Further results on quasiperiodic partially localized solutions of homogeneous elliptic equations on $\mathbb{R}^{N+1}$

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Abstract

We study positive partially localized solutions of the elliptic equation

$$\Delta_x u + u_{yy} + f(u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1}$$

where $N \geq 2$ and f is a C^1 function satisfying $f(0) = 0$ and $f'(0) < 0$. By partially localized solutions we mean solutions $u(x, y)$ which decay to zero as $|x| \to \infty$ uniformly in y. Our main concern is the existence of positive partially localized solutions which are quasiperiodic in y. The fact that such solutions can exist in equations of the above form was demonstrated in our earlier work: we proved that the nonlinearity f can be designed in such a way that equation (1) possesses positive partially localized quasiperiodic solutions with 2 frequencies. Our main contributions in the present paper are twofold. First, we improve the previous result by showing that positive partially localized quasiperiodic solutions with any prescribed number $n \geq 2$ of frequencies exist for some nonlinearities f. Second, we give a tangible sufficient condition on f which guarantees that equation (1) has such quasiperiodic solutions, possibly after f is perturbed slightly. The condition, with $n = 2$, applies, for example, to some combined-powers nonlinearities $f(u) = u^p + \lambda u^q - u$ with suitable exponents $p > q > 1$ and coefficient $\lambda > 0$.

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Contents

1 Introduction and main results

We consider the semilinear elliptic equation

$$\Delta u + u_{yy} + f(u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $N \geq 2$, Δ is the Laplace operator in $x \in \mathbb{R}^N$, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying

$$f(0) = 0, \; f'(0) < 0. \tag{1.2}$$

We are mainly interested in positive solutions of this equation which decay to 0 in the x-variables uniformly in y:

$$\lim_{|x|\to\infty} \sup_{y\in\mathbb{R}} u(x, y) = 0. \tag{1.3}$$

Henceforth, we refer to solutions satisfying (1.3) as *partially localized* solutions.

Partially localized solutions include in particular solutions which decay in the y variable as well, so they are *fully localized*. Positive fully localized solutions, frequently referred to as *ground states* of (1.1), are well understood: they are radially symmetric about some center in $\mathbb{R}^{N+1}$ and radially decreasing away from that center (see [18]). For basic results on the existence and nonexistence of ground states we refer the reader to [4]; theorems on uniqueness (up to translations) and nonuniqueness can be found in [1, 8, 9, 10, 11, 14, 25, 26, 29, 32, 33, 34, 38, 42].

A different class of partially localized solutions of (1.1) is obtained from ground states of the equation on $\mathbb{R}^N$,

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^N, \tag{1.4}$$

if they exist, by extending them to functions on $\mathbb{R}^N \times \mathbb{R}$ constant in y. This is a rather trivial remark, but ground states of (1.4) will play an important role below.

More interesting partially localized positive solutions which are not fully localized are solutions periodic (and nonconstant) in y. Their existence, as well as some structural properties, have been established in [2, 12, 28] for a large class of nonlinearities including the power nonlinearity $f(u) = u^p - u$ with $1 < p < (N+2)/(N-2)$.

Looking beyond periodic solutions, and considering that equation (1.1) has a formal Hamiltonian structure (cp. [20, 30, 36]), one naturally asks if positive partially localized solutions which are quasiperiodic (and not periodic) in y may exist for some nonlinearities f.

Let us recall the definition of a quasiperiodic solution. Given an integer $n \geq 2$, we say that a vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ is *nonresonant*, or, equivalently, that the numbers $\omega_1, \ldots, \omega_n$ are *rationally independent*, if

$$\omega \cdot \alpha \neq 0 \quad (\alpha \in \mathbb{Z}^n \setminus \{0\}). \tag{1.5}$$

Here $\omega \cdot \alpha$ is the usual dot product. A real function $u(x, y)$ on $\mathbb{R}^N \times \mathbb{R}$ is said to be *quasiperiodic* in y if there exist an integer $n \geq 2$, a nonresonant vector $\omega^* = (\omega_1^*, \ldots, \omega_n^*) \in \mathbb{R}^n$, and an injective function U defined on $\mathbb{T}^n$ (the n-dimensional torus) with values in the space of real-valued functions on $\mathbb{R}^N$ such that

$$u(x, y) = U(\omega_1^* y, \ldots, \omega_n^* y)(x) \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}). \tag{1.6}$$

The vector ω^* is called a *frequency vector* and its components the *frequencies* of u. Note that the nonresonance of the frequency vector is a part of our definition. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, then the image of the map $y \mapsto u(\cdot, y)$ is dense in an n-dimensional manifold diffeomorphic to $\mathbb{T}^n$.

The question whether positive quasiperiodic partially localized solutions can exist in equations of the above type was first addressed in our earlier paper [37]. We proved that for a carefully designed nonlinearity, equation (1) does have such quasiperiodic solutions with 2 frequencies. Restricting the number of frequencies to 2 in this result was not a matter of choice; the method used in the proof works in that case only. The nonlinearity f was found in [37] by an elaborate construction which served well the given purpose—finding quasiperiodic solutions for *some* nonlinearity satisfying (1.2)—but did not give any feasible way of showing the existence of quasiperiodic solutions in specific equations.

These shortcomings motivated our research documented in the present paper. In our new existence result, there is no restriction on the number of frequencies of quasiperiodic solutions. Moreover, what is perhaps more significant, we have found tangible sufficient conditions for the existence of positive quasiperiodic partially localized solutions of (1.1). This result allows us, among other applications, to find quasiperiodic solutions for nonlinearities which are arbitrarily small perturbations of some specific functions, such as the combined-powers nonlinearity $f(u) = u^p + \lambda u^q - u$ for suitable exponents $p > q > 1$ and coefficient λ. Under some natural conditions, we are also able to find such solutions within a specific class of equations (without needing a small perturbation).

We now give statements of our main results, starting with the following theorem addressing the possible number of frequencies of quasiperiodic solutions of (1.1). In this and the other three theorems stated in the introduction, the dimension N is fixed and it is assumed that $N \geq 2$.

Theorem 1.1. *Given any integer $n \geq 2$, there is a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0 > f'(0)$ such that equation* (1.1) *has a positive solution u satisfying* (1.3) *which is radially symmetric in x and quasiperiodic in y with n frequencies.*

A hypothesis in our second theorem involves a ground state of equation (1.4) (the equation in one less dimension). We need to recall some definitions. As noted above, any ground state ϕ is radially symmetric, so, possibly after a shift in $\mathbb{R}^N$, we can write $\phi = \phi(r)$, $r = |x|$. Consider now the Schrödinger operator $A(\phi) = -\Delta - f'(\phi(r))$, viewed as a self-adjoint operator on $L^2_{\text{rad}}(\mathbb{R}^N)$, the space consisting of all radial $L^2(\mathbb{R}^N)$-functions, with domain $H^2(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$. Since the potential $f'(\phi(r))$ has the limit $f'(\phi(\infty)) = f'(0) < 0$, the essential spectrum of $A(\phi)$ is contained in $[-f'(0), \infty)$ (cp. [40]). Therefore, the spectrum in $(-\infty, 0]$ consists of a finite number of isolated eigenvalues; these eigenvalues are all simple due to the radial symmetry. The *Morse index* of ϕ is defined as the number of negative eigenvalues of $A(\phi)$. We remark that we allow 0 to be an eigenvalue of $A(\phi)$, but only (strictly) negative eigenvalues count toward the Morse index. If 0 is an eigenvalue, the ground state is said to be *degenerate*, otherwise it is *nondegenerate*.

We will assume that for some integer $n \geq 2$ the following holds.

(G) Equation (1.4) has a ground state ϕ of Morse index n.

For a C^1 function $g : \mathbb{R} \to \mathbb{R}$, we denote

$$\|g\|_1 := \sup\{|g(u)|, |g'(u)| : u \in \mathbb{R}\}.$$

Theorem 1.2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with $f(0) = 0 > f'(0)$ such that* (G) *is satisfied for some $n \geq 2$. Then for any $\epsilon > 0$ there is a C^∞ function $\tilde{f}$ such that $\|f - \tilde{f}\|_1 < \epsilon$ and equation* (1.1) *with f replaced by $\tilde{f}$ has a positive solution u satisfying* (1.3) *which is radially symmetric in x and quasiperiodic in y with n frequencies.*

We emphasize that hypothesis (G) is a condition on the *eigenvalues* of the linearization at a ground state. Unlike the construction in [37], the hypothesis involves neither the corresponding eigenfunctions nor higher-order terms of the Taylor expansion of f at the ground state ϕ. This makes Theorem 1.2 much easier to apply; we show some interesting applications in a moment. On the other hand, the construction in [37] has its advantages when it does apply. Namely, it yields an uncountable family of positive partially localized quasiperiodic solutions (disregarding translations) of an equation of the form (1.1). Our present results do not have such a multiplicity statement (see Remark 2.4(iv) for an explanation). This is a relatively small price to pay for a much broader applicability of the new results.

We now give some applications of Theorem 1.2; the first one is a proof of Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 follows directly from Theorem 1.2 and a theorem of [34] which says that for any $n \geq 2$ (and $N \geq 2$) there is a smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying conditions (1.2) and (G). □

Besides [34], examples of functions satisfying conditions (1.2) and (G) (with $n = 2$) can also be found in [11, 14]. The most explicit among these examples is the combined-powers nonlinearity

$$f(u) = u^p + \lambda u^q - u, \tag{1.7}$$

where $1 < q < p < 5$ and $\lambda > 0$. As shown in [14], fixing a sufficiently large λ and then taking p sufficiently close to 5—note that 5 is the critical Sobolev exponent $(N+2)/(N-2)$ in dimension $N = 3$—one achieves that equation (1.4) with $N = 3$ has a ground state with Morse index 2 (in addition to two other ground states with Morse index 1). Thus, by Theorem 1.2, one can find quasiperiodic partially localized positive solutions for equation (1.1), where f is an arbitrarily small perturbation of a function of the form (1.7).

It is an interesting question whether partially localized quasiperiodic solutions can also be found for a combined-powers nonlinearity itself, that is, without a small perturbation. We believe that our techniques can be used to give a positive answer, although most exponents p, q have to be excluded due to smoothness requirements in our method. We state here one theorem for analytic nonlinearities (a related result for C^k nonlinearities with k large enough is given in the next section) and then discuss its possible applicability to combined-powers nonlinearities.

Consider an equation of the form (1.1) involving a real parameter $\lambda > 0$:

$$\Delta u + u_{yy} + f(u;\lambda) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.8}$$

Here, f is an analytic function on $\mathbb{R} \times J$, J being an open interval in $\mathbb{R}$, such that

$$f(0;\lambda) = 0, \; f_u(0;\lambda) < 0 \quad (\lambda \in J). \tag{1.9}$$

Also consider the corresponding equation for the ground states on $\mathbb{R}^N$:

$$\Delta u + f(u;\lambda) = 0, \quad x \in \mathbb{R}^N. \tag{1.10}$$

We assume that for some constants $\lambda_0, \hat{\lambda}_0 \in J$ with $\lambda_0 < \hat{\lambda}_0$ the following holds:

(GP) For each $\lambda \in [\lambda_0, \hat{\lambda}_0)$ equation (1.10) has a ground state ϕ^λ such that the following conditions are satisfied:

(c1) The map $\lambda \mapsto \phi^\lambda : [\lambda_0, \hat{\lambda}_0) \to L^\infty(\mathbb{R}^N)$ is continuous;

(c2) for each $\lambda \in (\lambda_0, \hat{\lambda}_0)$, ϕ^λ is a nondegenerate ground state with Morse index 2;

(c3) ϕ^{λ_0} is a degenerate ground state with Morse index 1.

