Positivity and symmetry of nonnegative solutions of semilinear elliptic equations on planar domains

P. Poláčik*

School of Mathematics, University of Minnesota

Minneapolis, MN 55455

Abstract. We consider the Dirichlet problem for the semilinear equation $\Delta u + f(u) = 0$ on a bounded domain $\Omega \subset \mathbb{R}^N$. We assume that Ω is convex in a direction e and symmetric about the hyperplane $H = \{x \in \mathbb{R}^N : x \cdot e = 0\}$. It is known that if $N \geq 2$ and Ω is of class C^2, then any nonzero nonnegative solution is necessarily strictly positive and, consequently, it is reflectionally symmetric about H and decreasing in the direction e on the set $\{x \in \Omega : x \cdot e > 0\}$. In this paper, we prove the same result for a large class of nonsmooth planar domains. In particular, the result is valid if any of the following additional conditions on Ω holds:

(i) Ω is convex (not necessarily symmetric) in the direction perpendicular to e,

(ii) Ω is strictly convex in the direction e,

(iii) Ω is piecewise-$C^{1,1}$.

Key words: semilinear elliptic equation, planar domain, positivity, symmetry of solutions

AMS Classification: 35J61, 35B06, 35B05

*Supported in part by NSF grant DMS-0900947

Contents

1 Introduction and the main results

Consider the elliptic problem

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and Ω is a bounded domain in $\mathbb{R}^N$, which is convex in one direction and reflectionally symmetric about a hyperplane orthogonal to that direction. Without loss of generality, changing the coordinate system if necessary, we assume that the direction is $e^1 := (1, 0, \dots, 0)$ (that is, Ω is convex in x_1, or, shortly, x_1-convex) and the symmetry hyperplane is given by

$$H_0 = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 = 0\}.$$

By a well-known theorem of Gidas, Ni, and Nirenberg [17] and its more general versions for nonsmooth domains, as given by Berestycki and Nirenberg [4] and Dancer [10], each positive solution u of (1.1), (1.2) is even in x_1:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega), \tag{1.3}$$

and, moreover, $u(x_1, x')$ decreases with increasing $|x_1|$:

$$u_{x_1}(x_1, x') < 0 \quad ((x_1, x') \in \Omega, x_1 > 0). \tag{1.4}$$

The method of moving hyperplanes, used in these results, was introduced by Alexandrov [1] and further developed and applied in a symmetry problem by

Serrin [28]. We refer the reader to [3, 22, 23, 24] for surveys of related results and references. Further extensions can be found in [5, 11] (equations with non-Lipschitz nonlinearities) or [8] (viscosity solutions), for example.

In this paper, we continue our investigation of nonnegative solutions. The question is whether the above symmetry and monotonicity theorem still holds if the positivity of the solution is relaxed to the assumption that the solution is nonnegative and not identical to zero.

In one dimension, the answer is no, as already noted in [17] and documented by the example $u'' + u - 1 = 0$ on $\Omega = (-(2k+1)\pi, (2k+1)\pi)$, $k \in \mathbb{N}$ (with the solution $u(x) = 1 + \cos x$). Of course, the case $N = 1$ is very special in that the boundary of Ω is not connected. So this example is not very indicative of what happens in higher dimension. If one allows the nonlinearity f to depend on $x' = (x_2, \dots, x_N)$, $f = f(x', u)$, which preserves the reflectional symmetry of the problem, then nonnegative solutions with nontrivial nodal sets in Ω do exist on some multidimensional domains, see [26] for specific examples. In [26], we further examined such nonnegative solutions and discovered that they have an interesting symmetry structure, similar to that of the solution $u(x) = 1 + \cos x$ in the one-dimensional example: each nonnegative solution is even in x_1 and, if it is not identical to zero, its nodal set divides Ω into a finite number of reflectionally symmetric subdomains in which the solution has the usual Gidas-Ni-Nirenberg symmetry and monotonicity properties (see Section 3 for more details). This result is valid for fully nonlinear equations

$$F(x, u, Du, D^2u) = 0, \quad x \in \Omega, \tag{1.5}$$

under suitable symmetry assumptions.

Examples of nonnegative solutions with nontrivial nodal sets, as given in [26], rely on the explicit dependence of f on spatial variables. But even with nonhomogeneous nonlinearities $f(x', u)$, there are domains such that no such example exists, regardless how $f(x', u)$ is chosen (see [26]). It is an interesting and natural question as to what kind of domains admit examples of solutions with nontrivial nodal sets and whether there are any such examples at all with spatially homogeneous equations. In this paper we focus on the latter problem.

For the homogeneous problem (1.1), (1.2), several results proving the nonexistence of nonnegative solutions with nontrivial nodal sets are available. In [7], the nonexistence is proved if Ω is a ball (see also the monographs [13, 16, 27] for a discussion and extensions of this result); in [9] it was proved for smooth domains which are convex in all directions; and in [21] the smoothness of $\partial\Omega$ together with a strict x_1-convexity was proved to be sufficient. Recently, we proved the nonexistence result for problem (1.1), (1.2) on a general C^2-domain [25]. For nonsmooth domains, a sufficient condition for the strict positivity of nonnegative nonzero solutions was given in [15]. It requires, roughly speaking, that for any $\delta > 0$ there be a fixed two-dimensional wedge W, such that if the tip of W is translated to any point of $\partial\Omega$ with $x_1 \geq \delta$, then W is contained in $\bar{\Omega}$. In the recent work [14], further sufficient conditions were derived from an extension of Serrin's result on overdetermined problems.

In this paper, we prove the positivity result for planar domains Ω, assuming, in addition to the symmetry and x_1-convexity, just a minor technical condition on Ω. Our result applies in particular to domains which are strictly x_1-convex or, more generally, domains whose boundary is piecewise $C^{1,1}$ near the points where the strict x_1-convexity fails.

To be more precise, we say that $\partial\Omega$ has a *step* at a point $y = (y_1, y_2) \in \partial\Omega$ if there exist a neighborhood $B \subset \mathbb{R}^2$ of y and $\epsilon > 0$ such that

$$B \cap \partial\Omega = \{(x_1, \zeta(x_1)) : x_1 \in (y_1 - \epsilon, y_1 + \epsilon)\}, \tag{1.6}$$

where $\zeta \in C(y_1 - \epsilon, y_1 + \epsilon) \cap C^2((y_1 - \epsilon, y_1) \cup (y_1, y_1 + \epsilon))$, $\zeta' \equiv 0$ on $(y_1 - \epsilon, y_1)$ and $\zeta' < 0$ on $(y_1, y_1 + \epsilon)$.

In particular, if $\partial\Omega$ has a step at y, then y is the right-end point of a horizontal portion of $\partial\Omega$. If Ω is strictly convex in x_1, $\partial\Omega$ contains no horizontal line segments (that is, segments parallel to the x_1-axis), hence it has no steps. In case $\partial\Omega$ does have steps, the following hypothesis requires $\partial\Omega$ to be piecewise $C^{1,1}$ near each of them.

(H1) If $\partial\Omega$ has a step at a point $y = (y_1, y_2) \in \partial\Omega$ and $\zeta : (y_1 - \epsilon, y_1 + \epsilon) \to \mathbb{R}$ is as in (1.6), then there is $\epsilon_1 \in (0, \epsilon]$ such that ζ'' is bounded on $(y_1, y_1 + \epsilon_1)$; in other words, $\zeta \in C^{1,1}(y_1, y_1 + \epsilon_1)$.

We also need a stronger regularity assumption on the function f near $u = 0$:

(H2) There are positive constants δ, α such that $f\big|_{[0,\delta)} \in C^{1,\alpha}[0, \delta)$.

