Convergence and quasiconvergence properties of solutions of parabolic equations on the real line: an overview

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Dedicated to Bernold Fiedler on the occasion of his 60th birthday.

Abstract. We consider semilinear parabolic equations $u_t = u_{xx} + f(u)$ on $\mathbb{R}$. We give an overview of results on the large time behavior of bounded solutions, focusing in particular on their limit profiles as $t \to \infty$ with respect to the locally uniform convergence. The collection of such limit profiles, or, the ω-limit set of the solution, always contains a steady state. Questions of interest then are whether—or under what conditions—the ω-limit set consists of steady states, or even a single steady state. We give several theorems and examples pertinent to these questions.

Key words: semilinear heat equation on the real line, asymptotic behavior, convergence, quasiconvergence, entire solutions
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Contents

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1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and u_0 is a bounded continuous function on $\mathbb{R}$.

Problem (1.1), (1.2) has a unique (classical) solution u defined on a maximal time interval $[0, T(u_0))$. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is *global.* In this overview paper, we discuss the behavior of bounded solutions as $t \to \infty$.

By standard parabolic regularity estimates, any bounded solution has compact orbit in $L^\infty_{loc}(\mathbb{R})$. In other words, any sequence $t_n \to \infty$ has a subsequence $\{t_{n_k}\}$ such that $u(\cdot, t_{n_k}) \to \varphi$, locally uniformly on $\mathbb{R}$, for some continuous function φ; we refer to any such function φ as a *limit profile of* u; the collection of all limits profiles of u is the ω-limit set of u:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi, \text{in } L^\infty_{loc}(\mathbb{R}), \text{ for some } t_n \to \infty\}. \tag{1.3}$$

The simplest possible large time behavior of a bounded solution is convergence to an equilibrium (a steady state): $u(\cdot, t) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$ for some solution of the equation $\varphi'' + f(\varphi) = 0$. By compactness, this is the case precisely when $\omega(u)$ consists of a single element φ. The convergence may hold in stronger topologies, but we take the convergence in $L^\infty_{loc}(\mathbb{R})$, the topology in which the orbit is compact, as the minimal requirement in the definition of convergence and quasiconvergence. A bounded solution u is said to be *quasiconvergent* if $\omega(u)$ consists entirely of steady states. Thus, quasiconvergent solutions are those bounded solutions that are attracted by steady states. This follows from the following well-known property of the ω-limit set:

$$\lim_{t\to\infty} \operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot, t), \omega(u)) = 0 \tag{1.4}$$

($L^\infty_{loc}(\mathbb{R})$ is a metric space, with metric derived from a countable family of seminorms). For large times, each quasiconvergent solution stays near steady states, from which it can be proved that $u_t(\cdot, t) \to 0$ in $L^\infty_{loc}(\mathbb{R})$, as $t \to \infty$. This makes quasiconvergent solutions hard to distinguish—numerically, for example—from convergent solutions; they move very slowly at large times. A central question in this paper is whether, or to what extent, is quasiconvergence a "general property" of equations of the form (1.1).

If equation (1.1) is considered on a bounded interval, instead of $\mathbb{R}$, and one of common boundary conditions, say Dirichlet, Neumann, Robin, or periodic is assumed, then each bounded solution is convergent [5, 34, 53]. In contrast, bounded solutions of (1.1) on $\mathbb{R}$ are not convergent in general even for the linear heat equation, that is, equation (1.1) with $f \equiv 0$. More specifically, if u_0 takes values 0 and 1 on suitably spaced long intervals with sharp transitions between them, then, as $t \to \infty$, $u(\cdot, t)$ will oscillate between 0 and 1, thus creating a continuum $\omega(u)$—connectedness in the metric space $L^\infty_{loc}(\mathbb{R})$ is another well-known property of the limit set—which contains the constant steady states 0 and 1 (see [7]). In the case of the linear heat equation, it is easy to show that each bounded solution is quasiconvergent; namely, its ω-limit set consists of constant steady states. This follows from the invariance property of the ω-limit set: $\omega(u)$ consists of *entire* solutions of (1.1), by which we mean solutions defined for all $t \in \mathbb{R}$. If u is bounded, then the entire solutions in $\omega(u)$ are bounded as well and, by the Liouville theorem for the linear heat equation, all such solutions are constant.