What we have in mind here is that there is a branch (ϕ^λ, λ), $\lambda \in (\lambda_0, \hat{\lambda}_0)$, of ground states of (1.8) of Morse index 2 emanating from a "bifurcation point" $(\phi^{\lambda_0}, \lambda_0)$ (note that the linearization of the equation at the degenerate ground state ϕ^{λ_0} has 0 as the second eigenvalue, as the Morse index of ϕ^{λ_0} is 1).

Theorem 1.3. *Assume that f is an analytic function on $\mathbb{R} \times J$ satisfying* (1.9)*, and* (GP) *holds for some $\lambda_0, \hat{\lambda}_0 \in J$ with $\lambda_0 < \hat{\lambda}_0$. Then there is a dense subset Λ of the interval $(\lambda_0, \hat{\lambda}_0)$ such that for each $\lambda \in \Lambda$ equation* (1.8) *has a positive solution u satisfying* (1.3) *which is radially symmetric in x and quasiperiodic in y with* 2 *frequencies.*

Note that this is not a local result: we are not making the given interval $[\lambda_0, \hat{\lambda}_0)$ smaller in the conclusion. We are able to make such a global statement due to the analyticity assumption. In the next section, we give a local version of this result for finitely differentiable nonlinearities $f(u; \lambda)$.

Parameter dependent functions satisfying (c1)–(c3) are not difficult to find (an example of a smooth function with these properties is used in [37]). In fact, they are likely to arise when one considers suitable homotopies between two equations of the form (1.4): one with a nondegenerate ground state of Morse index 2 and the other one with a unique ground state of Morse index 1. The nondegenerate ground state can often be continued up to a bifurcation point with a degenerate ground state, so there is a good chance that a part of the homotopy will give a function $f(u; \lambda)$ with the desired properties. We speculate that such a scenario plays out in equations with some nonlinearities (1.7) when λ is decreased or increased from a fixed value $\lambda = \hat{\lambda}_0$ for which a nondegenerate ground state of Morse index 2 exists. The analysis in [14] strongly suggests that this is indeed the case for suitable $p < 5$, $p \approx 5$, and $N = 3$. Note, however, that to make use of Theorem 1.3 we need f to be analytic, hence we are bound to take integer exponents p and q. With $p = 5$ (the critical exponent) and $q = 2$, say, [14] still provides some evidence, partly numerical in this case, that the family of ground states as in (GP) exists. If this is confirmed, Theorem 1.3 yields quasiperiodic partially localized positive solutions of some equations with combined-powers nonlinearities.

As will become transparent in Subsection 2.3, the only role of the degenerate ground state ϕ^{λ_0} in (GP)(c3) is to ensure that a certain function of λ is nonconstant, and this nonconstancy can be used in place of condition (c3). Specifically, assuming conditions (c1) and (c2) in (GP), let $\mu_1(\lambda) < \mu_2(\lambda)$ be the two negative eigenvalues of the Schrödinger operator $-\Delta - f_u(\phi^\lambda(x);\lambda)$ (acting on $L^2_{\rm rad}(\mathbb{R}^N)$), for $\lambda \in (\lambda_0, \hat\lambda_0)$. They are defined due to (c2). Now assume the following condition.

(c3)' The function $\lambda \mapsto \mu_1(\lambda)/\mu_2(\lambda)$ is nonconstant on $(\lambda_0, \hat\lambda_0)$.

Theorem 1.4. *Theorem 1.3 remains valid if condition* (c3) *in* (GP) *is replaced by condition* (c3)'.

It is not difficult to show that condition (c3) implies condition (c3)' (see Subsection 2.3). We have stated one of our theorems with (c3) as a hypothesis because it is more explicit than (c3)', and, as indicated above, it may be relatively easy to verify for equations of the form (1.1) for which one has some information about their ground states. Condition (c3)', on the other hand, is more general, and it only involves nondegenerate ground states. For this reason, (c3)' appears to be a robust condition which is likely to hold in a "typical" application. Its verification in specific equations may not be easy, however.

It is clear from the above results and discussion that the existence of a ground state of (1.4) with Morse index greater than 1 is an essential prerequisite for our results on quasiperiodic partially localized solutions. Now, for some important classes of nonlinearities, including for instance the function $f(u) = u^p - u$ with any $p > 1$, the ground state of (1.4) is unique up to translations if it exists (see [8, 10, 25, 26, 33, 42]). In that case, there is no ground state of (1.4) with Morse index greater than 1 (see [11] or the introduction in [34] for a discussion of this point). The same goes for any equation (1.1) if $N = 1$. By elementary considerations, the ground state of (the ordinary differential equation) (1.4) is unique up to translations and has Morse index 1. The problem whether positive quasiperiodic partially localized solutions can exist in such equations cannot be resolved by our current method.

We remark that it is likely that all positive partially localized solutions are radially symmetric in x about some center in $\mathbb{R}^N$, cp. [6, 17, 21], although this has not been proved in full generality yet. In our theorems, we only consider solutions that are radial in x.

Positive partially localized solutions are but one class of solutions of (1.1) which are not fully localized, and other types of interesting solutions have been studied by a number of authors. We mention saddle-shaped and multiple-end solutions [7, 13, 15, 16, 23], solutions with infinitely many bumps and/or fronts formed along some directions [28, 41], solutions periodic and/or discretely symmetric in the x-variables with homoclinic or heteroclinic transitions in the y variable [3, 31, 39], solutions whose limit profiles at infinity are given by ground states in lower dimensions [27], as well as solutions periodic in at least one variable and quasiperiodic in another variable [35].

We have organized the rest of this paper as follows. In the next section, we consider equations depending on parameters and give sufficient conditions for the existence of partially localized quasiperiodic solutions. In the same section, we give a proof of Theorems 1.3, 1.4, and related results in finite-differentiability settings. Theorem 1.2 is proved in Section 3.

2 Equations with parameters

In this section, we first recall a result from [35] dealing with a class of (possibly nonhomogeneous) elliptic problems with parameters. The theorem gives sufficient conditions for the existence of partially localized quasiperiodic solutions. We then show how equation (1.1) can be put in the context of such elliptic problems via the linearization of (1.1), or its parameter dependent version, at a ground state. We examine the behavior of negative eigenvalues of such a linearization as parameters are varied, which is a crucial ingredient in the proofs of our theorems. Finally, we specifically consider the case of a single parameter and prove existence of quasiperiodic solutions with 2 frequencies in some settings.

When considering a radial function h on $\mathbb{R}^N$, we often abuse the notation slightly and use the same symbol h in $h = h(x)$ (viewing h as a function of $x \in \mathbb{R}^N$) as well as in $h = h(r)$ (viewing h as a function of $r = |x|$).

2.1 A general setup

Consider the following equation with a parameter $s \in \mathbb{R}^d$, $s \approx 0$:

$$\Delta u + u_{yy} + a(x; s)u + f_1(x, u; s) = 0, \quad x \in \mathbb{R}^N,\ y \in \mathbb{R}. \tag{2.1}$$

Here f_1 is a function on $\mathbb{R}^N \times \mathbb{R} \times B$, B being an open neighborhood of the origin in $\mathbb{R}^d$, such that

$$f_1(x, 0; s) = \frac{\partial}{\partial u} f_1(x, u; s)\,\Big|_{u=0} = 0 \quad (x \in \mathbb{R}^N,\ s \in B). \tag{2.2}$$

To formulate our hypotheses on the functions a and g, we need to introduce some notation. We denote by $C_{\mathrm{b}}(\mathbb{R}^N)$ the space of all continuous bounded (real-valued) functions on $\mathbb{R}^N$ and by $C^k_{\mathrm{b}}(\mathbb{R}^N)$ the space of functions on $\mathbb{R}^N$ with continuous bounded derivatives up to order k, $k \in \mathbb{N} := \{0, 1, 2, \dots\}$. By $C_{\mathrm{rad}}(\mathbb{R}^N)$, $C^k_{\mathrm{rad}}(\mathbb{R}^N)$ we denote the subspaces of $C_{\mathrm{b}}(\mathbb{R}^N)$ and $C^k_{\mathrm{b}}(\mathbb{R}^N)$, respectively, consisting of the functions which are radially symmetric in x; $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ is the space of all radial $L^2(\mathbb{R}^N)$-functions, and for $k \in \mathbb{N}$, $H^k_{\mathrm{rad}}(\mathbb{R}^N) := H^k(\mathbb{R}^N) \cap L^2_{\mathrm{rad}}(\mathbb{R}^N)$ is the space of all radial $H^k(\mathbb{R}^N)$-functions. When needed, we assume that these spaces are equipped with the usual norms and take the induced norms on the subspaces.

Given integers $n > 1$ and $d \geq n - 1$, let B be an open neighborhood of the origin in $\mathbb{R}^d$. We assume that the functions a and f_1 satisfy the following hypotheses with some integers

$$K > 4n + 1, \quad m > \frac{N}{2}. \tag{2.3}$$

(S1) $a(\cdot; s) \in C^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ for each $s \in B$, and the map $s \in B \mapsto a(\cdot; s) \in C^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ is of class C^{K+1}.

(S2) $f_1 \in C^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times B)$, and for each $\vartheta > 0$ the function f_1 is bounded on $\mathbb{R}^N \times [-\vartheta, \vartheta] \times B$ together with all its partial derivatives up to order $K+m+4$. Also, (2.2) holds and $f_1(x, u; s)$ is radially symmetric in x.

The next hypotheses concern the Schrödinger operator $A_1(s) := -\Delta - a(x; s)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$.

(A1)(a) There exists $L < 0$ such that

$$\limsup_{|x|\to\infty} a(x; s) \le L \quad (s \in B).$$

(A1)(b) For all $s \in B$, $A_1(s)$ has exactly n nonpositive eigenvalues,

$$\mu_1(s) < \mu_2(s) < \cdots < \mu_n(s), \tag{2.4}$$

and $\mu_n(s) < 0$.

Hypotheses (A1)(a) and (A1)(b) will collectively be referred to as (A1). Hypothesis (A1)(a) guarantees that for all s the essential spectrum $\sigma_{ess}(A_1(s))$ is contained in $[-L, \infty)$ (see [40]). Since we work in the radially symmetric setting, the eigenvalues (2.4) are all simple, while $-L > 0$, hypothesis (S1) and the simplicity of the eigenvalues in (A1)(b) imply that $\mu_1(s), \ldots, \mu_n(s)$ are C^{K+1} functions of s (see [22]).

We further assume the following nondegeneracy condition. Consider the map $s \mapsto \omega(s) := (\omega_1(s), \ldots, \omega_n(s))^T$ ($\omega(s)$ is a column vector), where

$$\omega_j(s) := \sqrt{|\mu_j(s)|}, \quad j = 1, \ldots, n. \tag{2.5}$$

(ND) The $n \times (d+1)$ matrix $\begin{bmatrix} \nabla\omega(0) & \omega(0) \end{bmatrix}$ has rank n.

The following theorem is a minor reformulation of Theorem 2.5 of [35]. (We remark that condition (ND) also appears in a theorem of [43] on quasiperiodic solutions of elliptic equations on a 2-dimensional strip.)

