot, s) = 0$ for each $\alpha \in (0, \alpha^*)$ and

(b') $\lim_{s\to\infty} v_{\alpha^*}(\cdot, s) = w_{\lambda_0}$,

with the limits in $L^\infty(\mathbb{R}^N)$. Interpreted in terms of the solutions u_α, (a') and (b') give statements (i) and (ii) of Theorem 1.3. Hence the proof of Theorem 1.3 will be complete once we prove Lemma 4.4.

Proof of Lemma 4.4. Fix an entire solution $\zeta(\cdot, s) \in \omega(v_\alpha)$ for some $\alpha \in (0, \alpha^*]$. In acordance with our conventions, $\zeta(\cdot, s)$ being radially symmetric, we can view it as a function of $y \in \mathbb{R}^N$ or of $\rho = |y| \in (0, \infty)$. We first prove that there exist the $L^\infty(\mathbb{R}^N)$-limits

$$w^\pm := \lim_{s\to\pm\infty} \zeta(\cdot, s). \tag{4.8}$$

The proof involves rather standard arguments based on the intersection comparison principle. We only consider the limit at $-\infty$, the arguments for ∞ are analogous and even simpler (also they are very similar to arguments used in [19]). To start off, we consider the set J of all accumulation points of $\zeta(0,s)$ as $s \to -\infty$. We want to show that J is a singleton. Assume not and fix $\lambda \in (\inf J, \sup J) \setminus \{\lambda_0\}$. We use the stationary solution w_λ for comparison. Since ζ is nonnegative, the maximum principle implies that it is strictly positive everywhere.

Consider first the case $\lambda > \lambda_0$, so that $\zeta(\rho_\lambda, s) > 0 = w_\lambda(\rho_\lambda)$. Then $z_{[0,\rho_\lambda)}(\zeta(\cdot,s) - w_\lambda)$ is finite for each $s \in \mathbb{R}$, it is nonincreasing, and drops whenever $\zeta(0,s) = \lambda = w_\lambda(0)$. By the choice of λ, the latter happens for infinitely many values of s, hence, necessarily,

$$\lim_{s \to -\infty} z_{[0,\rho_\lambda)}(\zeta(\cdot,s) - w_\lambda) = \infty. \tag{4.9}$$

Take a sequence $s_k \to -\infty$. By standard compactness and limiting arguments, we may assume that he sequence $\zeta(\cdot, s_k + \cdot)$ converges in $C^{1,0}_{loc}(\mathbb{R}^N \times \mathbb{R})$ to a solution ζ_∞ of (1.12). Clearly, the sequence s_k can be chosen such that $\zeta_\infty \not\equiv 0$ on any set of the form $\mathbb{R}^N \times (-\infty, T)$ (for example, choose it such that $\zeta(0, s_k) = \lambda$). Then ζ_∞ is positive everywhere and, as above for ζ, one shows that $z_{[0,\rho_\lambda)}(\zeta_\infty(\cdot,s) - w_\lambda)$ is finite for each s. Moreover, we can choose s_0 so that $\zeta_\infty(\cdot, s_0) - w_\lambda$ has only simple zeros in $[0, \rho_\lambda]$, all of them being contained in $[0, \rho_\lambda)$. It then follows that for all k sufficiently large $\zeta(\cdot, s_k + s_0) - w_\lambda$ has the same - finite number of zeros, contradicting (4.9).

In the other case, $\lambda < \lambda_0$, the function w_λ satisfies $w_\lambda(\rho)\rho^{2\beta} \to A(\lambda) > 0$ as $\rho \to \infty$. Therefore, using (4.7) we find R_2 such that $\zeta(R_2, s) < w_\lambda(\rho_\lambda)/2$ for all s. We can then proceed similarly as in the previous case, replacing the interval $[0, \rho_\lambda]$ with $[0, R_2]$, and arrive at a contradiction.

The contradiction shows that J is a singleton, that is, $\zeta(0,s)$ has a limit as $s \to -\infty$. Denote the limit by λ. Clearly, $\lambda \in [0, \infty)$. We claim that $\zeta(\cdot, s) \to w_\lambda$ in $L^\infty(\mathbb{R}^N)$ as $s \to -\infty$. Assume this is not true. Then using compactness and limiting arguments again, we find a sequence $s_k \to -\infty$ such that $\zeta(\cdot, s_k + \cdot)$ converges in $C^{1,0}_{loc}(\mathbb{R}^N \times \mathbb{R})$ to a solution ζ_∞ of (1.12) and $\zeta_\infty(\cdot, 0) \not\equiv w_\lambda$. In particular, we can find $\rho > 0$ and a small ϵ such that $\zeta_\infty(\rho, s) \neq w_\lambda(\rho)$ for all $s \in (-\epsilon, \epsilon)$. The zero number $z_{[0,\rho)}(\zeta_\infty(\cdot,s) - w_\lambda)$ is then finite for $s \in (-\epsilon, \epsilon)$. However, the choice of λ implies that $\zeta_\infty(0,s) = \lambda = w_\lambda(0,s)$ for each s, thus the zero number has to drop at each s, a contradiction.