Theorem 1.1. *Assume that Ω is a bounded domain in $\mathbb{R}^2$, which is x_1-convex and symmetric about H_0, and which satisfies condition* (H1). *Let*

$f : [0,\infty) \to \mathbb{R}$ be a locally Lipschitz function satisfying (H2). *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a nonnegative solution of* (1.1), (1.2)*, then either $u \equiv 0$ (hence, necessarily, $f(0) = 0$) or else $u > 0$ and u has the symmetry and monotonicity properties* (1.3) *and* (1.4).

The symmetry of u, as stated in Theorem 1.1, follows by the results of [4, 10], once we know that u is strictly positive. Note that the result is not true in general if f is merely continuous, even if Ω is a ball; see [5, 17] for examples, also see [5, 12] for local symmetry results for continuous f.

Remark 1.2. (i) Hypothesis (H1) can be relaxed by replacing the $C^{1,1}$ regularity of ζ on $(y_1, y_1+\epsilon)$ with the $C^{1,\alpha}$ regularity, for some $\alpha \in (0,1)$. However, some parts of the proof of Theorem 1.1 would then require different and significantly more involved arguments. We did not find this generalization worthwhile.

(ii) One can prove Theorem 1.1 (and Theorem 1.3 below) for a slightly more general equations than (1.1), for example, quasilinear equations considered in [25], which are invariant under reflections in all directions, not just in the direction of the x_1-axis.

(iii) The only purpose of hypothesis (H2) is to ensure that the solution u is of class C^3 near its nodal set in Ω. This follows by usual interior Hölder estimates, since $f(u)$ is of class $C^{1,\alpha}$ near the nodal set.

As we discuss in the last section, several alternative hypotheses can be used in place of (H1) in Theorem 1.1. An example is the convexity of Ω in x_2 (note that no symmetry of Ω in x_2 is assumed).

Theorem 1.3. *Theorem 1.1 remains valid if hypothesis* (H1) *is replaced with the assumption that Ω is convex in x_2.*

We use the assumption that $N = 2$ at several places. For example, we use the fact that the intersections of Ω with hyperplanes orthogonal to e^1 have convex and reflectionally symmetric connected components (they are line segments in dimension 2). Also, we employ an equal-angle restriction on the intersection of nodal curves of solutions of linear elliptic equation. Although some ideas from this paper apply in higher dimensions, they do not lead to similarly general results.

The paper is organized as follows. In the next section, we introduce notation associated with the method of moving hyperplanes and state some results for linear equations that facilitate the application of the method. Theorems 1.1 and 1.3 are proved in Section 3. Section 4 contains remarks on alternative hypotheses that can be used in Theorems 1.1, 1.3.

In the remainder of the paper, *the standing hypotheses* are that the domain $\Omega \subset \mathbb{R}^2$ is bounded, x_1-convex, and symmetric about H_0; and that f is a locally Lipschitz function. Hypotheses (H1), (H2), or the x_2-convexity are assumed only when explicitly stated.

2 Preliminaries

For any $\lambda \in \mathbb{R}$ and any set $G \subset \mathbb{R}^2$, we set

$$\begin{aligned} H_\lambda &:= \{x \in \mathbb{R}^2 : x_1 = \lambda\}, \\ \Sigma_\lambda^G &:= \{x \in G : x_1 > \lambda\}, \\ \Gamma_\lambda^G &:= H_\lambda \cap G, \\ \ell^G &:= \sup\{x_1 \in \mathbb{R} : (x_1, x_2) \in G \text{ for some } x_2 \in \mathbb{R}\}. \end{aligned} \tag{2.1}$$

When $G = \Omega$ and there is no danger of confusion, we often omit the superscript Ω and simply write Σ_λ for Σ_λ^Ω, ℓ for ℓ^Ω, etc.

Let P_λ stand for the reflection in the hyperplane H_λ. Note that since Ω is convex in x_1 and symmetric about the hyperplane H_0, $P_\lambda(\Sigma_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$.

For any function z on $\bar{\Omega}$ and any $\lambda \in [0, \ell]$, we define z^λ and $V_\lambda z$ by

$$\begin{aligned} z^\lambda(x) &:= z(P_\lambda x) = z(2\lambda - x_1, x_2), \\ V_\lambda z(x) &:= z^\lambda(x) - z(x). \end{aligned} \tag{2.2}$$

The function $V_\lambda z$ is defined on

$$\bar{\Omega} \cap P_\lambda(\bar{\Omega}) = \bar{\Sigma}_\lambda \cup P_\lambda(\bar{\Sigma}_\lambda).$$

Below we rely on the following standard observations. If u is a solution of (1.1), then u^λ satisfies the same equation as u in $\Omega \cap P_\lambda(\Omega)$. Hence, for any $x \in \Omega \cap P_\lambda(\Omega)$ we have

$$\Delta(u^\lambda - u) + f(u^\lambda) - f(u) = 0.$$

Therefore, by Hadamard's formula, the function $v = V_\lambda u$ solves on $U = \Omega \cap P_\lambda(\Omega) \supset \Sigma_\lambda$ the linear equation

$$\Delta v + c(x)v = 0, \quad x \in U, \tag{2.3}$$

where $c \in L^\infty(U)$ depends on λ, but its absolute value is bounded (uniformly in λ) by the Lipschitz constant of $f\big|_{[0,\max_{x\in\Omega} u(x)]}$.

Since $u \geq 0$, the Dirichlet condition (1.2) gives

$$v(x) \geq 0 \quad (x \in \partial\Sigma_\lambda \setminus \Gamma_\lambda). \tag{2.4}$$

Of course, on the remaining part of $\partial\Sigma_\lambda$, Γ_λ, we have

$$v(x) = 0 \quad (x \in \Gamma_\lambda). \tag{2.5}$$

At several occasions, we will use reflections in other directions $e \in S^1$. For that purpose, we introduce a similar notation:

$$\begin{aligned} H_\lambda^e &:= \{x \in \mathbb{R}^2 : x \cdot e = \lambda\}, \\ \Sigma_\lambda^{G,e} &:= \{x \in G : x \cdot e > \lambda\}, \\ \Gamma_\lambda^{G,e} &:= H_\lambda^e \cap G, \\ \ell^{G,e} &:= \sup\{x \cdot e : x \in G\}, \\ P_\lambda^e &:= \text{ the reflection in } H_\lambda^e. \end{aligned} \tag{2.6}$$

For a function z defined on Ω (or another set), we use the notation $V_\lambda^e z(x) := z(P_\lambda^e x) - z(x)$; the domain of $V_\lambda^e z$ is $\mathrm{Dom}(z) \cap P_\lambda^e(\mathrm{Dom}(z))$.

In addition to the usual maximum principle and the Hopf boundary principle, we shall use suitable versions of other well-known results concerning linear equations (2.3), as summarized in Propositions 2.1-2.3. We assume that U is a bounded domain in $\mathbb{R}^2$, $\beta_0 = \|c\|_{L^\infty(U)}$, $|U|$ denotes the measure of U, and $B(x_0, r)$ denotes the open disk of radius r centered at x_0. We remark that while we generally consider classical solutions of (1.1), it is sufficient to consider weaker notions of solutions when dealing with the linear equation (2.3). Below a solution of (2.3) refers to a strong solutions (a function in $W^{2,2}_{loc}(U)$ satisfying (2.3) almost everywhere). It would make no difference if weak solutions were considered instead; since $c \in L^\infty(U)$, each weak solution of (2.3) is automatically a strong solution [18, Section 8.3].