Theorem 2.1. *Let K and m be as in* (2.3). *Assume that hypotheses* (S1), (S2), (A1), (ND) *are satisfied. Then there is an uncountable set $W \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $(\bar\omega_1, \ldots, \bar\omega_n) \in W$ the following holds: equation* (2.1) *has for some $s \in B$ a solution u such that* (1.3) *holds, and $u(x, y)$ is radially symmetric in x and quasiperiodic in y with frequencies $\bar\omega_1, \ldots, \bar\omega_n$.*

2.2 Spatially homogeneous equations

In this subsection, we assume that n, d, and ℓ are fixed integers satisfying

$$n > 1, \quad d \geq n-1, \quad \ell > N/2 + 4n + 7. \tag{2.6}$$

Denoting by B_δ the open ball around the origin in $\mathbb{R}^d$ of radius δ, we also assume

$$\begin{gathered}\text{for some } \delta > 0,\ f : \mathbb{R} \times B_\delta \to \mathbb{R} \text{ is a } C^\ell \text{ function such that} \\ f(0;s) = 0,\ f_u(0;s) < 0 \quad (s \in B_\delta).\end{gathered} \tag{2.7}$$

Our goal is to show how Theorem 2.1 can be applied in the spatially homogeneous equation

$$\Delta u + u_{yy} + f(u;s) = 0, \quad (x,y) \in \mathbb{R}^{N+1}, \tag{2.8}$$

where $s \in B_\delta$ serves as a parameter. The associated equation for the ground states on $\mathbb{R}^N$ is

$$\Delta u + f(u;s) = 0, \quad x \in \mathbb{R}^N. \tag{2.9}$$

We formulate two additional hypotheses. The first one concerns the equation for $s = 0$ only.

(G0) Equation (2.9) with $s = 0$ has a nondegenerate ground state ϕ^0 with Morse index n.

To formulate our second hypothesis, which involves equation (2.9) for $s \approx 0$, we need some preparation. Denote by $C_{\mathrm{rad},0}(\mathbb{R}^N)$ the closed subspace of $C_{\mathrm{rad}}(\mathbb{R}^N)$ consisting of the functions converging to 0 as $|x| \to \infty$; as usual we assume the induced norm (the supremum norm) on $C_{\mathrm{rad},0}(\mathbb{R}^N)$. Condition (G0) implies, upon an application of the implicit function theorem, that the following statement is valid (see Lemma 2.3 below for a more detailed statement), possibly after the radius $\delta > 0$ is shrunk.

(Gs) There is neighborhood U of ϕ^0 in $C_{\mathrm{rad},0}(\mathbb{R}^N)$ such that for each $s \in B_\delta$ equation (2.9) has a unique ground state ϕ^s in U; this ground state is nondegenerate with Morse index n; and the map $s \mapsto \phi^s : B_\delta \to C_{\mathrm{rad},0}(\mathbb{R}^N)$ is of class C^ℓ.

The fact that the ground states ϕ^s are nondegenerate and have Morse index n means that for each $s \in B_\delta$ the Schrödinger operator $-\Delta - f_u(\phi^s(x);s)$ (acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$) has exactly n negative eigenvalues

$$\mu_1(s) < \mu_2(s) < \cdots < \mu_n(s), \tag{2.10}$$

and 0 is not its eigenvalue. The eigenvalues are of class $C^{\ell-1}$ as functions of $s \in B_\delta$. Consider now the $n \times (d+1)$ matrix

$$M(s) := \begin{bmatrix} \nabla\omega(s) & \omega(s) \end{bmatrix}, \quad s \in B_\delta, \tag{2.11}$$

where $\omega(s) = (\omega_1(s), \ldots, \omega_n(s))^T$ and $\omega_j(s) := \sqrt{-\mu_j(s)}$ $(j = 1, \ldots, n)$. We impose on this matrix the same condition as in the previous subsection; this is our second hypothesis:

(ND0) The matrix $M(0)$ has rank n.

The main result of this subsection is the following theorem.

Theorem 2.2. *Assume that n, d, and ℓ satisfy* (2.6)*, f satisfies* (2.7)*, and* (G0) *and* (ND0) *hold. Then there is an uncountable set $W \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $(\bar{\omega}_1, \ldots, \bar{\omega}_n) \in W$ equation* (2.8) *has for some $s \in B_\delta$ a solution u such that* (1.3) *holds, and $u(x, y)$ is radially symmetric in x and quasiperiodic in y with frequencies $\bar{\omega}_1, \ldots, \bar{\omega}_n$.*

For the proof of the theorem, we need some regularity statements from the following lemma (the analyticity statement in this lemma will be needed in the next subsection).

Lemma 2.3. *Assume that n, d, and ℓ satisfy* (2.6)*, f satisfies* (2.7)*, and* (G0) *holds. Then, possibly after $\delta > 0$ is made smaller, there is a neighborhood U of ϕ^0 in $C_{\mathrm{rad},0}(\mathbb{R}^N)$ and a family ϕ^s, $s \in B_\delta$ such that the following statements are valid:*

- *(i) For each $s \in B_\delta$, ϕ^s is a unique ground state of* (2.9) *in U.*
- *(ii) The map $s \mapsto \phi^s \in C_{\mathrm{rad},0}(\mathbb{R}^N)$ is of class C^ℓ and it is analytic if the function $f : \mathbb{R} \times B_\delta \to \mathbb{R}$ is analytic.*
- *(iii) The function $(x, s) \mapsto \phi^s(x)$ is of class C^ℓ, and it is bounded on $\mathbb{R}^N \times B_\delta$ together with all its partial derivatives up to order ℓ.*
- *(iv) For each $s \in B_\delta$, the ground state ϕ^s is nondegenerate and has Morse index n.*

Proof. Set $a(x) := f_u(\phi^0(x); 0)$. By (G0), the operator $-\Delta - a(x)$ (considered on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$) has exactly n negative eigenvalues, all simple, and 0 is not its eigenvalue. Also, due to the decay of the ground states, $a(x) \to f_u(0; 0) < 0$ as $|x| \to \infty$, so the essential spectrum of $-\Delta - a(x)$ is contained in a half-line (κ, ∞) for some $\kappa > 0$. These properties are preserved under small L^∞ perturbations of the function a. Therefore, statement (iv) is a direct consequence of statement (ii), once the latter is established.

We now consider the operator $-\Delta - a(x)$ in a different setting, namely, as a closed operator on the space $X := C_{\mathrm{rad},0}(\mathbb{R}^N)$. The X-realization of $-\Delta - a(x)$ is the operator L with domain

$$D(L) := \{u \in \cap_{p>1} W^{2,p}_{loc}(\mathbb{R}^N) : u, \Delta u \in X\}.$$

given by $Lv = -\Delta v - av$. For this realization, it is still true that 0 is not in its spectrum (the essential spectrum is still away from 0 and 0 is not an eigenvalue by elliptic regularity), so L^{-1} is a bounded linear operator on X. We now rewrite equation (2.9) as an equation for $u \in X$:

$$H(u; s) := u - L^{-1}(\tilde{f}(u; s) - au) = 0, \tag{2.12}$$

where $\tilde{f}$ is the Nemytskii operator of the function f:

$$\tilde{f}(u; s)(x) := f(u(x); s) \quad (u \in X,\ s \in B_\delta,\ x \in \mathbb{R}^N).$$

It is well-known (and straightforward to prove) that the assumptions on f imply that $\tilde{f} : X \times B_\delta \to X$ is of class C^ℓ. Moreover, if f is analytic, so is $\tilde{f}$. This can be easily verified (cp. [5]) using bounds on the derivatives of analytic functions and the fact that the ranges of all functions contained in any ball in X are contained in a compact subset of $\mathbb{R}$. Clearly, $H(\phi^0; 0) = 0$ and $D_u H(\phi^0; 0)$ is the identity on X. Thus, the implicit function theorem applies to H, which yields a neighborhood U of ϕ^0 in $C_{\mathrm{rad},0}(\mathbb{R}^N)$ and—making $\delta > 0$ smaller if necessary—a family ϕ^s, $s \in B_\delta$, such that statements (i) and (ii) hold.

We now show by induction in $k = 0, 1, \dots, \ell$ that the following statement is valid. The function $\phi^s(x)$ is of class C^k on $\mathbb{R}^N \times B_\delta$ and all its partial derivatives of order k are bounded on $\mathbb{R}^N \times \bar{B}_\delta$ (with $\delta > 0$ made smaller if necessary). This will prove statement (iii) and complete the proof of the theorem.

For $k = 0$, the statement follows immediately from (ii).

Assume the statement is valid for some $k < \ell$. Let $\tilde{\delta}$ stand for any partial derivative with respect to $x_1, \dots, x_N, s_1, \dots, s_d$ of order k; that is, $\tilde{\delta}$ is a "product" of k elements from $\{\delta_{x_1}, \dots, \delta_{x_N}, \delta_{s_1}, \dots, \delta_{s_d}\}$. All we need to show is that the function $\tilde{\delta}\phi^s(x)$ is of class C^1 and has bounded first-order partial derivatives on $\mathbb{R}^N \times B_\delta$.

We use an integral representation of the solutions of the equation $u - \Delta u = h(x)$ on $\mathbb{R}^N$. Let $G(x)$ be the Green function for the elliptic operator $I - \Delta$ on $\mathbb{R}^N$. An explicit form (for dimensions $N = 2, 3$) or a Bessel potential form of G are available, but are not needed here. We recall some properties of G which are relevant for us. The function G is smooth in $\mathbb{R}^N \setminus \{0\}$, and the functions G, $\partial_{x_i} G$, $i = 1, \dots, N$ (classical derivatives on $\mathbb{R}^N \setminus \{0\}$) are integrable on $\mathbb{R}^N$. For any bounded continuous function h, the convolution integral

$$u(x) = \int_{\mathbb{R}^N} G(x - y) h(y)\, dy = \int_{\mathbb{R}^N} G(y) h(x - y)\, dy \tag{2.13}$$

defines a continuous function u which is a unique bounded weak solution of the equation $u - \Delta u = h$ on $\mathbb{R}^N$. Moreover, $u \in C^1_{\mathrm{b}}(\mathbb{R}^N)$ and for $i = 1, \dots, N$ one has

$$\partial_{x_i} u(x) = \int_{\mathbb{R}^N} \partial_{x_i} G(x - y) h(y)\, dy. \tag{2.14}$$

Except perhaps for the last statement, these are standard properties of Green's functions of general elliptic operators with constant coefficients (see, for example, [24, Chapter 1]). For the proof of (2.14) and the C^1 property of u, one can use the estimates on the derivatives of G (see Corollary 1.5.1 and Theorem 1.7.1 in [24]) and follow the arguments given in the proof of [19, Lemma 4.1].

Denoting

$$h(x;s) := f(\phi^s(x);s) + \phi^s(x), \tag{2.15}$$

and applying the above to ϕ^s, a bounded solution of $u - \Delta u = f(u;s) + u$, we obtain

$$\phi^s(x) = \int_{\mathbb{R}^N} G(x-y)h(y;s)\,dy = \int_{\mathbb{R}^N} G(y)h(x-y;s)\,dy. \tag{2.16}$$

Note that the induction hypothesis implies that h is of class C^k on $\mathbb{R}^N \times B_\delta$ and has all its partial derivatives of order k bounded on $\mathbb{R}^N \times B_\delta$.