In nonlinear equations, another different class of solutions of (1.1), as compared to the problems on bounded intervals, is given by traveling fronts – solutions of the form $U(x, t) = \phi(x-ct)$, where $c \in \mathbb{R}$ and ϕ is a C^2 monotone function. If $c \neq 0$, then the front moves with the constant speed c, hence, when looked at globally, it does not approach any equilibrium. However, from a different perspective, the traveling front still exhibits very simple dynamics: in $L^\infty_{loc}(\mathbb{R})$ it just approaches a constant steady state given by one of the limits $\phi(\pm\infty)$. There are solutions with much more complicated global dynamics, such as oscillations between traveling fronts with different speeds [52] (see also [27, 28, 30, 49]), whose local dynamics is similarly trivial. Thus, traveling fronts, while important for many other reasons, do not themselves give interesting examples of the local behavior. The simplicity of their local dynamics makes our central question even more compelling.

As it turns out, not all bounded solutions are quasiconvergent and we review below several examples illustrating this. On the other hand, there are interesting classes of initial data in (1.2) which yield quasiconvergent solutions and we review results showing this as well. These are the contents of Sections 2.3 and 2.4, respectively. In Sections 2.1, 2.2, we discuss related results on convergence to an equilibrium and convergence on average.

We consider bounded solutions of (1.1) only. This means that we will always assume that $|u| \leq c$ for some constant c. In terms of the initial value, the boundedness of the solution of (1.1), (1.2) is guaranteed if, for example, $a \leq u_0 \leq b$ for some constants a, b satisfying $f(a) \geq 0$, $f(b) \leq 0$. This follows from the comparison principle.

We focus almost exclusively on the one-dimensional problems, but at several places we mention extensions of theorems for (1.1), or the lack thereof, to the higher-dimensional problem

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N. \tag{1.6}$$

One of the most interesting open questions concerning multidimensional problems, the existence of at least one limit equilibrium, is mentioned in Section 2.2.

Below $C_b(\mathbb{R})$ and $C_0(\mathbb{R})$ denote the spaces of all continuous bounded functions on $\mathbb{R}$ and all continuous functions on $\mathbb{R}$ converging to 0 at $x = \pm\infty$, respectively. They are both equipped with the supremum norm. Further, $C_b^1(\mathbb{R})$ is the space of all functions f such that $f, f' \in C_b$. Its norm is

$$\|f\|_{C_b^1(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})}.$$

2 Overview of the results

2.1 Convergence to a steady state

In this section, we summarize results on the convergence of solutions of (1.1) to a steady state:

(S1) $\lim_{t\to\infty} u(\cdot,t) = \varphi$, in $L^\infty_{loc}(\mathbb{R})$, for some steady state φ of (1.1).

For the solution of (1.1), (1.2)—*assuming it is bounded*—(S1) has been proved in the following cases:

(I) $f(0) = 0$, $u_0 \ge 0$, and u_0 has compact support.

(II) $f(0) = 0$, $f'(0) < 0$, and the solution u is (bounded and) localized: $u(x,t) \to 0$, as $x \to \infty$, uniformly in $t \ge 0$ (u_0 may change sign in this case).

(III) $f(0) = 0$, $f'(0) < 0$, $u_0 \in C_0(\mathbb{R})$, $u_0 \ge 0$, and $\|u(\cdot,t)\|_{L^2(\mathbb{R})}$ stays bounded as $t \to \infty$.

(IV) $f(0) = 0$, $u_0 \ge 0$, and $u_0 = \phi_0 + \phi_1$, where $\phi_0, \phi_1 \in C(\mathbb{R})$, ϕ_0 is even and decreasing on $(0,\infty)$, and there are positive constants c and θ such that

$$\phi_0(x)e^{\theta|x|} \to c,\ \phi_1(x)e^{\theta|x|} \to 0, \quad \text{as } |x| \to \infty.$$

(V) f is *generic*; $u_0 \in C(\mathbb{R})$ has finite limits $a^\pm := u_0(\pm\infty)$ equal to zeros of f; and one of the following possibilities occurs:

$$a^- = a^+ \le u_0, \quad u_0 \le a^- = a^+, \quad a^- \le u_0 \le a^+, \quad a^- \ge u_0 \ge a^+.$$

In (I)–(IV), one can consider other zeros b of f in place of $b = 0$ and modify the assumptions on u_0 accordingly. For example, (I) applies, after the transformation $u \mapsto u+b$, when $f(b) = 0$, $u_0 \geq b$, and $u_0 - b$ has compact support. If the solution is localized, as in (II), then the convergence in (S1) clearly takes place in $L^\infty(\mathbb{R})$ and not just in $L^\infty_{loc}(\mathbb{R})$. In the cases (I)–(IV) (including the case (II), where u_0 may change sign), the limit steady state is either a constant function or it is a function of one sign which is a shift of an even function with unique critical point (a ground state at some level). The same is true in (V) if $a^- = a^+$. If $a^- \neq a^+$, the limit steady state is either a constant or a strictly monotone steady state (a standing front).