Thus we have proved the existence of the limits in (4.8), in fact, each of the limits is equal to the steady state w_λ, for some $\lambda \in [0,\infty)$. Since the limit have to be nonnegative and inherit the decay property of ζ, see (4.7), the only possible limits are $w_0 \equiv 0$ and w_{λ_0}.

We shall now treat the cases $\alpha = \alpha^*$ and $\alpha < \alpha^*$ separately.

Proof of statement (b). Take $\alpha = \alpha^*$. Lemma 4.3 implies that $0 \notin \omega(v_{\alpha^*})$. For each entire solution $\zeta(\cdot,s) \in \omega(v_{\alpha^*})$ the limits w^+ and w^- in (4.8) belong to $\omega(v_{\alpha^*})$, hence these limits coincide and are equal to w_{λ_0}. In particular, as $\omega(v_{\alpha^*})$ is compact and contains at least one such solution ζ, we have $w_{\lambda_0} \in \omega(v_{\alpha^*})$. To prove that $\omega(v_{\alpha^*}) = \{w_{\lambda_0}\}$, we need to rule out the existence of a homoclinic solution ζ in $\omega(v_{\alpha^*})$. Assume that $\zeta(\cdot,s) \in \omega(v_{\alpha^*})$ is a homoclinic to w_{λ_0}: $\zeta \not\equiv w_{\lambda_0}$ and $\zeta(\cdot,s) \to w_{\lambda_0}$ as $s \to \pm\infty$. The limits here are in $L^\infty(\mathbb{R}^N)$ a priori, but by parabolic estimates also in $C^1_{loc}(\mathbb{R}^N)$. We have $\zeta(0,\cdot) \not\equiv \lambda_0$ (otherwise, considering the zero number of $\zeta(\cdot,s) - w_{\lambda_0}$ on a suitable interval one shows $\zeta \equiv w_{\lambda_0}$, similarly as in the proof of the convergence properties (4.8) above). Hence we can choose $\lambda \in (\inf_s \zeta(0,s), \sup_s \zeta(0,s)) \setminus \{\lambda_0\}$. Consider the function $\zeta(\cdot,s) - w_\lambda$. We find $R > 0$ such that that $\zeta(R,s) - w_\lambda(R) \neq 0$ for each $s \in \mathbb{R}$ and $w_{\lambda_0}(R) - w_\lambda(R) \neq 0$. If $\lambda > \lambda_0$, we can take $R = \rho_\lambda$, the first zero of w_λ (cf. Proposition 2.1), since ζ and w_{λ_0} are both positive. In the case $\lambda < \lambda_0$, it is sufficent to take R large enough; the relations are then satisfied due to (4.7) and $\lim_{\rho\to\infty} \rho^{2\beta} w_{\lambda_0}(\rho) = 0 < \lim_{\rho\to\infty} \rho^{2\beta} w_\lambda(\rho)$ (see Proposition 2.1). Also note that w_{λ_0}, w_λ being two different solutions of the same second-order ODE, their difference has only simple zeros. We now examine the function $s \mapsto z_{[0,R)}(\zeta(\cdot,s) - w_\lambda)$. On the one hand, this nonincreasing function must be constant since for $s \approx \pm\infty$ it takes the value $z_{[0,R)}(w_{\lambda_0} - w_\lambda)$, due to the C^1 convergence of $\zeta(\cdot,s)$ to w_{λ_0}. On the other hand, our choice of λ implies that $\zeta(0,s) = \lambda = w_\lambda(0)$ for some s, hence the function cannot be constant by the dropping property. This contradiction rules out the homoclinic and completes the proof of statement (b).

Proof of statement (a). Take $\alpha \in (0,\alpha^*)$. By the same arguments as in the previous case, $\omega(v_\alpha)$ cannot contain a homoclinic to either w_{λ_0} or 0. Thus $\omega(v_\alpha) = \{0\}$ will be proved, if we show that $w_{\lambda_0} \notin \omega(v_\alpha)$.