Let us now return to the function $\tilde{\delta}\phi^s(x)$. Clearly, due to the integrability of G, we can differentiate the second integral in (2.16) to obtain

$$\tilde{\delta}\phi^s(x) = \int_{\mathbb{R}^N} G(y)\tilde{\delta}h(x-y;s)\,dy = \int_{\mathbb{R}^N} G(x-y)\tilde{\delta}h(y;s)\,dy.$$

By the above remarks, we can next take the derivatives with respect to x_i, $i \in \{1,\dots,N\}$, to obtain

$$\partial_{x_i}\tilde{\delta}\phi^s(x) = \int_{\mathbb{R}^N} \partial_{x_i}G(x-y)\tilde{\delta}h(y;s)\,dy = \int_{\mathbb{R}^N} \partial_{x_i}G(y)\tilde{\delta}h(x-y;s)\,dy.$$

Using the integrability of $\partial_{x_i}G$, the continuity and boundedness properties of $\tilde{\delta}h$, and the dominated convergence theorem, one shows easily that $\partial_{x_i}\tilde{\delta}\phi^s(x)$ is continuous and bounded on $\mathbb{R}^N \times B_\delta$.

We now deal with the derivatives $\partial_{s_j}\tilde{\delta}\phi^s(x)$, $j = 1,\dots,d$. We obtain the desired continuity and boundedness of these derivatives directly from statement (ii) if $\tilde{\delta}$ contains no derivatives with respect to the variables $x_1,\dots,x_N$. Otherwise, if $\tilde{\delta}$ contains at least one derivative δ_{x_i} for some i, we have, changing the order of the partial derivatives in $\tilde{\delta}$ if necessary, $\tilde{\delta}\phi^s(x) = \delta_{x_i}\hat{\delta}\phi^s(x)$, where $\hat{\delta}$ is a partial derivative of order $k-1$. Differentiating as above, we obtain, first,

$$\tilde{\delta}\phi^s(x) = \int_{\mathbb{R}^N} \partial_{x_i}G(y)\hat{\delta}h(x-y;s)\,dy,$$

and then

$$\partial_{s_j}\tilde{\delta}\phi^s(x) = \int_{\mathbb{R}^N} \partial_{x_i}G(y)\partial_{s_j}\hat{\delta}h(x-y;s)\,dy.$$

Arguing as above, we obtain the desired continuity and boundedness properties of these functions as well. □

Proof of Theorem 2.2. Modifying the nonlinearity $f(u; s)$ in $\{(u, s) : u < 0\}$ only, we will assume that the following additional condition holds:

$$f(u; s) > 0 \quad (u < 0,\ s \in B_\delta). \tag{2.17}$$

This is at no cost to generality as positive solutions are unaffected by such a modification. What we gain from this extra assumption is that all bounded solutions of (2.8) are nonnegative, as one can easily show by employing negative constant subsolutions. By the strong maximum principle, any nonnegative solution is either identical to zero or strictly positive.

With ϕ^s as in (Gs), set

$$a(x; s) := f_u(\phi^s(x); s), \tag{2.18}$$
$$f_1(x, u; s) := f(\phi^s(x) + u; s) - f(\phi^s(x); s) - a(x; s)u. \tag{2.19}$$

We verify that these functions satisfy the hypotheses of Theorem 2.1 with $K := 4n+2$, $m := \ell - 4n - 7 > N/2$ (cp. (2.6), (2.3)). Obviously, f_1 satisfies (2.2). Since ground states are radially symmetric in x, so are the functions a and f_1. Our choices of K and m yield $\ell = K + m + 5$; the regularity assumption on f and Lemma 2.3(iii) imply the regularity properties in (S1), (S2) with $B = B_{\delta/2}$ (so that $\bar{B} \subset B_\delta$). The decay of the ground states and the second condition in (2.7) imply that (A1)(a) holds, possibly after $\delta > 0$ is made smaller. Condition (A1)(b) holds, as already noted before the theorem (cp. (2.10)), and (ND0), which is a hypothesis of this theorem, is equivalent to (ND). Thus, the hypotheses of Theorem 2.1 are all satisfied.

Now, with a and f_1 as in (2.18), (2.19), $u = u(x, y)$ is a solution of (2.8) for some $s \in B_\delta$ if (and only if) $u = \phi^s + \tilde{u}$ for a solution $\tilde{u}$ of (2.1) (with the same s). Since ϕ^s is a radial (in x) function, independent of y and satisfying $\phi^s(x) \to 0$ as $|x| \to \infty$, the function $u(x, y)$ is quasiperiodic in y, radially symmetric in x, and decaying to 0 as $|x| \to \infty$ uniformly in y, if $\tilde{u}$ has all these properties. In this case, u and $\tilde{u}$ share the quasiperiodicity frequencies. Therefore, the conclusion of Theorem 2.2 follows from Theorem 2.1; we just need to note that the solutions obtained this way are positive. Indeed, they are bounded hence nonnegative due to (2.17), and, being quasiperiodic in the sense of our definition (in particular, not periodic), they are nonzero, hence strictly positive. □

Remark 2.4. (i) Note that hypotheses (G0), (ND0) are "local" (in fact, they are conditions on ϕ^s and $M(s)$ at $s = 0$ only, but because of the gradient involved in the definition of $M(s)$, we need to consider $\omega(s)$ for $s \approx 0$). Therefore, the conclusion of the theorem remains valid when $\delta > 0$ is shrunk arbitrarily. This is useful for density results such as Theorems 1.2 and 1.3.

(ii) The assumption that $M(0)$ has rank n can be replaced by the assumption that there is a sequence s_j converging to the origin such that $M(s_j)$ has rank n for $j = 1, 2, \ldots$. The conclusion of Theorem 2.2 and the previous remark remain valid under this weaker assumption. To see this, simply apply Theorem 2.2 for $j = 1, 2, \dots,$ with s_j taking up the role of the origin, that is, with the function $f(u; s_j + s)$ in place of $f(u; s)$.

(iii) In this paper, we do not have much use of the property that no two vectors in the frequency set W are linearly dependent. This is more meaningful in some scaling invariant problems, such as those considered in [35].

(iv) While Theorems 2.1, 2.2 state that an uncountable set of quasiperiodic solutions (whose frequencies form an uncountable set W) can be found within a given parametric family of equations, the theorems do not say anything about the multiplicity of solutions for any single equation. Since the parameters can take uncountably many values, the existence of uncountably many solutions for any single one of them is not guaranteed. This is the reason for the lack of any multiplicity statement in Theorems 1.1, 1.2.

For the verification of condition (ND0) in applications, some understanding of the partial derivatives of the functions $s \mapsto \mu_j(s)$ at $s = 0$ is needed. The rest of this subsection is devoted to a computation of these derivatives.

Denote by $\psi_1(\cdot; s), \dots, \psi_n(\cdot; s)$ the eigenfunctions of the operator $-\Delta - f_u(\phi^s(x); s)$ (acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$) associated with the eigenvalues $\mu_1(s), \dots, \mu_n(s)$, respectively, all normalized in the $L^2(\mathbb{R}^N)$ norm. This determines the eigenfunctions uniquely up to a sign. The signs can be chosen in such a way that the eigenfunctions are of class C^1 as $H^2_{\mathrm{rad}}(\mathbb{R}^N)$-valued functions of $s \in B_\delta$, and this is what we will assume below. We derive the following formulas:

Proposition 2.5. *Under the hypotheses of Theorem 2.2, the following relations hold for $i = 1, \dots, d$, $j = 1, \dots, n$:*

$$\frac{\partial \mu_j(s)}{\partial s_i}\Big|_{s=0} = -\int_{\mathbb{R}^N} \left(f_{uu}(\phi^0(x); 0)\dot{\phi}_i(x) + g_i'(\phi^0(x)) \right) (\psi_j(x; 0))^2 \, dx, \tag{2.20}$$

where

$$g_i(u) = \frac{\partial f(u; s)}{\partial s_i}\Big|_{s=0} \quad (u \in \mathbb{R},\ i = 1, \dots, d), \tag{2.21}$$

and $\dot{\phi}_i \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$ is the unique solution of the equation

$$\Delta \dot{\phi}_i + f_u(\phi^0(x); 0)\dot{\phi}_i + g_i(\phi^0(x)) = 0. \tag{2.22}$$

We remark that the existence and uniqueness of the solution $\dot{\phi}_i$ is a consequence of the nondegeneracy of the ground state ϕ^0. Note that $g_i(0) = 0$ and $g_i \in C^{\ell-1}$, which implies that $g_i(\phi^0(x))$ decays exponentially as $|x| \to \infty$, just like $\phi^0(x)$, and is therefore in $L^2_{\mathrm{rad}}(\mathbb{R}^N)$.

Proof of Proposition 2.5. We first simplify the notation slightly. Clearly, it is sufficient to consider the case $d = 1$ of just one parameter s (the others being fixed). Also, since only the first derivative of $f(u; s)$ with respect to s at $s = 0$ enters the computation and f is of class C^ℓ in all its arguments, it is sufficient to take the nonlinearity $f(u; s)$ in (2.9) in the form $f(u) + sg(u)$, where we have written $g_1 = g$.

Substituting $u = \phi^s$ in (2.9), we differentiate the equation with respect to s at $s = 0$, noting that this operation can be performed thanks to Lemma 2.3. We obtain the equation for $\dot{\phi} = d\phi^s/ds\,\big|_{s=0}$, which reads as (2.22) (with $g_i = g$):

$$\Delta\dot{\phi} + f'(\phi^0(x))\dot{\phi} + g(\phi^0(x)) = 0. \tag{2.23}$$

Next, consider the equation for the eigenfunction $\psi_j(\cdot; s)$:

$$\Delta\psi_j + \big(f'(\phi^s(x)) + sg'(\phi^s(x))\big)\psi_j + \mu_j(s)\psi_j = 0. \tag{2.24}$$

Differentiating with respect to s at $s = 0$, we obtain

$$\begin{aligned}\Delta\dot{\psi}_j + f'(\phi^0(x))\dot{\psi}_j + \mu_j(0)\dot{\psi}_j \\ + \big(f''(\phi^0(x))\dot{\phi}(x) + g'(\phi^0(x))\big)\psi_j(x; 0) + \dot{\mu}_j\psi_j(x; 0) = 0,\end{aligned} \tag{2.25}$$

where

$$\dot{\psi}_j = \frac{d\psi_j(\cdot; s)}{ds}\,\Big|_{s=0}, \quad \dot{\mu}_j = \frac{d\mu_j(s)}{ds}\,\Big|_{s=0}.$$

Also, by the L^2 normalization of $\psi_j(\cdot; s)$,

$$\int_{\mathbb{R}^N} \psi_j(x; 0)\dot{\psi}_j(x)\,dx = 0.$$

Multiplying equation (2.25) by $\psi_j(x; 0)$ and integrating by parts over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} \Big(f''(\phi^0(x))\dot{\phi}(x) + g'(\phi^0(x))\Big)\,(\psi_j(x; 0))^2\,dx + \dot{\mu}_j = 0.$$

This verifies formula (2.20). □

In the radial variable, the integrals in (2.20) read as follows:

$$\frac{\partial\mu_j(s)}{\partial s_i}\,\Big|_{s=0} = \sigma_N \int_0^\infty \Big(f_{uu}(\phi^0(r); 0)\dot{\phi}_i(r) + g_i'(\phi^0(r))\Big)\,(\psi_j(r; 0))^2 r^{N-1}\,dr, \tag{2.26}$$

where σ_N is the surface area of the unit sphere in $\mathbb{R}^N$.