In (V), "f is generic" means that f is taken from an open and dense subset of the space $C^1_b(\mathbb{R})$. This set depends on whether $a^- = a^+$ or $a^- \neq a^+$, but in both cases it can be characterized by explicit conditions involving a class of traveling fronts, namely, traveling fronts appearing in a so-called minimal propagating terrace. The references for these generic results are [44, Section 2.5] for $a^- \neq a^+$ and [36] for $a^- = a^+$.

In the case (I), the convergence result was proved in [10]; earlier theorems under additional conditions can be found in [14, 16, 54]. The same result, with an additional information on the limit steady states and an extension to higher dimensions, was proved differently in [11].

Case (II) was considered in [18]; the convergence was proved there in the more general setting of time-periodic nonlinearities. Clearly, the localization property of u is a strong assumption. Unlike the boundedness, which is often easy to verify using super and sub-solutions (see the introduction), the assumption that u is localized is rather implicit; bounding u by time-independent and decaying super and sub-solutions would typically lead to $u(\cdot, t) \to 0$ as $t \to \infty$ and the convergence statement becomes trivial. However, the localization can often be verified for positive threshold solutions, that is, positive solutions on the boundary of the domain of attraction of the asymptotically stable steady state 0 (the stability is guaranteed by the assumption $f'(0) < 0$). Threshold solutions for reaction diffusion equations on $\mathbb{R}$ have been studied and proved to be convergent by several authors, see [4, 10, 14, 15, 16, 43, 18, 35, 40, 54] (related results in higher space dimension can be bound, for example, in [41] and references therein).

The proofs of the convergence results in the cases (III), (IV) can be found in [35]; in fact, [35] contains more general sufficient conditions for the convergence, of which (III) and (IV) are special cases.

We finish this section with brief remarks on convergence properties of bounded positive solution in higher space dimensions. Assuming that $f(0) = 0$, $f'(0) < 0$, and either u satisfies additional boundedness conditions in an integral norm or is localized, the convergence is proved in [1, 23] (earlier results under more restrictive conditions were given in [8, 17]). Convergence theorems for a class of asymptotically autonomous equations can be found in [6, 9, 23]. Assuming $f(0) = 0$, the locally uniform convergence to an equilibrium for nonnegative bounded solutions with compact initial support was established in [11]. For initial data which do not have compact support, bounded positive solutions, even localized ones, can behave in a more complicated manner [46, 47, 48].

2.2 Existence of a limit steady state

We next recall the following general result, valid for each bounded solution of (1.1) (with no extra conditions on u_0):

(S2) There is a sequence $t_n \to \infty$ such that $u(\cdot, t_n) \to \varphi$, in $L^\infty_{loc}(\mathbb{R})$, for some steady state φ of (1.1).

In other words, for any bounded solution u, the limit set $\omega(u)$ contains at least one steady state. This result was proved in [24] (see also [25]). In fact, more general nonlinearities, namely, nonlinearities depending on x, $f = f(x, u)$, are treated in [24] and the result is valid for equations on $\mathbb{R}^2$. The validity of the result for equations on $\mathbb{R}^N$ for $N \geq 3$ is open.

In [24], (S2) is derived from another statement, which is of independent interest. It says that on average each bounded solution approaches a set of steady states. To formulate this more precisely, we introduce a different ω-limit set, $\tilde{\omega}(u)$, as follows. We say that $\varphi \in \tilde{\omega}(u)$ if for each neighborhood $\mathcal{V}$ of φ in $L^\infty_{loc}(\mathbb{R})$ one has

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \chi_{\mathcal{V}}(u(\cdot, t))\, dt > 0$$

($\chi_{\mathcal{V}}$ stands for the characteristic function of $\mathcal{V}$). It is shown in [24] (for dimensions 1 and 2) that $\tilde{\omega}(u)$ is nonempty and consists entirely of steady states.