Proposition 2.1. *Let $v \in W^{2,2}_{loc}(U)$ be a solution of* (2.3)*.*

(i) If $v \geq 0$ in U, then either $v \equiv 0$ or $v > 0$ in U.

(ii) There is $\delta_0 > 0$ depending only on β_0 such that the conditions $|U| < \delta_0$ and $\liminf_{x\to\partial U} v \geq 0$ imply $v \geq 0$ in U.

(iii) If $v \equiv 0$ in a nonempty open subset of U, then $v \equiv 0$ in U.

Note that no sign condition on the coefficient c is needed in Proposition 2.1. Statement (i) is the standard strong maximum principle for nonnegative solutions. Statement (ii) is the maximum principle for small domains (see [4, 6]). Statement (iii) is the well-known weak unique continuation theorem.

Proposition 2.2. *Assume that U is a connected component of $\Sigma_\lambda^{W,e}$ for some bounded $C^{1,1}$ domain W, some $\lambda \in \mathbb{R}$, and some direction $e \in S^1$ which is tangent to ∂W at a point $x^0 \in \partial W \cap \bar{U} \cap H_\lambda^e$. If v is a solution of (2.3) such that*

$$v \geq 0 \text{ in } U,\ v \in C^2(\bar{U}), \tag{2.7}$$

and $v(x^0)$, $Dv(x^0)$, $D^2v(x^0)$ all vanish, then $v \equiv 0$ in U.

This is a version of the corner point lemma proved in [28], but we have to justify our hypotheses. First we note that the condition $c \equiv 0$ in [28, Lemma 2] can be removed, thanks to the fact that v is allowed to be a supersolution there (cf. p. 316 in [28]). The main difference of the above statement from [28, Lemma 2] is that W is assumed to be of class $C^{1,1}$, rather than C^2. The C^2-assumption is used in the proof of [28, Lemma 2] to guarantee that W satisfies the interior ball condition at x_0. This remains valid under the weaker $C^{1,1}$ assumption (cf. [2]). The rest of the proof of [28, Lemma 2] applies without change.

The next lemma is a form of the "sweeping principle."

Proposition 2.3. *Let $v, w \in W^{2,2}_{loc}(U)$ be two solutions of (2.3) such that $v > 0$ in U and $\limsup_{x\to\partial U} w \leq 0$. Then either $w \leq 0$ in U or there is $\beta > 0$ such that $\beta v \equiv w$ in U.*

Proof. First we claim that for a sufficiently large $\sigma > 0$ we have $\sigma v > w$ in U. Since $v, w \in W^{2,2}_{loc}(U) \subset C(U)$, for any compact set $K \subset U$, we achieve $\sigma v > w$ in K by choosing $\sigma = \sigma(K)$ large enough. Proposition 2.1(iii) implies that if we do this with a set K such that $U \setminus K$ has a small enough measure, then $\sigma v > w$ in U. Let now $\beta = \inf\{\sigma > 0 : \sigma v(x) \geq w(x)\ (x \in U)\}$. If $\beta = 0$, then $w \leq 0$ in U. If $\beta > 0$, then $\beta v - w \geq 0$, so, by Proposition 2.1(iii), either $\beta v \equiv w$ in U or $\beta v - w > 0$ in U. However, the latter would imply that if $K \subset U$ is a compact set, then $\sigma v - w > 0$ in K for all $\sigma \leq \beta$ close enough to β. Choosing again K such that $U \setminus K$ has a small enough measure, we find $0 < \sigma < \beta$ such that $\sigma v - w > 0$ in U, contradicting the definition of β. Thus the possibility $\beta v - w > 0$ in U is ruled out and the lemma is proved. $\square$

We finish this section with the following result, essentially a corollary to Proposition 2.2, which will come handy in the next section. Since this is a local result, the symmetry hypothesis on Ω plays no role.

Lemma 2.4. *Let $u \in C^2(\Omega) \cup C(\bar{\Omega})$ be a nonnegative solution of* (1.1)*. Let W be a $C^{1,1}$ subdomain of Ω, x^0 a point on ∂W, and e a unit vector, tangent to ∂W at x^0. Set $\lambda = e \cdot x^0$ (so that $x^0 \in H^e_\lambda$). Assume that there is a ball B centered at x^0 such that the following conditions are satisfied:*

(i) $V^e_\lambda u$ is defined on $\bar{\Sigma}^{W,e}_\lambda \cap \bar{B}$ and $V^e_\lambda u \in C^2(\bar{\Sigma}^{W,e}_\lambda \cap \bar{B})$,

(ii) $V^e_\lambda u \geq 0$ on $\Sigma^{W,e}_\lambda \cap B$,

(iii) $u = 0$ and $\nabla u = 0$ on $\bar{\Sigma}^{W,e}_\lambda \cap \partial W \cap B$.

Then necessarily $V^e_\lambda u \equiv 0$ on the (unique) connected component of $\Sigma^{W,e}_\lambda$ whose closure contains x^0.

Remark 2.5. If $x^0 \in \Omega$ and the ball B is such that $\bar{B} \subset \Omega$, then, obviously, $V^e_\lambda u$ is defined everywhere in $\bar{B}$, regardless of the direction e, and $V^e_\lambda u \in C^2(\bar{B})$. Thus condition (i) holds trivially for small balls. However, in case $x^0 \in \partial\Omega$, for $V^e_\lambda u$ to be defined on $\bar{\Sigma}^{W,e}_\lambda \cap \bar{B}$, one must have $P^e_\lambda(\bar{\Sigma}^{W,e}_\lambda \cap \bar{B}) \subset \bar{\Omega}$. Also, the C^2 regularity of $V^e_\lambda u$ up to the boundary of $\Sigma^{W,e}_\lambda \cap B$ is not automatically guaranteed if $x^0 \in \partial\Omega$.

In applications below, W is chosen such that $\bar{\Sigma}^{W,e}_\lambda \cap \partial W \cap B$ is a nodal curve of u in Ω. As $u \geq 0$ in Ω, condition $\nabla u = 0$ in (iii) is then automatically satisfied.

Proof of Lemma 2.4. To apply Proposition 2.2, we first choose a smaller $C^{1,1}$ domain $\tilde{W} \subset W \cap B$ which shares with W its boundary in a neighborhood of x^0 and is such that $H^e_\lambda \cap \tilde{W}$ is a line segment. Let U be the connected component of $\Sigma^{\tilde{W},e}_\lambda$ whose closure contains x^0. Set $v = V^e_\lambda u$. Clearly, conditions (i), (ii) imply that the hypotheses of Proposition 2.2 up to (2.7) are satisfied. We use (iii) to show that $v(x^0)$, $Dv(x^0)$, $D^2v(x^0)$ all vanish.

First, since $x^0 \in H^e_\lambda$, we have $v(x^0) = V^e_\lambda u(x^0) = 0$. For the same reason, if η is a unit vector orthogonal to e, then we have the following relations for the directional derivatives:

$$(D_\eta v)(x^0) = (V_\lambda^e(D_\eta u))(x^0) = 0,$$
$$(D_{\eta\eta} v)(x^0) = (V_\lambda^e(D_{\eta\eta} u))(x^0) = 0.$$

Next, on the closure of $H_\lambda \cap \tilde{W}$, we have $D_e v = -2D_e u$ and, similarly, $D_{e\eta} v = -2D_{e\eta} u$. Consequently, by condition (iii), $D_e v(x^0) = 0$. Further, differentiating the second identity in (iii) along ∂W, we obtain $D_{e\eta} v(x_0) = -2D_{e\eta} u(x_0) = 0$. Finally, the derivative $D_{ee} v(x) = D_{ee} u(P_\lambda^e x) - D_{ee} u(x)$ vanishes for $x \in H_\lambda^e$, consequently, $D_{ee} v(x^0) = 0$. Since e, η are orthonormal, these computations show that $v(x^0) = 0$, $Dv(x^0) = 0$, $D^2 v(x^0) = 0$.