Remark 2.6. Clearly, condition (ND0) is satisfied if the matrix $\big[\nabla\omega(0)\big]$ has rank n, and this is the case, due to the relations $\omega_j(s) := \sqrt{-\mu_j(s)}$, if the $n \times d$ matrix

$$\left[\frac{\partial\mu_j(s)}{\partial s_i}\,\Big|_{s=0}\right]_{j,i},$$

whose entries are given in (2.26), has rank n.

2.3 Two frequencies, one parameter

Obviously, for the matrix in (2.11) to have rank n the number of parameters has to be at least $n-1$, thus the assumption $d \geq n-1$ in the previous subsection. When $n = 2$, that is, when quasiperiodic solutions with two frequencies are sought, just one parameter is sufficient, which has some advantages. In this subsection, we prove a few results, including Theorem 1.3, specific to the case $n = 2$.

For now, we continue to assume the hypotheses from the first paragraph of the previous section (cp. (2.6), (2.7)), taking $n = 2$ and $d = 1$. Also, we assume condition (G0), define the eigenvalues $\mu_1(s)$, $\mu_2(s)$ as in (2.10), and take $\omega_j(s) := \sqrt{-\mu_j(s)}$ ($j = 1, 2$).

Consider first of all the determinant of the 2×2 matrix $M(s)$ in (2.11):

$$\begin{aligned} \det M(s) = \det \begin{bmatrix} \omega_1'(s) & \omega_1(s) \\ \omega_2'(s) & \omega_2(s) \end{bmatrix} &= \omega_1(s)\omega_2(s) \left(\frac{\omega_1'(s)}{\omega_1(s)} - \frac{\omega_2'(s)}{\omega_2(s)} \right) \\ &= \omega_1(s)\omega_2(s) \left(\log \frac{\omega_1(s)}{\omega_2(s)} \right)' \\ &= \frac{\sqrt{\mu_1(s)\mu_2(s)}}{2} \left(\log \frac{\mu_1(s)}{\mu_2(s)} \right)'. \end{aligned} \tag{2.27}$$

In view of this expression, Theorem 2.2 for $n = 2$ implies the following result.

Theorem 2.7. *Assume that the hypotheses of Theorem 2.2 are satisfied with $n = 2$, $d = 1$, and with* (ND0) *replaced by the following condition: the function $s \mapsto \mu_1(s)/\mu_2(s)$ is not constant on any interval $(-\epsilon, \epsilon)$ with $\epsilon \in (0, \delta)$. Then there is a sequence $\bar{s}_j \to 0$ such that for $j = 1, 2, \dots$ the following holds. Equation* (2.8) *with $s = \bar{s}_j$ has a positive solution u satisfying* (1.3) *such that $u(x, y)$ is radially symmetric in x and quasiperiodic in y (with 2 frequencies).*

Proof. By (2.27), the assumption on the function $s \mapsto \mu_1(s)/\mu_2(s)$ implies that there is a sequence $s_j \to 0$ such that $\det M(s_j) \neq 0$, that is, $M(s_j)$ has rank 2. The conclusion of the theorem now follows from Theorem 2.2 and Remarks 2.4(i),(ii): we choose a sequence $\delta_j \to 0$, with $0 < \delta_j < \delta$ and apply Theorem 2.2 with $f(u; s_j + s)$, δ_j in place of $f(u; s)$, δ, respectively. We then take $\bar{s}_j$ to be any number $s \in B_{\delta_j}$ as in the conclusion of Theorem 2.2. □

Using the previous theorem, we now prove Theorems 1.3, 1.4, and related local results for C^ℓ nonlinearities.

We consider equations (1.8), (1.10) with $f(u; \lambda)$ satisfying (1.9). Assume for now that $f(u; \lambda)$ is of class C^ℓ with $\ell > N/2+15$ (as in (2.6)), and the following assumption (GP), copied here from the introduction, is satisfied:

(GP) There are positive constants $\lambda_0 < \hat{\lambda}_0$ such that for each $\lambda \in [\lambda_0, \hat{\lambda}_0)$ equation (1.10) has a ground state ϕ^λ such that the following conditions are satisfied:

(c1) The map $\lambda \mapsto \phi^\lambda : [\lambda_0, \hat{\lambda}_0) \to L^\infty(\mathbb{R}^N)$ is continuous;

(c2) for each $\lambda \in (\lambda_0, \hat{\lambda}_0)$, ϕ^λ is a nondegenerate ground state with Morse index 2;

(c3) ϕ^{λ_0} is degenerate ground state with Morse index 1.

Theorem 2.8. *Under the above assumptions, there is a sequence $\bar{\lambda}_j$ in $(\lambda_0, \hat{\lambda}_0)$ such that $\bar{\lambda}_j \to \lambda_0$ and for $j = 1, 2, \ldots$ the following holds. Equation* (1.8) *with $\lambda = \bar{\lambda}_j$ has a positive solution u satisfying* (1.3) *such that $u(x, y)$ is radially symmetric in x and quasiperiodic in y.*

Proof. Denote by $\bar{\mu}_1(\lambda)$, $\bar{\mu}_2(\lambda)$, with $\bar{\mu}_1(\lambda) < \bar{\mu}_2(\lambda)$, the two nonpositive eigenvalues of the Schrödinger operator $-\Delta - f_u(\phi^\lambda(x); \lambda)$ (acting on L^2_{rad}). By (c2) and (c3), $\bar{\mu}_2(\lambda) < 0$ for $\lambda \in (\lambda_0, \hat{\lambda}_0)$ and $\bar{\mu}_2(\lambda) = 0$. By (c1), $\bar{\mu}_1(\lambda)$, $\bar{\mu}_2(\lambda)$ are continuous as functions of λ, on the interval $[\lambda_0, \hat{\lambda}_0)$. Moreover, there is a constant $\gamma < 0$ such that $\bar{\mu}_1(\lambda) \le \gamma < 0$.

In view of the nondegeneracy of the ground states in (GP) and the continuity in (c1), the implicit function theorem implies (cp. Lemma 2.3) that the map in (c1) is of class C^ℓ on the open interval $(\lambda_0, \hat{\lambda}_0)$, which in turn implies that the map $\lambda \mapsto f_u(\phi^\lambda(\cdot); \lambda) \in C_{\mathrm{b}}(\mathbb{R}^N)$ is of class $C^{\ell-1}$. It then follows that the functions $\bar{\mu}_1(\lambda)$, $\bar{\mu}_2(\lambda)$ are of class $C^{\ell-1}$ on $(\lambda_0, \hat{\lambda}_0)$. These functions being continuous on $[\lambda_0, \hat{\lambda}_0)$, the relations $\bar{\mu}_1(\lambda) < \gamma$, $\bar{\mu}_2(\lambda_0) = 0$, and $\bar{\mu}_2(\lambda) < 0$ for $\lambda \in (\lambda_0, \hat{\lambda}_0)$ clearly imply that there is a sequence λ_j in $(\lambda_0, \hat{\lambda}_0)$ such that $\lambda_j \to \lambda_0$ and $\bar{\mu}_1(\lambda)/\bar{\mu}_2(\lambda)$ has a nonzero derivative at $\lambda = \lambda_j$, for $j = 1, 2, \ldots$. Hence, for each j, Theorem 2.7 applies to the equation

$$\Delta u + u_{yy} + f(u; \lambda_j + s) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.28}$$

$s \approx 0$. Indeed, the eigenvalues $\mu_1(s)$, $\mu_2(s)$ of $-\Delta - f_u(\phi^{\lambda_j+s}(x); \lambda_j + s)$ clearly coincide with $\bar{\mu}_1(\lambda_j + s)$, $\bar{\mu}_2(\lambda_j + s)$, so the hypotheses of Theorem 2.7 are satisfied. Denote by $\bar{s}^j_k$, $k = 1, 2, \ldots$, the sequence from Theorem 2.7. Passing to a subsequence, we may assume that $\bar{s}^j_j \to 0$ as $j \to \infty$. Choosing the resulting approximating values $\bar{\lambda}_j := \lambda_j + \bar{s}^j_j$ so that $|\lambda_j - \bar{\lambda}_j| \to 0$, we obtain a sequence $\bar{\lambda}_j$ for which the conclusion of the theorem holds. □

Note that the only use of condition (c3) in the previous proof was to guarantee the existence of a sequence λ_j in $(\lambda_0, \hat{\lambda}_0)$ such that $\lambda_j \to \lambda_0$ and $\bar{\mu}_1(\lambda)/\bar{\mu}_2(\lambda)$ has a nonzero derivative at $\lambda = \lambda_j$, for $j = 1, 2, \ldots$. Obviously, this is also guaranteed if instead of (c3) one assumes the following condition:

(c3)” For any $\epsilon \in (0, \hat{\lambda}_0 - \lambda_0)$ the function $\lambda \mapsto \bar{\mu}_1(\lambda)/\bar{\mu}_2(\lambda)$ is nonconstant on the interval $(\lambda_0, \lambda_0 + \epsilon)$.

Thus, we obtain the following local version of Theorem 1.4:

Theorem 2.9. *Theorem 2.8 remains valid if condition* (c3) *in* (GP) *is replaced by condition* (c3)”.

We conclude this section with the proof of Theorems 1.3, 1.4.

Proof of Theorems 1.3, 1.4. We use similar arguments as in the proof of Theorem 2.8 combined with the analyticity of $f(u;\lambda)$.

Under the analyticity assumption, the functions $\bar{\mu}_1(\lambda)$, $\bar{\mu}_2(\lambda)$ are analytic on $(\lambda_0, \hat{\lambda}_0)$. As in the proof of Theorem 2.8, the assumption (GP) of Theorem 1.3 implies that the function $\bar{\mu}_1(\lambda)/\bar{\mu}_2(\lambda)$ is not constant on $(\lambda_0, \hat{\lambda}_0)$; in Theorem 1.4 this is assumed directly in condition (c3)'. In either case, by the analyticity, $\bar{\mu}_1(\lambda)/\bar{\mu}_2(\lambda)$ is not constant on any interval. Therefore, we can again apply Theorem 2.7 to equation (2.28), only this time we can take arbitrary $\lambda_j \in (\lambda_0, \hat{\lambda}_0)$. This implies that there indeed exists a dense subset $\Lambda \subset (\lambda_0, \hat{\lambda}_0)$ as in the conclusion of Theorem 1.3. $\square$

3 Proof of Theorem 1.2

Assuming that $f : \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses of Theorem 1.2, we prove the conclusion of the theorem in two steps carried out in the following two subsections. First, we show that f can be perturbed slightly (with respect to the C^1 norm) in such a way that after the perturbation condition (G) is still satisfied, with the same n, and in addition the ground state in (G) is *nondegenerate.* After a further perturbation, maintaining the previous properties, we can also assume that f is of class C^∞. In the second step, we put f in an n-parameter family of functions $f(u) + s_1 g_1(u) + \cdots + s_n g_n(u)$, $s = (s_1, \ldots, s_n) \approx 0$. We show that smooth functions $g_1, \ldots, g_n$ can always be chosen such that the matrix $M(0)$ defined as in Subsection 2.2 has rank n. This will make Theorem 2.2 applicable and the desired conclusion will follow.