2.3 Examples of non-quasiconvergent solutions

In this section, we discuss bounded solutions which are not quasiconvergent:

(S3) $\omega(u)$ contains functions which are not steady states of (1.1).

An early evidence of the existence of such a solution was given in [12] for the nonlinearity $f(u) = u(1-u^2)$. The solution constructed there oscillates between the constant steady states -1, 1, while repeatedly annihilating pairs of kinks coming in from $\pm\infty$. The construction of [12] strongly suggests that the solution is not quasiconvergent and, more precisely, its ω-limit set contains a nonstationary solution which, in $L^\infty_{loc}(\mathbb{R})$, is a heteroclinic connection from -1 to 1 and another solution which is a heteroclinic connection from 1 to -1. This can indeed be verified rigorously, as shown in [42], at least if the initial data are chosen carefully. Further examples of non-quasiconvergence solutions were given in [42, 43] for *bistable* nonlinearities, that is, functions f satisfying the following conditions:

(BS) For some $\alpha < 0 < \gamma$ one has $f(\alpha) = f(0) = f(\gamma) = 0$, $f'(\alpha) < 0$, $f'(\gamma) < 0$, $f < 0$ in $(\alpha, 0)$, $f > 0$ in $(0, \gamma)$.

We say that a bistable nonlinearity f is *balanced* or *unbalanced* if, respectively,

$$\int_\alpha^\gamma f(s)\,ds = 0 \quad\text{or}\quad \int_\alpha^\gamma f(s)\,ds > 0. \tag{2.1}$$

It is well known that for any balanced bistable nonlinearity the stationary equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}, \tag{2.2}$$

has a solution v such that $\alpha < v < \gamma$, v is decreasing, and $v(-\infty) = \gamma$, $v(\infty) = \alpha$; of course $v(-x)$ is then a solution which is increasing and $v(\infty) = \gamma$, $v(-\infty) = \alpha$. We refer to such solutions as *standing waves* of (1.1). In the unbalanced case, (2.2) has a solution v such that $\alpha < v < \gamma$ and $v - \alpha \in C_0(\mathbb{R})$; we refer to this v as a *ground state* (more precisely, it is a ground state at level α).

Non-quasiconvergent solutions with additional properties, as indicated, have been found in the following cases (u stands for the solution of (1.1), (1.2)):

(I) f is bistable and balanced: there is $u_0 \in C(\mathbb{R})$ with $\alpha \le u_0 \le \gamma$, such that $\omega(u)$ contains the constant steady states α, γ and no other steady states.

(II) f is bistable and balanced: there is $u_0 \in C_0(\mathbb{R})$ with $\alpha \le u_0 \le \gamma$, such that $\omega(u)$ contains the constant steady states α, γ, as well as functions which are not steady states of (1.1).

(III) f is bistable and balanced: there is $u_0 \in C_0(\mathbb{R})$ with $\alpha \leq u_0 \leq \gamma$ such that $\omega(u)$ contains an increasing standing wave, a decreasing standing wave, as well as functions which are not steady states of (1.1).

(IV) f is bistable and unbalanced: there is $u_0 \in C_0(\mathbb{R})$ with $\alpha \leq u_0 \leq \gamma$ such that $\omega(u)$ contains the steady state α, a ground state ϕ at level α, as well as functions which are not steady states of (1.1).

Note that in (I) the non-quasiconvergence of the solution is guaranteed by the connectedness of $\omega(u)$ in $L^\infty_{loc}(\mathbb{R})$.

The proofs of (I), (II), and (III) can be found in [42]. The proof of (I) consists, essentially, of the construction from [12] done with some care so that the properties stated in (I) and in the discussion above can be rigorously verified. The proofs in (II) and (III) are more involved as u_0 is required to be in $C_0(\mathbb{R})$; unlike in (I), where u_0 is alternatingly equal to α and γ on large intervals. Thus (II), (III) show that large oscillation, or, oscillations with amplitudes bounded below by a positive constant, are not necessary for these constructions. It is necessary that u_0 changes sign, however. One of the results in the next section shows that if $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$ (or $u_0 \leq 0$), then the solution u is quasiconvergent.

The result in (IV) was first proved with with the weaker condition $u_0 \in C_b(\mathbb{R})$ [42]; then later with $u_0 \in C_0(\mathbb{R})$ by a more elaborate construction [43]. The fact, that the nonlinearity in (IV) is unbalanced shows another interesting fact. The presence of non-quasiconvergent solutions is not an exceptional phenomenon, it occurs for a robust class of nonlinearities (of course, the middle zero of f is put at 0 just for convenience, it can take any value between α, γ).