By Proposition 2.2, $V_\lambda^e u \equiv 0$ in U. By Proposition 2.1(iii), one also has $V_\lambda^e u \equiv 0$ in the connected component of $\Sigma_\lambda^{W,e}$ containing U, which proves the lemma. □

3 Proofs of the main Theorems

Throughout the section we assume that the standing hypotheses are satisfied and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a nonnegative solution of (1.1), (1.2).

If $f(0) \geq 0$, then the maximum principle implies that either $u \equiv 0$ or $u > 0$ and there is nothing to prove. We therefore assume that

$$f(0) < 0. \tag{3.1}$$

This condition in particular implies that $u \not\equiv 0$.

We start by quoting the following result of [26].

Theorem 3.1. *There exist $m \in \mathbb{N}$ and constants $\lambda_1, \dots, \lambda_m$ with the following properties:*

(i) $0 = \lambda_m < \lambda_{m-1} < \dots < \lambda_1 < \ell$.

(ii) For $i = 1, \dots, m$, $V_{\lambda_i} u \equiv 0$ on a connected component of Σ_{λ_i}. In particular, as Σ_0 is connected, $V_0 u \equiv 0$ in Σ_0, that is, u is even in x_1.

(iii) There are open mutually disjoint open sets $G_i \subset \Omega$, $i = 1, \dots, m$, with G_m possibly empty, such that the following statements are true:

(a) $\emptyset \neq G_i \subset \Sigma_0 \quad (i = 1, \dots, m-1)$.

(b) $\bar{\Omega} = \bar{G}_m \cup \bigcup_{i=1}^{m-1}(\bar{G}_i \cup P_0(\bar{G}_i))$.

(c) *For* $i = 1, \ldots, m$, *the set* G_i *is convex in* x_1 *and* $P_{\lambda_i}(G_i) = G_i$.

(d) *For* $i = 1, \ldots, m$, *one has* $u > 0$ *in* G_i, $u = 0$ *on* ∂G_i, $V_{\lambda_i} u \equiv 0$ *in* G_i, *and* $u_{x_1} < 0$ *in* $\Sigma_{\lambda_i}^{G_i}$.

(iv) $\Sigma_{\lambda_1}^{G_1}$ *is the union of (some) connected components of* $\Sigma_{\lambda_1} = \Sigma_{\lambda_1}^{\Omega}$.

Up to (iii), these are the statements of [26, Theorem 2.2]. Statement (iv), although not explicitly included in that theorem, follows from the definitions of λ_1 and the set G_1 in [26, Sections 4.2, 4.3]. Namely,

$$\lambda_1 := \inf \{\mu \in (0, \ell] : V_\lambda u(x) > 0 \ \text{ for all } x \in \Sigma_\lambda \text{ and } \lambda \in [\mu, \ell]\}, \tag{3.2}$$

and $\Sigma_{\lambda_1}^{G_1}$ is the union of all connected components of Σ_{λ_1} on which $V_\lambda u$ vanishes identically (G_1 is determined from this by the symmetry requirement $P_{\lambda_1} G_1 = G_1$).

The results of [26] are proved for fully nonlinear equations under suitable symmetry conditions. They apply to the semilinear problem (1.1), (1.2) without any additional assumptions on f ((H2) is not needed).

If $\lambda_1 = 0$ (hence $m = 1$), (ii) and (iii) give the usual symmetry and monotonicity properties of u and, in fact, u is positive in Ω in that case. The proofs of Theorems 1.1 and 1.3 consist in showing that, under the given hypotheses, $\lambda_1 > 0$ cannot occur.

We continue assuming that $\lambda_1 > 0$. From now on, we also assume hypothesis (H2) to hold.

In Subsection 3.1, we draw several conclusions from the assumption $\lambda_1 > 0$. In Subsections 3.2, 3.3, we show that these conclusions lead to a contradiction under the hypotheses of Theorems 1.1, 1.3, respectively.

In the proof of Theorem 1.1, we do not need the global description of the nodal structure of u, as given in Theorem 3.1. We only need the symmetry properties of the function u and the set G_1. In the proof of Theorem 1.3, we use in addition the fact that the nodal set of u contains all local minima of u. More precisely, the following result is a direct consequence of statements (b)-(d) of Theorem 3.1.

Corollary 3.2. *If* $y = (y_1, y_2)$ *is a point in* Ω *such that for some* $\epsilon > 0$ $u_{x_1}(\cdot, y_2) < 0$ *on* $(y_1 - \epsilon, y_1)$ *and* $u_{x_1}(\cdot, y_2) > 0$ *on* $(y_1, y_1 + \epsilon)$, *then* $u(y) = 0$.

For the remainder of this section, we fix a connected component G of the set G_1. By statements (a) and (iv) of Theorem 3.1, $G \subset \Sigma_0$ and $\Sigma_{\lambda_1}^G$ is a connected component of Σ_{λ_1}. In particular,

$$\partial G \cap \bar{\Sigma}_{\lambda_1} \subset \partial\Omega. \tag{3.3}$$

By statement (d), $V_{\lambda_1} u \equiv 0$ in G and

$$V_\lambda u > 0 \text{ in } \Sigma_\lambda^G \text{ for each } \lambda > \lambda_1. \tag{3.4}$$

For later reference, we also formulate a similar condition for the opposite direction $-e^1$:

$$V_{-\lambda}^{-e^1} u > 0 \text{ in } \Sigma_{-\lambda}^{G,-e^1} \text{ for each } \lambda < \lambda_1. \tag{3.5}$$

This follows from (3.4) and statement (d) of Theorem 3.1.

3.1 Consequences of $\lambda_1 > 0$

We first examine the structure of ∂G in more detail. Set

$$S := \partial G \cap \Omega.$$

Below, we refer to lines parallel to the x_1-axis as horizontal and to their perpendicular lines as vertical.

Lemma 3.3. *There are numbers $a < b$ and a C^2-function $\mu : (a, b) \to (-\infty, \lambda_1)$ such that*

$$S = \{(\mu(x_2), x_2) : x_2 \in (a, b)\} \tag{3.6}$$

and one has

$$\partial G = S \cup P_{\lambda_1}(S) \cup J_a \cup J_b, \tag{3.7}$$

where J_a, J_b are closed connected subsets of the horizontal lines $T_a := \mathbb{R} \times \{a\}$, $T_b = \mathbb{R} \times \{b\}$, respectively (thus each of them is either a point or a closed line segment).

Proof. Since the domain G is convex in x_1 and symmetric about H_{λ_1}, it is contained in the strip

$$T_{(a,b)} := \mathbb{R} \times (a, b), \tag{3.8}$$

where $a < b$ are such that $H_{\lambda_1} \cap G = \{\lambda_1\} \times (a, b)$. By the x_1-convexity and symmetry of G, the set $J_a := \partial G \cap T_a$ is either a closed segment, symmetric about H_{λ_1}, or the set consisting of the single point (λ_1, a). An analogous statement applies to $J_b := \partial G \cap T_b$.

Take now any point $x \in \partial G \cap \Sigma_{\lambda_1} \setminus (T_a \cup T_b)$. The x_1-convexity and symmetry of Ω, and the assumed relation $\lambda_1 > 0$ imply $P_{\lambda_1} x \in \Omega$, hence $P_{\lambda_1} x \in S$. It follows that

$$P_{\lambda_1}(S \setminus (T_a \cup T_b)) = \partial G \cap \bar{\Sigma}_{\lambda_1} \setminus (T_a \cup T_b),$$

which proves (3.7).