We remark that in our approach, the two nondegeneracy properties—the nondegeneracy of the ground state in condition (G) and the full-rank property of the matrix in (ND0)—are obtained by a direct perturbation argument. An alternative approach could be to prove that such properties are in some sense generic for functions f in the considered class. Typically, the parametric Smale-Sard theorem, or transversality theorem, is used in such an approach (see, for example, [12, Section 4], for an application of transversality in the verification of a nondegeneracy condition in a bifurcation problem involving elliptic equations with a similar structure). The question whether the genericity of the two nondegeneracy properties can be established in our setting could be of independent interest, but we have not pursued it. One of the reasons we decided to take the direct perturbation route is that not only did we need to achieve that an approximation $\tilde{f}$ of a given function f has nondegenerate ground states, we had to make sure that one of those ground states has the same Morse index as ϕ, the ground state given in condition (G). This would not be guaranteed by the genericity result.

3.1 Nondegeneracy of the ground state

Throughout this subsection we assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying conditions (1.2) and (G) for some $n \geq 2$. We also recall the notation introduced in Section 1: $\|g\|_1 = \sup\{|g(u)|, |g'(u)| : u \in \mathbb{R}\}$.

Lemma 3.1. *For any $\epsilon > 0$ there is a C^1 function $\tilde{f}$ such that $\tilde{f}(0) = 0 > \tilde{f}'(0)$, $\|f - \tilde{f}\|_1 < \epsilon$, and condition* (G) *is satisfied with f replaced by $\tilde{f}$ and with the additional property that the ground state in* (G) *is nondegenerate.*

We prepare the proof of this lemma by some preliminary observations. Let ϕ be a ground state of (1.4), as in (G). In spherical coordinates, the (radial) function ϕ satisfies the equation

$$\phi_{rr} + \frac{N-1}{r}\phi_r + f(\phi) = 0, \quad r > 0. \tag{3.1}$$

Also, $\phi_r(0) = 0$, $\phi_r(r) < 0$ for $r > 0$, and $\phi(r)$, $\phi_r(r)$ decay exponentially to 0 as $r \to \infty$. Differentiating (3.1), we see that $W := -\phi_r$ satisfies

$$w_{rr} + \frac{N-1}{r}w_r + \left(a(r) - \frac{N-1}{r^2}\right) w = 0, \quad r > 0, \tag{3.2}$$

with

$$a(r) := f'(\phi(r)). \tag{3.3}$$

Further, by the above properties of ϕ, we have $W(0) = 0$, $W(r) = -\phi_r(r) > 0$ for $r > 0$, and $W(r)$ decays exponentially to 0 as $r \to \infty$. The last property and equation (3.2) imply that W_r decays exponentially as well. Note that $\lim_{r\to\infty} a(r) = f'(0) < 0$. We will keep the notation a and W for the functions introduced above in the remainder of this subsection.

The function W can viewed as an eigenfunction (corresponding to the eigenvalue $\nu = 0$) of the following (singular) eigenvalue problem in the radial variable r:

$$w_{rr} + \frac{N-1}{r}w_r + \left(a(r) - \frac{N-1}{r^2} + \nu\right) w = 0, \quad r > 0, \tag{3.4}$$

$$w(0) = 0, \quad w(r) \to 0 \text{ as } r \to \infty. \tag{3.5}$$

In the variable $x \in \mathbb{R}^N$, W also represents an eigenfunction of a (regular) eigenvalue problem. Namely, the function $V(x) := W(|x|)x_1/|x| = -\phi_{x_1}(x)$ is a positive eigenfunction of the operator $-\Delta - a(r)$ on the half-space $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_1 > 0\}$ with Dirichlet boundary condition on $\partial\mathbb{R}^N_+$; equivalently, $V(x)$ can be viewed as an eigenfunction of the operator $-\Delta - a(r)$ considered on the closed subspace $L^2_o(\mathbb{R}^N)$ of $L^2(\mathbb{R}^N)$ consisting of all functions odd in x_1 with domain $H^2_o(\mathbb{R}^N) := H^2(\mathbb{R}^N) \cap L^2_o(\mathbb{R}^N)$. We also define $v(x) := w(|x|)x_1/|x|$ where w is a constant multiple of W such that v is positive on the half space $\mathbb{R}^N_+$ and v is normalized in the $L^2(\mathbb{R}^N)$-norm.

The functions v and w defined this way are assumed to be fixed in the rest of this subsection.

Now, the ground state ϕ is degenerate if and only if 0 is also an eigenvalue of $-\Delta - a(r)$ in the radial space. The main idea of the proof of Lemma 3.1 consists in the following. We first find a perturbation $\tilde{a}$ of the function a such that the perturbed operator $-\Delta - \tilde{a}(r)$ still has 0 as an eigenvalue for the operator on $L^2_o(\mathbb{R}^N)$, but 0 is no longer an eigenvalue in the radial space. We then use a reverse construction, finding a nonlinearity $\tilde{f}$ and a ground state $\tilde{\phi}$ of

$$\Delta u + \tilde{f}(u) = 0, \quad x \in \mathbb{R}^N, \tag{3.6}$$

such that $\tilde{a}(r) = \tilde{f}'(\tilde{\phi}(r))$. The resulting function $\tilde{f}$ will have all the desired properties if $\tilde{a}$ is close enough to a.

The reverse construction is described in the following results of [34].

Lemma 3.2. *Assume the following hypotheses.*

(a) $\tilde{a}(r)$ is a continuous function on $[0,\infty)$ which converges to a negative limit as $r \to \infty$.

(b) $\tilde{w} \in C^1([0,\infty))$ is a positive solution of (3.2), with a replaced by $\tilde{a}$, which satisfies the following conditions:

(i) $\tilde{w}(0) = 0$, $\tilde{w}_r(0) > 0$,

(ii) $e^{\beta r}\tilde{w}(r) \to 0$, $e^{\beta r}\tilde{w}_r(r) \to 0$ *as* $r \to \infty$ *for some* $\beta > 0$.

Then

$$\tilde{\phi}(r) := \int_r^\infty \tilde{w}(t)\,dt, \quad r = |x| \geq 0, \tag{3.7}$$

defines a ground state of (3.6) for a C^1 function $\tilde{f}$ that satisfies (1.2) and for which

$$\tilde{f}'(\tilde{\phi}(r)) = \tilde{a}(r) \quad (r \geq 0). \tag{3.8}$$

On the interval $[0, \tilde{\phi}(0)]$, $\tilde{f}$ is given explicitly by

$$\tilde{f}(z) = \int_0^z \tilde{a}(\xi(\tau))d\tau, \tag{3.9}$$

where $\xi : [0,\infty) \to (0, \tilde{\phi}(0)]$ is the inverse of $\tilde{\phi}$.

Of course, we need to guarantee that the function $\tilde{f}$ resulting from the reverse construction is a small perturbation of f if $\tilde{a}(r)$ is a small perturbation of $a(r) = f'(\phi(r))$. This is the purpose of the following lemma.

Lemma 3.3. *Given any $\epsilon > 0$ there is $\delta > 0$ such that the following statement is valid. Let $\tilde{a}$ be any function satisfying the hypotheses of Lemma 3.2 together with the relation $\|\tilde{a} - a\|_{L^\infty(0,\infty)} < \delta$, and let the positive solution $\tilde{w}$ in Lemma 3.2(b) be normalized so that*

$$\int_0^\infty \tilde{w}(r)\, dr = \phi(0). \tag{3.10}$$

Then, with $\tilde{\phi}$ and $\tilde{f}$ as in (3.7), (3.9), the function $\tilde{f}$ can be extended from $[0, \tilde{\phi}(0)]$ to $\mathbb{R}$ in such a way that $\|f - \tilde{f}\|_1 < \epsilon$.

In the proof of Lemma 3.3, and then again in the proof of Lemma 3.4, we will use some perturbation results from [37, Section 4], which we now recall.

As noted above, $\nu = 0$ is an eigenvalue of the operator $-\Delta - a(r)$ considered on $L^2_o(\mathbb{R}^N)$ with domain $H^2_o(\mathbb{R}^N)$. The corresponding eigenfunction $-\phi_{x_1}$ is positive in $\mathbb{R}^N_+$, which means that $\nu = 0$ is the principal eigenvalue. Here and below, the principal eigenvalue refers to an eigenvalue below the essential spectrum admitting an eigenfunction which is positive in the half space $\mathbb{R}^N_+$. It is well known that, if it exists, such an eigenvalue is unique and simple (also, being below the essential spectrum, it is an isolated eigenvalue). Therefore, if $\mathcal{U}$ is a sufficiently small neighborhood of a in $C_{\mathrm{rad}}(\mathbb{R}^N)$, then for each $\tilde{a} \in \mathcal{U}$ the principal eigenvalue $\nu(\tilde{a})$ of the operator $-\Delta - \tilde{a}(r)$ on $L^2_o(\mathbb{R}^N)$ (with domain $H^2_o(\mathbb{R}^N)$) is well defined, and the corresponding eigenfunction—referred to as the principal eigenfunction—is defined uniquely up to a scalar multiple. Denoting by $v(\tilde{a})$ the principal eigenfunction which is normalized in the $L^2(\mathbb{R}^N)$-norm and positive in $\mathbb{R}^N_+$, the functions

$$\tilde{a} \mapsto \nu(\tilde{a}) : \mathcal{U} \to \mathbb{R}, \quad \tilde{a} \mapsto v(\tilde{a}) : \mathcal{U} \to H^2_o(\mathbb{R}^N)$$

are both smooth. Combining this result with elliptic regularity, we obtain that, for any $p \geq 2$, $v(\tilde{a})$ depends continuously (and smoothly) on $\tilde{a} \in \mathcal{U}$ as a $W^{2,p}(\mathbb{R}^N)$-valued function, and therefore also as a $C^1_{\mathrm{b}}(\mathbb{R}^N)$-valued function.

Since we are dealing with radial potentials, the above results can be interpreted in terms of the eigenvalue problem

$$w_{rr} + \frac{N-1}{r} w_r + \left(\tilde{a}(r) - \frac{N-1}{r^2} + \nu \right) w = 0, \quad r > 0, \tag{3.11}$$

$$w(0) = 0, \quad w(r) \to 0 \text{ as } r \to \infty. \tag{3.12}$$

Indeed, using separation of variables in spherical coordinates on the eigenvalue problem $\Delta v + \tilde{a}(|x|)v + \nu v = 0$ in $\mathbb{R}^N_+$, one shows that the principal eigenfunction $v(\tilde{a})$ (positive and normalized in the $L^2(\mathbb{R}^N)$-norm, as above) can be written as

$$v(\tilde{a})(x) = \frac{x_1}{r} w(\tilde{a})(r) \quad (x = (x_1, \ldots, x_N) \in \mathbb{R}^N,\ r = |x| > 0), \tag{3.13}$$

where $w(\tilde{a})$ satisfies (3.11), (3.12) with $\nu = \nu(\tilde{a})$. Obviously, the function $w(\tilde{a})$ in (3.13) is determined uniquely by $v(\tilde{a})$.

On the other hand, if we are given a positive solution $\tilde{w}$ of (3.11), (3.12) with $\tilde{a}$ close enough to a and ν close enough to 0, then necessarily $\nu = \nu(\tilde{a})$ and $\tilde{w}$ is a scalar multiple of the function $w(\tilde{a})$ defined by $w(\tilde{a})(r) = v(\tilde{a})(r, 0, \dots, 0)$ (so that relation (3.13) holds). To verify this statement, note that suitable proximity relations $\tilde{a} \approx a$ and $\nu \approx 0$ in particular guarantee that $\tilde{a} + \nu$ is bounded from above by a negative constant on an interval $[R, \infty)$. This implies that the solution of (3.11), (3.12) is unique up to a scalar multiple and it decays exponentially to 0 as $r \to \infty$ together with its derivative $\tilde{w}'$. The function $v(x) = \tilde{w}(|x|)x_1/|x|$ then gives an eigenfunction of the operator $-\Delta - \tilde{a}(r)$ on $L^2_o(\mathbb{R}^N)$ with the eigenvalue ν and the positivity of $\tilde{w}$ implies the relations $\nu = \nu(\tilde{a})$ and $\tilde{w} = cw(\tilde{a})$ for some $c > 0$.