As mentioned above, the ω-limit set always consists of entire solutions. There is a vast variety of entire solutions, including spatially periodic heteroclinic orbits between steady states (see [19, 20] and references therein), traveling waves, and many types of "nonlinear superpositions" of traveling waves and other entire solutions (see [2, 3, 26, 29, 38, 39] and references therein). It is not clear which of these entire solutions can actually occur in the ω-limit set of a bounded solution of (1.1). On the other hand, it is also an interesting question what kind of entire solutions occur in $\omega(u)$ in the above examples of non-quasiconvergent solutions.

We already mentioned above that [42] shows that a heteroclinic loop between the constant steady states α and γ can occur in $\omega(u)$. By a heteroclinic loop we mean a pair of heteroclinic entire solutions—one connecting α to γ and another one connecting γ to α. Very likely, these heteroclinic solutions are the two-front entire solutions studied in detail in [3].

The result in the case (IV) hints at the existence of another entire solution – a rather curios one. Namely, as $t \to \infty$ the solution $u(\cdot, t)$ in (IV) must repeatedly visit small neighborhoods of ϕ, α, and ϕ again. This is indicative of the existence of a "heteroclinic loop" between the steady states ϕ, α. The existence of a solution connecting ϕ to α is well known and rather easy to establish: there is an entire solution $y(x,t)$ monotonically decreasing in t, such that $y(\cdot,t) \to \phi$ as $t \to -\infty$ and $y(\cdot,t) \to \alpha$ as $t \to \infty$, with the uniform convergence in both cases. The existence of a connection in the opposite direction, from α to ϕ, is more interesting; in view of the asymptotic stability of α and instability of ϕ it even seems to be impossible at the first glance. Such a connection does in fact exist, however, one must remember that the convergence to the limit steady states is not required to be uniform, only locally uniform, because that is convergence used in the definition of $\omega(u)$. A heteroclinic solution connecting α to φ was found in [37] and it takes a form of an entire solution U with an interesting spatial structure (see Figure 1). For $t \approx -\infty$, $U(\cdot, t)$ has two humps, coming from spatial infinity, one from $-\infty$, the other one from $+\infty$. As t increases, the humps move toward the origin $x = 0$, eventually "colliding" and mixing up, after which just one

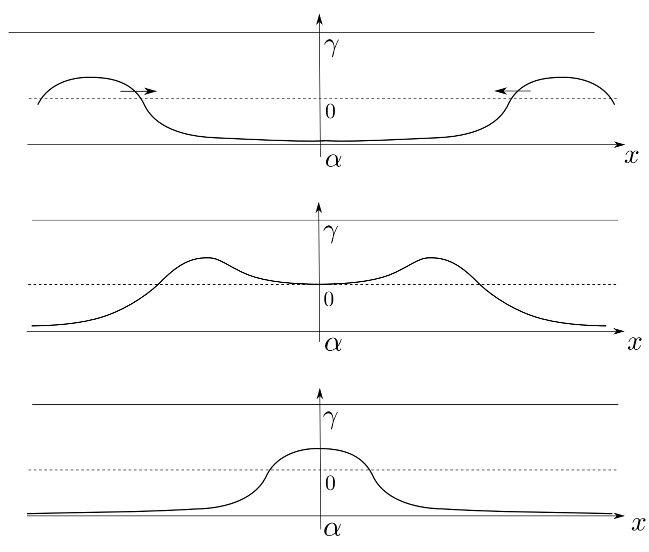

Figure 1: The shape of the entire solution $U(\cdot, t)$ for $t \approx -\infty$, $t = 0$, and $t \approx \infty$ (top to bottom).

hump forms as the solution approaches the ground state as $t \to \infty$. The presence of the moving humps, or, pulses, is perhaps the most interesting feature of this solution. It is well known that, unlike in reaction diffusion systems (see, for example, [13, 21, 31, 32, 33, 50, 51]), scalar equations (1.1) do not admit traveling pulses, that is, localized profiles moving with a constant nonzero speed. In accord with this, the humps in the solution $U(\cdot, t)$ do not move with constant speed; they slow down as $t \to -\infty$.