Next we prove that S is a C^2 submanifold of Ω. Since $u = 0$ on $\partial G \supset S$ and $u \geq 0$ in Ω, we have

$$\nabla u = 0 \text{ on } S. \tag{3.9}$$

Fix any $x^0 \in S$. Since

$$\Delta u(x^0) = -f(u(x^0)) = -f(0) > 0,$$

we have $u_{x_i x_i}(x^0) \neq 0$, where $i = 1$ or $i = 2$. By the implicit function theorem, there is a neighborhood B of x^0 such that $\{x \in B : u_{x_i}(x) = 0\} = \gamma$, where γ is a one-dimensional C^2 submanifold of Ω (here we use the fact that $u \in C^3$ near its zeros, cp. Remark 1.2(iii)). Obviously, $S \cap B \subset \gamma$. We have thus proved that the following is true:

(SM) Locally, near any of its points, S is a subset of a C^2 curve (that is, a connected one-dimensional submanifold) in Ω.

Next we observe that the x_1-convexity of Ω implies that for each $x \in G$, the horizontal line T passing through x intersects $\partial G \cap \bar{\Sigma}_{\lambda_1}$ at exactly one point or at a closed segment of T. The same is therefore true for S. In particular, if two points of S lie on the same horizontal line, then a segment on that line must be contained in S. However, the following claim in particular implies that there is no horizontal line segment M in S. Hence no two points of S lie on the same horizontal line. This readily implies that (3.6) is valid for some function μ, therefore, by (SM), S is a C^2 curve itself. Also note that the function μ takes values in $(-\infty, \lambda_1)$, since $S \subset G \setminus \bar{\Sigma}_{\lambda_1}$. To show that μ is a C^2-function, we now only need to verify that the tangent lines to S are nowhere horizontal. This also follows from the following claim. Hence, once we prove the claim, the proof of Lemma 3.3 will be complete.

Claim. If M is a C^2 curve contained in S and x^0 is any point in M, then the tangent line to M at x^0 is not horizontal. In particular, S cannot contain a horizontal line segment.

We prove this by contradiction. Assume that the tangent line to M at x^0 is horizontal, that is, $e := -e^1$ is tangent to M at x^0. Set $\lambda := e^1 \cdot x^0 < \lambda_1$. Choose $\epsilon > 0$ so small that $B := B(x^0, \epsilon)$ has the following properties:

$$\bar{B} \subset \Omega \setminus \bar{\Sigma}_{\lambda_1}, \tag{3.10}$$

$$\bar{B} \cap S = \bar{B} \cap M \tag{3.11}$$

(the latter is possible by statement (SM)). In particular, $\partial G \cap \bar{B} \subset S$. It is then easy to find a C^2 subdomain $W \subset G$ such that $W \cap \bar{B} = G \cap \bar{B}$.

By (3.10) and (3.5), the function $V^e_{-\lambda} u$ is (strictly) positive in $\Sigma^{G,e}_{-\lambda} \supset \Sigma^{W,e}_{-\lambda} \cap B$. Since $\partial W \cap \bar{B} \subset S$ is a nodal line of u, the hypotheses of Lemma 2.4 are satisfied (cp. Remark 2.5). However, the strict positivity of $V^e_{-\lambda} u$ contradicts the conclusion of Lemma 2.4. $\square$

Lemma 3.4. *Let a, b be as in Lemma 3.3 and $T_{(a,b)}$ as in* (3.8)*. There is a function $\tilde{u} \in C^2(T_{(a,b)}) \cap C(\bar{T}_{(a,b)})$ such that $\tilde{u} \equiv u$ in $\bar{\Omega} \cap \bar{T}_{(a,b)}$ and $\Delta \tilde{u} + f(\tilde{u}) \equiv 0$ in $T_{(a,b)}$.*

Proof. Obviously, $\bar{\Omega} \cap \bar{T}_{(a,b)}$ is x_1-convex. Using the symmetry of $u\big|_G$ about H_{λ_1} and reflecting in H_{λ_1}, we find an extension u^* of $u\big|_{\Omega \cap T_{(a,b)}}$ to $(\Omega \cap T_{(a,b)}) \cup P_{\lambda_1}(\Omega \cap T_{(a,b)})$, which solves there the same equation (1.1). Moreover, by Theorem 3.1, u is also symmetric about the hyperplanes H_0, $H_{\pm\lambda_1}$, hence u^* is symmetric about H_0, $H_{\pm\lambda_1}$, $H_{2\lambda_1}$. Clearly, we can use these symmetries and further reflections to eventually obtain the extension to $T_{(a,b)}$, as needed. $\square$

The previous lemma allows us to apply to $u\big|_G$ the method of moving hyperplanes in the direction $e^2 := (0, 1)$. The point is that, although a priori we do not know if G itself is x_2-convex or not, $T_{(a,b)}$ is of course x_2-convex. We apply the moving hyperplanes to the extension $\tilde{u}$, while still employing the Dirichlet condition satisfied by u on ∂G. That way we will establish the following additional symmetries of G and $u\big|_G$.

Lemma 3.5. *Let a, b, and μ be as in Lemma 3.3 and $\theta = (a + b)/2$. Then the following statements are valid:*

(i) *$\mu' > 0$ on (θ, b) and $\mu \equiv \mu(2\theta - \cdot)$ on (a, b) (that is, G is symmetric about $H^{e^2}_\theta$),*

(ii) *$u(P^{e^2}_\theta x) = u(x)$ for each $x \in G$ and $u_{x_2}(x) < 0$ for each $x \in G$ with $x_2 > \theta$.*

Proof. Let $\tilde{u}$ be as in Lemma 3.4. For $\lambda \in [\theta, b)$, consider the function $v := V_\lambda^{e^2}\tilde{u}$. It solves a linear equation (2.3) on $T_{(\lambda,b)}$ whose coefficient c is bounded uniformly with respect to λ. Since u (and hence $\tilde{u}$) vanishes on ∂G and $\tilde{u} \geq 0$ everywhere in $T_{(a,b)}$, we have $v \geq 0$ on the boundary of $T_{(\lambda,b)} \cap G$. For $\lambda \approx b$, $T_{(\lambda,b)} \cap G$ has small measure. Hence, by Proposition 2.1(ii), $V_\lambda^{e^2}\tilde{u} \geq 0$ in $T_{(\lambda,b)} \cap G$. By Proposition 2.1(i), either $V_\lambda^{e^2}\tilde{u} > 0$ or $V_\lambda^{e^2}\tilde{u} \equiv 0$ in $T_{(\lambda,b)} \cap G$. The latter cannot happen if $\lambda > \theta$. Indeed, we have $\tilde{u} = u > 0$ in G, whereas $V_\lambda^{e^2}\tilde{u} \equiv 0$ in conjunction with $\tilde{u} = u = 0$ on $\partial G \supset J_b$ would imply that $\tilde{u}$ vanishes at $P_\lambda^{e^2}((\lambda_1, b)) \in G$. Hence $V_\lambda^{e^2}\tilde{u} > 0$ in $T_{(\lambda,b)} \cap G$ for all $\lambda \approx b$. Set

$$\nu_0 := \inf\{\nu \in [\theta, b) : V_\lambda^{e^2}\tilde{u} > 0 \text{ in } T_{(\lambda,b)} \cap G \text{ for all } \lambda \in [\nu, b)\}. \tag{3.12}$$

Then $V_{\nu_0}^{e^2}\tilde{u} \geq 0$ in $T_{(\nu_0,b)}$. As above, if $\nu_0 > \theta$, then we obtain the strict inequality. Choose a compact set $K \subset G \cap T_{(\nu_0,b)}$. For $\lambda < \nu_0$ close enough to ν_0, we have $V_\lambda^{e^2}\tilde{u} > 0$ in K. Choosing K suitably, we achieve that $T_{(\lambda,b)} \cap G \setminus K$ has small measure for $\lambda \approx \nu_0$. Then it follows from Proposition 2.1(ii) that $V_\lambda^{e^2}\tilde{u} > 0$ in $T_{(\lambda,b)} \cap G \setminus K$, hence in $T_{(\lambda,b)} \cap G$, in contradiction to the definition of ν_0. Thus necessarily $\nu_0 = \theta$.