Below, for $\tilde{a} \approx a$ (the function in (3.3)), the *principal eigenvalue* of (3.11), (3.12) refers to the eigenvalue $\nu(\tilde{a})$. Also, the aforementioned fixed functions v and w satisfy $v = v(a)$, $w = w(a) = -\phi_r$.

Proof of Lemma 3.3. Note that the normalization (3.10) implies that $\tilde{\phi}(0) = \phi(0)$. It is clearly sufficient to prove that $|f - \tilde{f}|$, $|f' - \tilde{f}'|$ are uniformly small on the interval $[0, \phi(0)]$; an extension of $\tilde{f}$ such that $|f - \tilde{f}|$, $|f' - \tilde{f}'|$ are uniformly small on $\mathbb{R}$ is then easy to construct.

To estimate $|f'(u) - \tilde{f}'(u)|$ for $u \in [0, \phi(0)]$, we can instead estimate $|f'(\tilde{\phi}(r)) - \tilde{f}'(\tilde{\phi}(r))|$ for $r \in [0, \infty)$.

First, we estimate $\|\phi - \tilde{\phi}\|_{L^\infty(0,\infty)}$. By assumption, the function $\tilde{w}$ is a positive solution of (3.11), (3.12) with $\nu = 0$. As noted above, this means, if $\|\tilde{a} - a\|_{L^\infty(0,\infty)}$ is small enough, that ($\nu(\tilde{a}) = 0$ and) $\tilde{w} = cw(\tilde{a})$, where $w(\tilde{a})$ is the function introduced in (3.13) and the constant factor c is determined from the normalization (3.10). By the continuous dependence of $w(\tilde{a})$ on $\tilde{a}$, $\|w(\tilde{a}) - w\|_{L^\infty(0,\infty)}$ is small if $\|\tilde{a} - a\|_{L^\infty(0,\infty)} < \delta$ with a sufficiently small δ. In addition, we have the following universal estimate

$$w(\tilde{a})(r) \le e^{-\theta r} w(\tilde{a})(R) \quad (r \ge R) \tag{3.14}$$

if $\delta > 0$ is small enough. Here θ and R are positive constants independent of $\tilde{a}$ (they depend on δ). This follows from an easy computation which shows that for some $\theta, R > 0$ the function $e^{-\theta r}$ is a supersolution of equation (3.11) on $[R, \infty)$, provided $\|\tilde{a} - a\|_{L^\infty(0,\infty)}$ is small enough and ν is close enough to 0. Using (3.14) and the smallness of $\|w(\tilde{a}) - w\|_{L^\infty(0,\infty)}$ in (3.7), one shows easily that for any $\epsilon_1 > 0$ there is $\delta > 0$ such that $\|\tilde{a} - a\|_{L^\infty(0,\infty)} < \delta$ implies $\|\phi - \tilde{\phi}\|_{L^\infty(0,\infty)} < \epsilon_1$.

Now,

$$\begin{aligned} |f'(\tilde{\phi}(r)) - \tilde{f}'(\tilde{\phi}(r))| &\le |f'(\tilde{\phi}(r)) - f'(\phi(r))| + |f'(\phi(r)) - \tilde{f}'(\tilde{\phi}(r))| \\ &= |f'(\tilde{\phi}(r)) - f'(\phi(r))| + |a(r) - \tilde{a}(r)|. \end{aligned}$$

By the uniform continuity of f' on $[0, \phi(0)] = [0, \tilde{\phi}(0)]$, the last sum can be made arbitrarily small by choosing $\delta > 0$ small enough.

We have thus obtained the desired smallness estimate on $|f'(u) - \tilde{f}'(u)|$. The smallness of $|f(u) - \tilde{f}(u)|$ now follows from the mean value theorem and the relations $f(0) = \tilde{f}(0) = 0$. □

We now construct a suitable approximation of the function $a(r) = f'(\phi(r))$ in the case that the ground state ϕ is degenerate.

Lemma 3.4. *Assume that the ground state ϕ is degenerate. Then for any $\delta > 0$ there is a function $\tilde{a}$ satisfying the hypotheses of Lemma 3.2 together with the relation $\|\tilde{a} - a\|_{L^\infty(0,\infty)} < \delta$ such that the Schrödinger operator $-\Delta - \tilde{a}(r)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ (with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$) has exactly n negative eigenvalues and 0 is not its eigenvalue.*

Proof. Since ϕ is a degenerate ground state of Morse index n, the operator $-\Delta - a(r)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ has exactly n negative eigenvalues and 0 is its $(n+1)$th eigenvalue. Let ψ_{n+1} be an eigenfunction corresponding to the eigenvalue 0. Thus, in the radial variable, ψ_{n+1} satisfies the equation

$$\psi_{rr} + \frac{N-1}{r}\psi_r + a(r)\psi = 0, \quad r > 0. \tag{3.15}$$

Comparing this equation to (3.2), it is clear that the functions ψ_{n+1}, w—and thus also the functions ψ^2_{n+1}, w^2—are linearly independent over any interval in $(0,\infty)$. Therefore, we can choose a smooth function $b(r)$ on $(0,\infty)$ with compact support such that

$$\int_0^\infty b(r)\psi^2_{n+1}(r)r^{N-1}\,dr < 0, \quad \int_0^\infty b(r)w^2(r)r^{N-1}\,dr = 0. \tag{3.16}$$

Fixing such a function b, we take $a(r) + \tau b(r)$ as a perturbed potential. First consider the operator $-\Delta - a(r) - \tau b(r)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$. By standard perturbation results, there is $\epsilon_0 > 0$ such that for all sufficiently small τ the spectrum of this operator in the interval $(-\infty, \epsilon_0)$ consists of $n+1$ eigenvalues $\mu_1(\tau) < \mu_2(\tau) < \dots \mu_{n+1}(\tau)$, these eigenvalues depend smoothly on τ, and $\mu_{n+1}(0) = 0$. The derivative $\mu'_{n+1}(0)$ is computed by differentiating the equation for the corresponding eigenfunction, similarly as the derivatives of the functions $\mu_j(s)$ were computed in the proof of Proposition 2.5. We obtain (cp. [37, Lemma 4.5]) that, up to a positive scalar factor, $-\mu'_{n+1}(0)$ is given by the first integral in (3.16).

Next consider the principal eigenvalue $\nu(\tau) := \nu(a + \tau b)$ of (3.11), (3.12) with $\tilde{a} = a + \tau b$. We have $\nu(0) = 0$ and, as computed in [37, Lemma 4.5], $-\nu'(0)$ is given, up to a positive scalar factor, by the second integral in (3.16).

Thus, conditions (3.16) give $\mu'_{n+1}(0) > 0 = \nu'(0)$. Therefore, for all sufficiently small $\tau > 0$ we have $\mu_{n+1}(\tau) > \nu(\tau)$.

Take now the shifted potential $\tilde{a}(r) := a(r) + \tau b(r) - \nu(\tau)$. Clearly, the principal eigenvalue of (3.11), (3.12) is 0. The corresponding positive solution $w(\tilde{a})(r)$ of (3.11), (3.12) satisfies conditions (b) of Lemma 3.2. For small τ, condition (a) of Lemma 3.2

is obviously satisfied as well. Further, for τ small enough the first $n+1$ eigenvalues of the operator $-\Delta - \tilde{a}(r)$ (on L^2_{rad}) are

$$\mu_1(\tau) - \nu(\tau) < \cdots < \mu_n(\tau) - \nu(\tau) < \mu_{n+1}(\tau) - \nu(\tau)$$

and they exhaust the spectrum of this operator in $(-\infty, \epsilon_0/2)$. Since $\mu_n(0) < \mu_{n+1}(0) = 0$ and $\nu(0) = 0$, for sufficiently small $\tau > 0$ we have $\mu_n(\tau) - \nu(\tau) < 0 < \mu_{n+1}(\tau) - \nu(\tau)$. So $\tilde{a}$ has all the properties required in the conclusion of Lemma 3.4, and, of course, $\tilde{a}$ is close (in L^∞-norm) to a for $\tau \approx 0$. The proof is complete. $\square$

Proof of Lemma 3.1. There is nothing to prove if the ground state ϕ itself is nondegenerate, simply take $\tilde{f} \equiv f$. If ϕ is degenerate, we take a function $\tilde{a}$ as in Lemma 3.4 to construct $\tilde{f}$ as in Lemma 3.2, and we extend it to $[0, \infty)$ using Lemma 3.3. This function $\tilde{f}$ satisfies all the given requirements. $\square$

3.2 Completion of the proof

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a given C^1 function satisfying conditions (1.2) and (G) for some $n \geq 2$. We are seeking a perturbation of f satisfying the conclusion of Theorem 1.2.

Due to Lemma 3.1, perturbing f slightly we may assume without loss of generality that the ground state in (G) is nondegenerate. By the implicit function theorem, further small (in the C^1-norm) perturbations of f will not alter condition (G) or the nondegeneracy property. Thus, again without loss of generality, we may assume that f also satisfies the following conditions:

$$f \in C^\infty(\mathbb{R}) \text{ and for some } \alpha > 0,\ \delta_0 > 0 \text{ one has } f(u) = -\alpha u \quad (|u| < \delta_0). \tag{3.17}$$

A function f with all the above properties is assumed to be fixed for the remainder of this subsection.

We will find a perturbation $\tilde{f}$ of this function f, as needed for the proof of Theorem 1.2, among functions of the form

$$f(u) + \sum_{i=1}^{n} s_i g_i(u), \tag{3.18}$$

where the g_i, to be specified below, are C^∞ functions on $\mathbb{R}$ vanishing at $u = 0$ and $s = (s_1, \ldots, s_n) \approx 0 \in \mathbb{R}^n$. We take the nonlinearity (3.18) in equations (2.8), (2.9) in lieu of $f(u; s)$. This clearly fits the framework of Subsection 2.2 with $d = n$. Our goal is to apply Theorem 2.2, hence we want to choose the functions g_i in such a way that condition (ND0) holds. We will work with the sufficient condition for (ND0) as given in Remark 2.6. In the present case—with the nonlinearity $f(u; s)$ in (2.8), (2.9) replaced by (3.18)—the sufficient condition requires that the $n \times n$ matrix with the following entries is nonsingular:

$$\int_0^\infty \left(f''(\phi(r))\dot{\phi}_i(r) + g_i'(\phi(r)) \right) (\psi_j(r))^2 r^{N-1}\, dr, \quad i, j = 1, \dots, n. \tag{3.19}$$

Here, ϕ is the ground state as in (Gs), $\psi_1, \dots, \psi_n$ are the normalized eigenfunctions of the operator $-\Delta - f'(\phi(x))$ (acting on $L^2_{\text{rad}}(\mathbb{R}^N)$) associated with its negative eigenvalues $\mu_1 < \cdots < \mu_n$, and, for $i = 1, \dots, n$, $\dot{\phi}_i \in H^2_{\text{rad}}(\mathbb{R}^N)$ is the unique solution of the equation

$$\Delta \dot{\phi}_i + f'(\phi(x))\dot{\phi}_i + g_i(\phi(x)) = 0 \tag{3.20}$$

(cp. Proposition 2.5). Functions g_i with the all desired properties are provided by the following lemma.