It is easy to verify that the function $\frac{\alpha^*}{\alpha} u_\alpha$ is a subsolution of (1.1) with the same initial condition as u_{α^*}. Hence $u_{\alpha^*} \geq \frac{\alpha^*}{\alpha} u_\alpha$ and therefore $v_{\alpha^*} \geq \frac{\alpha^*}{\alpha} v_\alpha$. Since we already know that $v_{\alpha^*}(\cdot,s) \to w_{\lambda_0}$ as $s \to \infty$, we conclude that each $\psi \in \omega(v_\alpha)$ satisfies $\frac{\alpha^*}{\alpha}\psi \leq w_{\lambda_0}$, in particular, $w_{\lambda_0} \notin \omega(v_\alpha)$. □

5 Appendix

In this section we extend the symmetry result of [15] to solutions of the following equation

$$u_t = \Delta u + \frac{1}{2} x \cdot \nabla u + ku + f(u), \quad x \in \mathbb{R}^N, \ t \le t_0. \tag{5.1}$$

Here $N \ge 2$, k is a positive constant, $t_0 \in \mathbb{R}$, and $f : [0, \infty)$ is a C^1-function satisfying

$$f(u) = O(u^\sigma) \text{ as } u \searrow 0 \quad \text{for some } \sigma > 1. \tag{5.2}$$

The constants appearing in (5.1), (5.2) are fixed for the whole section.

We consider (classical) solutions of (5.1) defined for all $t \in (-\infty, t_0]$ (ancient solutions). As we refer to [15] for parts of the proof of our result, we adopt the notation of [15], rather than that of the previous sections. Like the notation, the results here are independent of the previous sections.

Theorem 5.1. *Under the above assumptions, let u be a nonnegative bounded solution of* (5.1) *satisfying*

$$u(x,t) = o(|x|^{-2k}) \textit{ as } |x| \to \infty, \textit{ uniformly in } t \le t_0. \tag{5.3}$$

Then for each $t \le t_0$ the function $u(\cdot, t)$ is radially symmetric.

In the special case of steady state solutions u, this is Theorem 2.1 of [15]. To extend it to the setting of ancient solutions, we follow the steps of the proof of [15]. The only essential difference from [15] is that in place of the maximum principle for elliptic equations we need a maximum principle for ancient solutions of parabolic equations, as given in Lemma 5.2 below. In its formulation, $L(x)$ is the elliptic operator defined by

$$L(x)v = \Delta v + \frac{1}{2} x \cdot \nabla v - \frac{2\beta}{|x|^2} x \cdot \nabla v \quad (x \in \mathbb{R}^N \setminus \{0\}), \tag{5.4}$$

where β is a positive constant.

Lemma 5.2. *Given positive constants β, ρ, assume that v is a continuous function on $(\mathbb{R}^N \setminus B_\rho) \times (-\infty, t_0]$ such that v is bounded from above, $v \in C^{2,1}(\mathbb{R}^N \setminus \bar{B}_\rho) \times (-\infty, t_0])$, and the following inequalities are satisfied*

$$v_t \le L(x)v, \qquad (x,t) \in (\mathbb{R}^N \setminus \bar{B}_\rho) \times (-\infty, t_0], \tag{5.5}$$

$$v(x,t) \le 0, \qquad (x,t) \in \partial B_\rho \times (-\infty, t_0]), \tag{5.6}$$

$$\limsup_{\substack{|x| \to \infty \\ t \le t_0}} v(x,t) \le 0. \tag{5.7}$$

Then $v \le 0$ *on* $(\mathbb{R}^N \setminus B_\rho) \times (-\infty, t_0]$.

Proof. Assume v is positive somewhere. Then by (5.5)-(5.7) and the maximum principle (see e.g. [20, Proposition 52.4]), there exists $t_1 < t_0$ such that

$$M(t) := \sup_{|x|>\rho} v(x,t) > 0 \quad (t \le t_1).$$

Also by the maximum principle, noting that $L(x)$ has no zero order term, $M(t)$ is a nonincreasing function. Hence there exists $M := \lim_{t\to-\infty} M(t)$ and, in view of the boundedness of v, $M \in (0,\infty)$. By (5.7), there exists $\rho_1 > \rho$ such that

$$m := \max\big(0, \sup_{\substack{|x|>\rho_1 \\ t\le t_1}} v(x,t)\big) < M.$$

Replacing t_1 by a smaller number if necessary, we may assume that $M(t) > m$ for all $t \le t_1$. Set $w = v - m$. Then w satisfies

$$w_t \le L(x)w, \qquad \rho < |x| < \rho_1,\ t \le t_1, \tag{5.8}$$

$$w(x,t) \le 0, \qquad |x| \in \{\rho, \rho_1\},\ t \le t_1, \tag{5.9}$$

$$M - m \ge \max_{\rho\le|x|\le\rho_1} w(x,t) = M(t) - m > 0, \quad t \le t_1. \tag{5.10}$$