By continuity, $V_\theta^{e^2}\tilde{u} \geq 0$ in $T_{(\theta,b)} \cap G$ and, as usual, the Hopf boundary principle applied to $V_\lambda^{e^2}\tilde{u}$ gives $u_{x_2}(x) < 0$ for each $x \in G$ with $x_2 = \lambda > \theta$. Applying an analogous moving hyperplane procedure in the opposite direction, we obtain $V_\theta^{e^2}\tilde{u} \leq 0$ in $P_\theta^{e^2}(T_{(a,\theta)} \cap G) = T_{(\theta,b)} \cap P_\theta^{e^2}(G)$. Consequently, $V_\theta^{e^2}\tilde{u} \equiv 0$ in $T_{(\theta,b)} \cap G \cap P_\theta^{e^2}(G)$. Since this open set is nonempty (it contains the vertical segment $\{(\lambda_1, x_2) : x_2 \in (\theta, b)\}$), by unique continuation we have $V_\theta^{e^2}\tilde{u} \equiv 0$ in $T_{(a,b)}$. This and the relations $\tilde{u} \equiv u > 0$ in G and $\tilde{u} = u = 0$ on ∂G, readily imply that G is symmetric about $H_\theta^{e^2}$, which gives $\mu \equiv \mu(2\theta - \cdot)$ on (a, b).

It remains to prove that $\mu'(\lambda) > 0$ for each $\lambda \in (\theta, b)$. For any $\lambda \in (\theta, b)$, we have $V_\lambda^{e^2}\tilde{u} > 0$ in $T_{(\lambda,b)} \cap G$. We first rule out the relation $\mu'(\lambda) < 0$. Assume it holds. Then one easily finds a point $x \in T_{(\lambda,b)} \cap G$ in a vicinity of $(\mu(\lambda), \lambda)$ such that $P_\lambda^{e^2}x \in \partial G$. But then, since $\tilde{u}(x) > 0 = \tilde{u}(P_\lambda^{e^2}x)$, we have $V_\lambda^{e^2}\tilde{u}(x) < 0$, a contradiction. Thus $\mu'(\lambda) \geq 0$. The equality would mean that e^2 is tangent to $S \subset \partial G$ at $(\mu(\lambda), \lambda)$. This is easily excluded using Lemma 2.4, Remark 2.5, and the relation $V_\lambda^{e^2}\tilde{u} > 0$ in $T_{(\lambda,b)} \cap G$. □

Some arguments in the next section rely on regularity of the function u (or, more precisely, the restriction of u to $\bar{G}$) up to the boundary of G.

Basic regularity properties are established in the following lemma. We use the simplified notation

$$u_i = u_{x_i} = D_i u, \quad u_{ij} = u_{x_i x_j}.$$

Lemma 3.6. *(i) $u \in W^{2,p}(G)$ for each $p \in [1, \infty)$ and $u_1, u_2 \in C^\alpha(\bar{G})$ for each $\alpha \in (0,1)$.*

(ii) $u_1 \in W_0^{1,2}(G)$ and it is a weak solution of the problem

$$\Delta v + f'(u(x))v = 0, \quad x \in G, \tag{3.13}$$

$$v = 0, \quad x \in \partial G. \tag{3.14}$$

Consequently, $u_1 \in W_{loc}^{2,2}(G)$ and it is a strong solution of (3.13).

(iii) If J_a, J_b consist of single points, then statement (ii) holds with u_1 replaced by u_2.

Proof. We have $u \in W_0^{1,2}(G)$. Indeed, since $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a classical solution of (1.1), (1.2), and $u = 0$ on ∂G, it coincides on G with the unique weak $W_0^{1,2}(G)$-solution of (the Dirichlet problem for) the equation

$$\Delta u = -h(x), \quad x \in G,$$

with $h(x) = f(u(x))$.

From $u \in W_0^{1,2}(G)$ it follows that the function $\bar{u}$ defined by

$$\bar{u}(x) = \begin{cases} u(x), & x \in \bar{G}, \\ 0, & x \in \bar{T}_{(a,b)} \setminus \bar{G}, \end{cases}$$

belongs to $W_0^{1,2}(T_{(a,b)})$. Clearly, the function $\bar{h}$ defined by

$$\bar{h}(x) = \begin{cases} f(u(x)), & x \in \bar{G}, \\ 0, & x \in \bar{T}_{(a,b)} \setminus \bar{G}, \end{cases}$$

belongs to $L^\infty(T_{(a,b)})$. Since $\nabla u \equiv 0$ on $\partial G \cap T_{(a,b)}$ (cp. (3.9), (3.7)), an integration by parts shows that for each C^1 function ϕ with compact support in $T_{(a,b)}$ one has

$$\int_{T_{(a,b)}} \nabla \bar{u} \nabla \phi = \int_{T_{(a,b)}} \bar{h} \phi,$$

hence $\bar{u}$ is a weak solution of the Dirichlet problem for

$$\Delta\bar{u} = -\bar{h}(x), \quad x \in T_{(a,b)}.$$

Now, the boundary of $T_{(a,b)}$ being smooth, we obtain by standard elliptic regularity results that $\bar{u} \in W^{2,p}(T_{(a,b)} \cap B)$ for each $p \in [1,\infty)$ and each ball B in $\mathbb{R}^2$. Consequently, by the Sobolev imbedding theorem $\bar{u}_1, \bar{u}_2 \in C^\alpha(\bar{T}_{(a,b)} \cap B)$ for each $\alpha \in (0,1)$ and each ball B. This proves statement (i).

Further, for the derivatives of $\bar{u}$ we have $\bar{u}_1, \bar{u}_2 \in W^{1,2}(T_{(a,b)}) \cap C(\bar{T}_{(a,b)})$ and $\bar{u}_1 \equiv \bar{u}_2 \equiv 0$ in $T_{(a,b)} \setminus \bar{G}$. If the sets J_a and J_b are segments, rather than single points, then also $\bar{u}_1 = 0$ on $J_a \cup J_b$, which shows that the trace of $\bar{u}_1$ is zero. Hence (thanks to the regularity of the boundary of $T_{(a,b)}$), $\bar{u}_1 \in W_0^{1,2}(T_{(a,b)})$. Since $\bar{u}_1$ is the zero extension of $u_1\big|_G$, this means that $u_1 \in W_0^{1,2}(G)$. The fact that u_1 is a weak solution of (3.13), (3.14), is proved easily using (1.1). Since the function $x \mapsto f'(u(x))$ belongs to $L^\infty(G)$, u_1 is also a strong solution of (3.13) (see [18, Section 8.3]). This proves statement (ii).

If J_a, J_b consists of single points, the trace of $\bar{u}_2$ is zero also. The above arguments then apply to $\bar{u}_2$ equally well and one obtains (iii). □

3.2 Proof of Theorem 1.1

We derive a contradiction from the conclusions obtained in the previous subsection, assuming that (H1) holds (in addition to the standing hypotheses and (H2)). Actually, we use different arguments depending on whether the set J_b in (3.7) is a single point or a line segment and hypothesis (H1) is only needed in the latter case.