Lemma 3.5. *There exist functions $g_i \in C^\infty(\mathbb{R})$, $i = 1, \dots, n$, each with compact support contained in $(0, \phi(0))$, such that the $n \times n$ matrix with entries (3.19) is nonsingular.*

Before proving this lemma, we will use it to prove Theorem 1.2.

Proof of Theorem 1.2. Taking functions g_i as in Lemma 3.5, Theorem 2.2 applies to the nonlinearity (3.18). This implies (cp. Remark 2.4(i)), that we can find arbitrarily small $s_1, \dots, s_n$ such that the function

$$\tilde{f}(u) = f(u) + \sum_{i=1}^n s_i g_i(u)$$

satisfies the conclusion of Theorem 1.2, save possibly for the smallness (in the C^1 norm) of $f - \tilde{f}$. Since the g_i are compactly supported, we can make $\|f - \tilde{f}\|_1$ arbitrarily small by taking $s_1, \dots, s_n$ smaller if necessary. The theorem is thus proved. $\square$

The rest of this section is devoted to the proof of Lemma 3.5. We first reformulate the desired properties of the functions g_i in terms of the following functions on $(0, \infty)$:

$$b_i(r) := g_i(\phi(r)) \quad (i = 1, \dots, n). \tag{3.21}$$

Note that since $\phi'(r) < 0$, the functions g_i can be determined from (3.21) if b_i are defined first, and that is how we will proceed in the proof.

Proof of Lemma 3.5. Suppose for a while that $b_1, \dots, b_n$ are smooth, compactly supported functions on $(0, \infty)$ such that the following conditions are satisfied:

(B1) For $i = 1, \dots, n$, denoting by $\bar{\phi}_i \in H^2_{\text{rad}}(\mathbb{R}^N)$ the unique solution of the equation

$$\Delta \bar{\phi}_i + f'(\phi(x))\bar{\phi}_i + b_i(|x|) = 0, \tag{3.22}$$

the supports of the functions $\bar{\phi}_i(r)$ and $f''(\bar{\phi}(r))$ (both viewed as functions of $r \in (0, \infty)$) are disjoint.

(B2) The $n \times n$ matrix with entries

$$\int_0^\infty \frac{b_i'(r)}{\phi_r(r)}(\psi_j(r))^2 r^{N-1}\,dr, \quad i,j=1,\dots,n, \tag{3.23}$$

is nonsingular.

Then there are uniquely defined smooth functions g_i, with compact support in $(0,\phi_0(0))$, satisfying relations (3.21), namely $g_i(u)=b_i(\xi(u))$, where $\xi:(0,\phi(0)]\to\mathbb{R}$ is the inverse function to ϕ. For such functions g_i,

$$g_i'(\phi(r))=\frac{b_i'(r)}{\phi_r(r)},$$

and, in view of (B1), the integrals (3.19) coincide with (3.23). Therefore, the functions g_i have all the properties stated in Lemma 3.5.

It remains to prove the existence of smooth functions b_i on $(0,\infty)$ with compact supports such that conditions (B1), (B2) are satisfied.

Fix two numbers $r_1>r_0>0$, where r_0 is sufficiently large so that $0<\phi(r_0)<\delta_0$ with δ_0 as in (3.17) (recall that $\phi(r)\to 0$ as $r\to\infty$). Note that

$$f'(\phi(r))=-\alpha, \quad f''(\phi(r))=0 \quad (r\ge r_0). \tag{3.24}$$

The functions $\tilde{b}_i$ will be chosen such that their supports are contained in (r_0,r_1).

Let us first reformulate condition (B1) in a more explicit way. The homogeneous equation corresponding to equation (3.22) reads, in the radial variable, as follows

$$v_{rr}+\frac{N-1}{r}v_r+f'(\phi(r))v=0, \quad r>0. \tag{3.25}$$

We choose two linearly independent solutions $\varphi(r)$, $\psi(r)$ of this equation such that $\psi(r)\to 0$, $|\varphi(r)|\to\infty$ as $r\to\infty$. The existence of such solutions follows from the behavior of $f'(\phi(r))$. In fact, by (3.24), $f'(\phi(r))=-\alpha$ for $r\in(r_0,\infty)$, and therefore we can choose $\psi(r)$ to coincide on (r_0,∞) with the function $r^{1-N/2}K_{N/2-1}(r\sqrt{\alpha})$, where $K_{N/2-1}$ is the modified Bessel function of the second kind. This function has the following asymptotics as $r\to\infty$:

$$K_{N/2-1}(r\sqrt{\alpha})=Ce^{-r\sqrt{\alpha}}r^{-1/2}(1+\mathcal{O}(1/r)) \tag{3.26}$$

with some positive constant C. For $\varphi(r)$ we choose a linearly independent solution with

$$r_0^{N-1}(\psi(r_0)\varphi'(r_0)-\psi'(r_0)\varphi(r_0))=1.$$

Note that this implies that the Wronskian of the two solutions satisfies

$$r^{N-1}(\psi(r)\varphi'(r)-\psi'(r)\varphi(r))=1 \quad (r>0)$$

(as one can easily verify by differentiation).

Consider now the solution v_i of the nonhomogeneous equation

$$v_{rr} + \frac{N-1}{r} v_r + f'(\phi(r))v = -b_i(r), \quad r > 0, \tag{3.27}$$

satisfying the initial conditions $v_i(r_0) = v_i'(r_0) = 0$. By the variation of constants formula, this solution is given by

$$v_i(r) = \psi(r) \int_{r_0}^{r} \eta^{N-1} b_i(\eta) \varphi(\eta)\, d\eta - \varphi(r) \int_{r_0}^{r} \eta^{N-1} b_i(\eta) \psi(\eta)\, d\eta. \tag{3.28}$$

If

$$\int_{r_0}^{r_1} \eta^{N-1} b_i(\eta) \psi(\eta)\, d\eta = 0 \tag{3.29}$$

and the support of b_i is contained in (r_0, r_1), then on the interval (r_1, ∞)

$$v_i(r) = \psi(r) \int_{r_0}^{r_1} \eta^{N-1} b_i(\eta) \varphi(\eta)\, d\eta \tag{3.30}$$

and thus $v_i(r)$ decays to zero exponentially due to the decay of ψ; and on the interval $(0, r_0]$, where $b_i \equiv 0$, the initial conditions imply that $v_i \equiv 0$. This means that, first, v_i coincides with the unique solution $\bar{\phi}_i \in H^2_{\text{rad}}(\mathbb{R}^N)$ of (3.22); and, second, this solution has its support disjoint from the support of the function $f''(\phi(r))$, as required in (B1).

So conditions (3.29), $i = 1, \ldots, n$, are sufficient for (B1); we take these conditions as requirements on the functions b_i to be met together with condition (B2). First, we use integration by parts in (3.29), so both (B2) and (3.29) are stated in terms of the derivatives b_i':

$$\int_{r_0}^{r_1} b_i'(r) \left(\int_{r_0}^{r} \eta^{N-1} \psi(\eta)\, d\eta \right) dr = 0. \tag{3.31}$$

We are now ready to define the functions b_i. It is more convenient to first choose the derivatives of these functions. We thus need to choose smooth functions $\tilde{b}_i$ with supports in (r_0, r_1) such that conditions (3.31) and (B2) hold with b_i' replaced by $\tilde{b}_i$ and, in addition,

$$\int_{r_0}^{r_1} \tilde{b}_i(r)\, dr = 0 \quad (i = 1, \ldots, n). \tag{3.32}$$

(Note that this last condition guarantees that $\tilde{b}_i = b_i'$ for a smooth function b_i with compact support in (r_0, r_1).) Let us explain why such a choice of functions $\tilde{b}_i$ is possible. We will verify shortly that the functions

$$1; \quad \int_{r_0}^{r} \eta^{N-1} \psi(\eta)\, d\eta; \quad \frac{(\psi_j(r))^2}{\phi_r(r)} r^{N-1},\ j = 1, \ldots, n; \tag{3.33}$$

are linearly independent over the interval $[r_0, r_1]$. Therefore, we can choose functionals on $L^2(r_0, r_1)$, represented by functions $\hat{b}_i \in L^2(r_0, r_1)$, $i = 1, \dots, n$ (which we extend as 0 outside (r_0, r_1)), taking the following values at the functions in (3.33):

$$\int_0^\infty \hat{b}_i(r) \frac{(\psi_j(r))^2}{\phi_r(r)} r^{N-1}\, dr = \delta_{ij} \quad (i, j = 1, \dots, n), \tag{3.34}$$

$$\int_{r_0}^{r_1} \hat{b}_i(r) \left(\int_{r_0}^{r} \eta^{N-1} \psi(\eta)\, d\eta \right) dr = \int_{r_0}^{r_1} \tilde{b}_i(r)\, dr = 0, \tag{3.35}$$

δ_{ij} being the Kronecker symbol. Relations (3.34) mean, in other words, that the matrix with entries

$$\int_0^\infty \frac{\hat{b}_i(r)}{\phi_r(r)} (\psi_j(r))^2 r^{N-1}\, dr, \quad i, j = 1, \dots, n,$$

is actually the identity matrix. By approximations, one now easily shows the existence of smooth functions $\tilde{b}_i$, $i = 1, \dots, n$, with supports in (r_0, r_1) which are still $L^2(r_0, r_1)$-orthogonal to the first two functions in (3.33) and such that condition (B2) holds with b_i' replaced by $\tilde{b}_i$. Such functions $\tilde{b}_i$ have all the needed properties.

To show that the functions (3.33) are linearly independent, first observe that by (3.24) the equations satisfied by the eigenfunctions ψ_j, $j = 1, ..., n$, reduce on (r_0, ∞) to

$$\psi_j'' + \frac{N-1}{r} \psi' + (\mu_j - \alpha)\psi = 0. \tag{3.36}$$

Thus, similarly as the function ψ above (see the paragraph containing (3.25)), the function ψ_j coincides on (r_0, ∞) with a nonzero scalar multiple of the function

$$r^{1-N/2} K_{N/2-1} \left(r\sqrt{-\mu_j + \alpha} \right).$$

The function ϕ_r is a negative decaying solution of equation (3.2) with $a(r) = f'(\phi(r))$. On (r_0, ∞) this equation coincides with the equation

$$w_{rr} + \frac{N-1}{r} w_r + \left(-\alpha - \frac{N-1}{r^2} \right) w = 0.$$

Therefore, on (r_0, ∞), ϕ_r is a nonzero scalar multiple of the function $r^{1-N/2} K_{N/2}(r\sqrt{\alpha})$. The modified Bessel function $K_{N/2}$ has the same asymptotics (3.26) as $K_{N/2-1}$.

It follows that the functions (3.33) are analytic on (r_0, ∞) and, except for the constant function 1, they decay to 0 exponentially with different exponential rates. Using this, it is easy to show that these functions are linearly independent over (r_0, ∞), hence, by analyticity, over any subinterval of (r_0, ∞). Indeed, take a linear combination of the functions in (3.33) and assume it is identical to zero. Then clearly the coefficient of the function with the slowest decay must be zero. Applying this reasoning inductively, each coefficient of the linear combination can be shown to be equal to zero. This implies the linear independence of the functions (3.33). $\square$

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