2.4 Quasiconvergence theorems

We now give sufficient conditions, in terms of the initial data, for the solution of (1.1), (1.2) to be quasiconvergent:

(S4) $\omega(u)$ consists of steady states of (1.1).

The most common way to prove the quasiconvergence of a solution is by means of a Lyapunov functional. For equation (1.1), the following energy functional is used frequently:

$$E(v) := \int_{-\infty}^{\infty} \big(\frac{v_x^2(x)}{2} - F(v(x)) \big)\, dx, \qquad F(v) := \int_0^v f(s)\, ds. \tag{2.3}$$

Of course, for this functional to be defined along a solution, one needs assumptions on f and u. Thus, if $f(0) = 0$ and $u_0 \in H^1(\mathbb{R})$, it can be proved that $E(u(\cdot, t))$ is a (finite) nonincreasing function on the existence time interval of the solution u of (1.1), (1.2). If u is bounded and the function $t \mapsto \|u(\cdot, t)\|_{L^2(\mathbb{R})}$ is bounded as well, then it can be proved that $t \mapsto E(u(\cdot, t))$ is bounded and u is quasiconvergent (the proof of this statement in a more precise form for equations (1.1) and (1.5) can be found in [16]).

For solutions which are not assumed to be bounded in an integral norm, the energy E is not very useful; in fact, as we have seen in the previous section, such solutions may not be quasiconvergent. Nonetheless, quasiconvergence has been proved for some classes of solutions, without the use of any Lyapunov functional.

Specifically, (S4) holds in the following cases:

(I) (Localized nonnegative initial data) $f(0) = 0$, $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$, and the solution u is bounded.

(II) (Front-like initial data) $u_0 \in C(\mathbb{R})$ and for some zeros $\alpha < \gamma$ of f one has $\alpha \leq u_0 \leq \gamma$, $u_0(-\infty) = \gamma$, $u_0(\infty) = \alpha$.

The quasiconvergence result in the case (II) is proved in [44, Section 2.4]. As shown there, the set $\omega(u)$ consists of constant steady states and standing waves of (1.1). There is also an extension of this result to the multidimensional problem (1.5), (1.6) [45]. There, the initial data u_0 are of the front-like type in the sense that

$$\lim_{x_N \to -\infty} u_0(x', x_N) = \gamma, \quad \lim_{x_N \to \infty} u_0(x', x_N) = \alpha,$$

where the limits are uniform in $x' := (x_1, \dots, x_{N-1})$.

In the case (I), the quasiconvergence result is proved in [35] and it says, more precisely, that $\omega(u)$ consist of steady states φ whose planar trajectories $\{(\varphi(x), \varphi'(x)) : x \in \mathbb{R}\}$ belong to a *chain* of the ODE (2.2). By a chain we mean a connected subset of $\mathbb{R}^2$ consisting of equilibria, heteroclinic orbits, and at most one homoclinic orbit of (2.2) (see Figure 2).

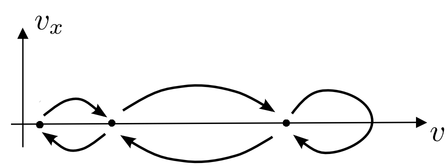

Figure 2: A chain in the phase-plane diagram of equation (2.2)

We are not aware of any extension of this result to the spatial dimension $N = 2$ (unless, one assumes that the support of u_0 is compact, as in the convergence results discussed in Section 2.1). In dimensions $N = 3$ and higher, the result—quasiconvergence of bounded solutions with initial data in $C_0(\mathbb{R}^N)$—is not valid, not even when the solutions are localized. This was shown in [48], where equation (1.5) with $f(u) = u^p$ and a suitable Sobolev-supercritical exponent p is considered. The existence of nonnegative bounded localized solutions which are not quasiconvergent is shown in that paper. The ω-limit sets of such solutions contain the trivial steady state and other entire solutions which are not steady states. Very likely, these entire solutions are homoclinic solutions which were found in [22]. It is also interesting that the non-quasiconvergent solutions in [48] are radially symmetric, hence, they are solutions of the "one-dimensional" problem:

$$\begin{aligned} &u_t = u_{rr} + \frac{N-1}{r} u_r + f(u), && r > 0,\, t > 0, \\ &u_r(0,t) = 0, && t > 0. \end{aligned} \tag{2.4}$$

This is another illustration of a well documented fact that, while equation (2.4) shares many properties with (1.1), sometimes the presence of the term $(N-1)u_r/r$ makes a big difference.

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