CASE A: J_b is a single point.

By the symmetry of G about $\theta = (a+b)/2$ (Lemma 3.5), J_a is a single point, as well. By Lemma 3.6, $u_1, u_2 \in W_0^{1,2}(G) \cap C(\bar{G})$ and they are (strong) solutions of (3.13).

We use a comparison argument with the functions $v(x) := -u_1(x)$ and $w(x) := -(x_2-\theta)u_1(x)+(x_1-\lambda_1)u_2(x)$. Note that, after putting the origin at the point (λ_1, θ) and letting ρ, φ be the polar coordinates, w coincided with the angular derivative u_φ of u. Clearly, $w \in W_0^{1,2}(G) \cap C(\bar{G})$ and it is also a solution of (3.13). Since (λ_1, θ) is the intersection of H_{λ_1} and $H_\theta^{e^2}$, the lines of symmetry of u, we have $w = 0$ on ∂G_{00}, where $G_{00} := \Sigma_{\lambda_1}^G \cap \Sigma_\theta^{G,e^2}$ is the "upper-right quarter" of G. Since $v > 0$ in $\Sigma^G_{\lambda_1} \supset G_{00}$, applying Proposition 2.3 to the functions v, w, and then to v, $-w$, we obtain that $w \equiv 0$ in G_{00}. Consequently, by unique continuation, $w \equiv 0$ in G. This means that $u\big|_G$ is radially symmetric around (λ_1, θ). Since $u > 0$ in G and $u = 0$ on ∂G, G must be a ball centered at (λ_1, θ).

To derive a contradiction, we now use similar arguments as in [25]. The function w satisfies $\Delta w + f'(u(x))w = 0$ in the whole of Ω, hence $w \equiv 0$ in Ω by unique continuation. Thus u is constant on connected components of $\partial B \cap \Omega$, where B is any disk centered at (λ_1, θ). If B has radius slightly larger than the radius of G, then ∂B intersects $\partial\Omega$ and hence, by the Dirichlet boundary condition, $u = 0$ on a connected component of $\partial B \cap \Omega$. Taking all such balls B, we obtain that $u \equiv 0$ on a nonempty open subset of Ω. From (1.1) we then conclude that $f(0) = 0$ a contradiction to (3.1).

CASE B: J_b is a line segment.

In this case we need further regularity of u. This is the only place where we need (H1).

Lemma 3.7. *Assume that J_b is a line segment. Then $u \in C^2(\bar{G})$.*

Proof. It follows from the symmetry of G in H_{λ_1} and Lemmas 3.3, 3.5 that $\partial\Omega$ has a step at the point $y := P_{\lambda_1}((\mu(b), b))$, the upper right "corner" of $\bar{G}$ (note that $\partial\Omega$ coincides with ∂G near this point). By (H1), there is a ball around y such that $\partial G \cap B$ consists of the horizontal segment $J_b \cap B$ and of a $C^{1,1}$ curve ending at y. Of course, this curve is contained in $T_{(a,b)}$, hence y is a corner point of $\bar{G}$ of opening at most π, in the sense of Definition 2.4 of [20]. By Proposition 2.8 of [20], u_1 is $C^{1,\alpha}$ near y, for each $\alpha \in (0, 1)$. Using the symmetries of u about H_{λ_1} and $H^{e^2}_\theta$, we obtain that u_1 is also $C^{1,\alpha}$ near the other three corners of $\bar{G}$. Consequently, by standard interior and boundary regularity results, $u_1 \in C^{1,\alpha}(\bar{G})$. Thus u_{11} and $u_{12} = u_{21}$ extend to continuous functions on $\bar{G}$ and the same is true of $u_{22}(x) = -f(u(x)) - u_{11}(x)$. $\square$

For $x \in S \cup P_{\lambda_1}(S) \subset \partial G$, let $\tau(x) = (\tau_1(x), \tau_2(x))$ denote the unit tangent vector to ∂G at x with $\tau_2(x) > 0$. Let $y := P_{\lambda_1}((\mu(b), b))$, as in the proof of Lemma 3.7. Then $y = (\lambda, b)$ for some λ. As J_b is a segment, we have $\lambda > \lambda_1$, which implies

$$V_\lambda u > 0 \text{ in } \Sigma^G_\lambda. \tag{3.15}$$

Assume first that $\tau_2(x) \to 0$ as $x \to y$. Then G is a $C^{1,1}$-domain (near y, hence globally by symmetry) and e^1 is tangent to ∂G at y. Moreover,

$\Sigma_\lambda^G \cap \partial G = P_{\lambda_1} S$, hence $u = 0$ and $\nabla u = 0$ on this set. Applying Lemma 2.4 with $W = G$ (which is legitimate by Lemma 3.7), we get a contradiction to (3.15).

Next consider the opposite case: there is a sequence $x^n \in S$ such that $P_{\lambda_1}(x^n) \to y$ and $\tau_2(x^n) = \tau_2(P_{\lambda_1}(x^n)) \to s \neq 0$ (the relation $\tau_2(P_{\lambda_1}(x)) = \tau_2(x)$ follows from the symmetry of G about H_{λ_1}). Differentiating the relations $u_1 = 0$, $u_2 = 0$ along S, we obtain

$$u_{11}\tau_1 + u_{12}\tau_2 = u_{21}\tau_1 + u_{22}\tau_2 = 0 \text{ on } S. \tag{3.16}$$

Of course, $u_{12} = u_{21}$ on S and thus (3.16) and (1.1) imply the following identities on S

$$0 = u_{11}\tau_1^2 - u_{22}\tau_2^2 = u_{11}\tau_1^2 + u_{11}\tau_2^2 + \tau_2^2 f(u) = u_{11} + \tau_2^2 f(0),$$

where we have also used the fact that $u = 0$ on S. Evaluating these identities along the sequence x^n and taking the limit, using the continuity of u_{11} (Lemma 3.7), we obtain

$$u_{11}(P_{\lambda_1}(y)) = -s^2 f(0) \neq 0.$$

On the other hand, $u_{11} = 0$ on the horizontal segment J_b. Since $P_{\lambda_1}(y) \in J_b$, we have a contradiction.

We have thus derived a contradiction in all cases and the proof of Theorem 1.1 is complete.

3.3 Proof of Theorem 1.3

Assume, in addition to the standing hypotheses and (H2), that Ω is x_2-convex.

Set $W = \Omega \cap P_\theta^{e^2}(\Omega)$ with $\theta = (a+b)/2$, as in Lemma 3.5. Then W is a domain, which is convex in both x_1 and x_2, and it is symmetric about the lines H_0 and $H_\theta^{e^2}$. Since $V_\theta^{e^2} u \equiv 0$ in G, unique continuation implies that $V_\theta^{e^2} u \equiv 0$ in the whole of W. This and (1.2) readily imply that $u = 0$ on ∂W. Therefore, we can use known symmetry results with respect to the direction e^2; in particular, a statement analogous to Corollary 3.2 holds. Specifically, if $y = (y_1, y_2) \in W$ and $\epsilon > 0$ are such that $u_2(y_1, \cdot) < 0$ on $(y_2 - \epsilon, y_2)$ and $u_2(y_1, \cdot) > 0$ on $(y_2 + \epsilon, y_2)$, then $u(y) = 0$. We are going to verify that this conclusion applies to all points $y \in H_\theta^{e^2}$ close to the point $(\mu(\theta), \theta) \in S \cap W$.