Let λ_1 and $\varphi_1 > 0$ stand for the principal eigenvalue and eigenfunction of the problem

$$\begin{aligned} L(x)\varphi = \lambda\varphi, &\quad x \in B_{\rho_1+1} \setminus \bar{B}_{\rho/2}, \\ w = 0, &\quad x \in \partial B_{\rho_1+1} \cup \partial B_{\rho/2}. \end{aligned}$$

As $L(x)$ has no zero order term, the strong maximum principle implies that $\lambda_1 < 0$. Choose $c > 0$ so large that $c\varphi_1 > M - m$ on $\bar{B}_{\rho_1} \setminus B_\rho$. Then for any $\tau < t_1$ the function $z(x,t) = ce^{\lambda_1(t-\tau)}\varphi_1(x)$ is a solution of $z_t = L(x)z$ on $(B_{\rho_1} \setminus \bar{B}_\rho) \times (\tau, t_1]$ and it dominates w on the parabolic boundary of that set. The comparison principle therefore yields

$$w(x,t_1) \le ce^{\lambda_1(t_1-\tau)}\varphi_1(x) \quad (\rho \le |x| \le \rho_1).$$

Since this is true for any τ, we can take $\tau \to -\infty$ to obtain

$$0 < M(t_1) - m = \max_{\rho\le|x|\le\rho_1} w(x,t_1) \le 0.$$

This contradiction shows that, as stated in the lemma, v cannot assume any positive value. $\square$

We now show how Lemma 5.2 and a modification of arguments of [15] are used in the proof Theorem 5.1. In the remainder of the section we assume that the hypotheses of Theorem 5.1 are satisfied. We first derive a stronger decay estimate on the solution u.

Proposition 5.3. *For each $\beta > 0$ one has*

$$u(x,t) = o(|x|^{-\beta}) \text{ as } |x| \to \infty, \text{ uniformly in } t \le t_0. \tag{5.11}$$

We start with the following preliminary statement.

Lemma 5.4. *If* (5.11) *holds for some $\beta > 2k$, then it holds for each $\beta > 0$.*

Proof. Let $\beta > 2k$ be such that (5.11) holds. Set $v(x,t) := |x|^\beta u(x,t)$ and, for an arbitrary $m > 0$, $w(x) := |x|^{-m}$. Let $L(x)$ be as in (5.4). The computation of [15, Proof of Lemma 2.1] shows that for a large enough R_0 the functions v, w satisfy

$$v_t - L(x)v \le 0 \le -L(x)w \quad (|x| \ge R_0,\, t \le t_0).$$

We further choose a sufficiently large C so that $v(x,t) \le Cw(x)$ for each $x \in \partial B_{R_0}$, $t \le t_0$. Then, using also (5.11), we conclude that Lemma 5.2 applies to $v - Cw$ and it gives $v(x,t) \le Cw(x)$ for each $x \in \mathbb{R}^N \setminus B_{R_0}$, $t \le t_0$. This show that (5.11) holds with β replaced by $\beta + m$. Since m was arbitrary, we obtain the conclusion. □

Proof of Proposition 5.3. Set $\beta := 2k$ and note that (5.11) holds for this β due to (5.3). Also set $\delta := \min\{1, k(\sigma - 1)\}$ (with σ as in (5.2)), $v(x,t) := |x|^\beta u(x,t)$, and $w(x) := |x|^{-\delta}$. Let again $L(x)$ be as in (5.4). The same arguments as in the previous proof, only this time one uses the computations of [15, Proof of Proposition 2.1] in place of those in [15, Proof of Lemma 2.1], show that if R_0 and C are sufficiently large, then $v(x,t) \le Cw(x)$ for each $x \in \mathbb{R}^N \setminus B_{R_0}$, $t \le t_0$. Consequently, (5.11) holds for $\beta = 2k + \delta$ and using Lemma 5.4 we conclude that it holds for each $\beta > 0$. □

Proof of Theorem 5.1. Choose a constant α with

$$\alpha > k + \max\{|f'(s)| : 0 \le s \le \|u\|_{L^\infty(\mathbb{R}^N \times (-\infty, t_0))}\}.$$

Fix an arbitrary $\tau < t_0$ and set

$$w(x,t) := (t-\tau)^{-\alpha} u\Big(\frac{x}{\sqrt{t-\tau}}, t - \tau\Big) \quad ((x,t) \in \mathbb{R}^N \times (\tau, t_0)).$$

The function w solves a suitable parabolic equation as in [15]. Observe that it is time-dependent regardless of whether u depends on t or not. For this reason, the arguments for the symmetry of u as given in [15, Section 2] apply equally well in our more general setting, as long as (5.11) is valid with $\beta = \alpha$. The only modification needed in the actual proof is that in all estimates and in the process of moving hyperplanes carried out in [15, Section 2], the time interval $(0, T]$ is replaced by $(\tau, t_0]$. □

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