First recall that $u_2 = 0$ on S. By the symmetry of u, $u_2 = 0$ also on $H_\theta^{e^2} \cap \Omega$. We claim that there is a ball B centered at $(\mu(\theta), \theta) \in S$ such that $S \cup H_\theta^{e^2}$ exhausts the nodal set of u_2 in B:

$$\{x \in B : u_2(x) = 0\} \subset S \cup H_\theta^{e^2}. \tag{3.17}$$

To prove this, we use the following well-known equal-angle property of the nodal set of u_2 (which is a solution of a linear equation (2.3)). There is a ball B centered at $(\mu(\theta), \theta)$ such that $\{x \in B : u_2(x) = 0\}$ consists of a finite number, say k, of C^1-curves ending at $(\mu(\theta), \theta)$ and having tangents at $(\mu(\theta), \theta) \in S$, and the tangents divide B into k angles of equal size (see, for example, [19] or [20, Theorem 2.1]). Also, u_2 must have different signs in any two neighboring sectors in B determined by these nodal curves (otherwise, u_2 or $-u_2$ is nonnegative, but not strictly positive, in the union of these sectors and the separating nodal curve, contradicting the strong maximum principle).

Now, by Lemma 3.5, the nodal set of u_2 in G coincides with $H_\theta^{e^2} \cap G$. Since $H_\theta^{e^2}$ is orthogonal to $S \subset \bar{G}$ at $(\mu(\theta), \theta)$, the above equal-angle result implies that there can be no nodal curves of u_2 in B other than those given by S and $H_\theta^{e^2}$. This proves the claim.

Let B be as in the above claim. Then $u_2\big|_B$ has exactly four nodal domains (that is, the connected components of $B \setminus u_2^{-1}(0)$), in two of them $u_2 < 0$ and in the remaining two $u_2 > 0$. By Lemma 3.5, $u_2 < 0$ for each $x \in B \cap G$ with $x_2 > \theta$. This determines the signs of u_2 in the remaining three nodal domains: in particular, $u_2 > 0$ for each $x \in B \setminus \bar{G}$ with $x_2 > \theta$ and $u_2 < 0$ for each $x \in B \setminus \bar{G}$ with $x_2 < \theta$. These relations show that the analog of Corollary 3.2, as mentioned above, indeed applies to all points on the horizontal segment $J := H_\theta^{e^2} \cap B \setminus \bar{G}$. Hence $u = 0$ on J and consequently $u_1 = 0$ on J.

Let us now consider the nodal set of u_1 in B. It contains J and S, therefore, by the equal-angle result, there must also be a nodal curve of u_1 ending at the point $(\mu(\theta), \theta) \in S$ and contained in G. However, since $u_1 > 0$ in $G \setminus \Sigma_{\lambda_1}^G$, there can be no such curve and we have a contraction.

This proves Theorem 1.3.

4 Concluding remarks

Let us briefly discuss other conditions which can be used in Theorem 1.1 in place of (H1).

First note that the case when the set J_b is a single point, rather than a segment, was treated in Subsection 3.2 without using (H1). Therefore, even without assuming (H1), the following conclusion can be derived from the assumption that (1.1), (1.2) has a nonnegative solution u with a nontrivial nodal set in Ω. There is a subdomain $G \subset \Omega$, which is symmetric about the two lines H_{λ_1} and $H_\theta^{e^2}$ and such that $\partial G \cap \partial\Omega$ consists of two horizontal line segments J_a, J_b, and the curve $\gamma = P_{\lambda_1}(S)$, where S has the symmetry and monotonicity properties described in (3.6) and Lemma 3.5. Thus the assumption that there is no such symmetric part of $\partial\Omega$ involving horizontal segments is also sufficient for Theorem 1.1. Or, we can require that if such a symmetric part exists and the connecting curve γ happens to be of class C^2 (locally), then γ is of class class $C^{1,1}$ up to its end points. The arguments used in Subsection 3.2 apply in this situation.

References

[1] A. D. Alexandrov, *A characteristic property of spheres*, Ann. Math. Pura Appl. **58** (1962), 303–354.

[2] R. Alvarado, D. Brigham, V. Mazya, M. Mitrea, and E. Ziade, *On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf-Oleinik boundary point principle*, J. Math. Sciences **176** (2011), 281–360.

[3] H. Berestycki, *Qualitative properties of positive solutions of elliptic equations*, Partial differential equations (Praha, 1998), Chapman & Hall/CRC Res. Notes Math., vol. 406, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 34–44.

[4] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), 1–37.

[5] F. Brock, *Continuous rearrangement and symmetry of solutions of elliptic problems*, Proc. Indian Acad. Sci. Math. Sci. **110** (2000), 157–204.

[6] X. Cabré, *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **48** (1995), 539–570.

[7] A. Castro and R. Shivaji, *Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric*, Comm. Partial Differential Equations **14** (1989), 1091–1100.

[8] F. Da Lio and B. Sirakov, *Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations*, J. European Math. Soc. **9** (2007), 317–330.

[9] L. Damascelli, F. Pacella, and M. Ramaswamy, *A strong maximum principle for a class of non-positone singular elliptic problems*, NoDEA Nonlinear Differential Equations Appl. **10** (2003), no. 2, 187–196.

[10] E. N. Dancer, *Some notes on the method of moving planes*, Bull. Austral. Math. Soc. **46** (1992), 425–434.

[11] J. Dolbeault and P. Felmer, *Symmetry and monotonicity properties for positive solutions of semi-linear elliptic PDE's*, Comm. Partial Differential Equations **25** (2000), no. 5-6, 1153–1169.

[12] ———, *Monotonicity up to radially symmetric cores of positive solutions to nonlinear elliptic equations: local moving planes and unique continuation in a non-Lipschitz case.*, Nonlinear Anal. **58** (2004), 299–317.

[13] Y. Du, *Order structure and topological methods in nonlinear partial differential equations. Vol. 1*, Series in Partial Differential Equations and Applications, vol. 2, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[14] J. Földes, *On Serrin's symmetry result in nonsmooth domains and its applications*, preprint.

[15] ———, *On symmetry properties of parabolic equations in bounded domains*, J. Differential Equations **250** (2011), 4236-4261.

[16] L. E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge Tracts in Mathematics, vol. 128, Cambridge University Press, Cambridge, 2000.

[17] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[19] P. Hartman and A. Wintner, *On the local behavior of solutions of non-parabolic partial differential equations*, Amer. J. Math. **75** (1953), 449–476.

[20] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini, *Nodal domains and spectral minimal partitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 101–138.

[21] P. Hess and P. Poláčik, *Symmetry and convergence properties for non-negative solutions of nonautonomous reaction-diffusion problems*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 573–587.

[22] B. Kawohl, *Symmetrization - or how to prove symmetry of solutions to a PDE*, Partial differential equations (Praha, 1998), Chapman & Hall/CRC Res. Notes Math., vol. 406, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 214–229.

[23] W.-M. Ni, *Qualitative properties of solutions to elliptic problems*, Handbook of Differential Equations: Stationary Partial Differential Equations, vol. 1 (M. Chipot and P. Quittner, eds.), Elsevier, 2004, pp. 157–233.

[24] P. Poláčik, *Symmetry properties of positive solutions of parabolic equations: a survey*, Recent progress on reaction-diffusion systems and viscosity solutions (W.-Y. Lin Y. Du, H. Ishii, ed.), World Scientific, 2009, pp. 170–208.

[25] ———, *Symmetry of nonnegative solutions of elliptic equations via a result of Serrin*, Comm. Partial Differential Equations **36** (2011), 657–669.

[26] ———, *On symmetry of nonnegative solutions of elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire **29** (2012), 1–19.

[27] P. Pucci and J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, 73, Birkhäuser Verlag, Basel, 2007.

[28] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318.