Symmetry of nonnegative solutions of elliptic equations via a result of Serrin

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Abstract. We consider the Dirichlet problem for semilinear elliptic equations on a smooth bounded domain Ω. We assume that Ω is symmetric about a hyperplane H and convex in the direction orthogonal to H. Employing Serrin's result on an overdetermined problem, we show that any nonzero nonnegative solution is necessarily strictly positive. One can thus apply a well-known result of Gidas, Ni and Nirenberg to conclude that the solution is reflectionally symmetric about H and decreasing away from the hyperplane in the orthogonal direction.

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1 Introduction and the main result

We consider the elliptic semilinear problem

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.1}$$
$$u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and Ω is a bounded domain in $\mathbb{R}^N$ which is convex in one direction and reflectionally symmetric about a hyperplane orthogonal to that direction. We choose the coordinate system such that the direction is $e_1 := (1, 0, \dots, 0)$ (that is, Ω is convex in x_1) and the symmetry hyperplane is given by

$$H_0 = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 = 0\}.$$

By a celebrated theorem of Gidas, Ni, and Nirenberg [19] and its generalization to nonsmooth domains given by Berestycki and Nirenberg [4] and Dancer [10], each positive solution u of (1.1), (1.2) is even in x_1:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega), \tag{1.3}$$

and, moreover, $u(x_1, x')$ is decreasing with increasing $|x_1|$:

$$u_{x_1}(x_1, x') < 0 \quad ((x_1, x') \in \Omega, x_1 > 0). \tag{1.4}$$

This result was proved using the method of moving hyperplanes introduced by Alexandrov [2] and further developed and applied in a symmetry problem by Serrin [29]. We refer the reader to the surveys [3, 22, 25, 26] or the more recent paper [8], for perspectives on this theorem, generalizations, related results, and many other references. Extensions to equations with non-Lipschitz nonlinearities can be found in, for example, [5, 12] and references therein.

As noted in [19], the above symmetry and monotonicity theorem is not valid in general if the solution $u \not\equiv 0$ is assumed to be nonnegative, rather than strictly positive. A counterexample in one dimension is given by the function $u(x) = 1 + \cos x$ considered as a solution of $u'' + u - 1 = 0$ on $\Omega = (-(2k+1)\pi, (2k+1)\pi)$, $k \in \mathbb{N}$. Of course, the case $N = 1$ is very special in that the boundary of Ω is not connected. This raises a natural question whether the symmetry theorem of [19] holds for nonnegative nonzero solutions if $N \geq 2$. Partial results have been obtained towards the answer to the question. In [7], it is proved that if Ω is a ball, then all nonnegative nonzero solutions are strictly positive (see also the monographs [14, 16] for the proof and a discussion of this result; an extension to quasilinear radial problems can be found in [28]). More generally, the same strict positivity result for problem (1.1), (1.2) on a C^2 domain Ω holds if the unit normal vector field on $\partial\Omega \setminus H_0$ is nowhere parallel to H_0 (see [20]) or if Ω is convex in all directions (see [9]). For nonsmooth domains, a sufficient condition for the strict positivity of nonnegative nonzero solutions was given in [15]. It requires, roughly speaking, that for any $\delta > 0$ there be a two-dimensional wedge W, such that if the tip of W is translated to any point of $\partial\Omega$ with $x_1 \geq \delta$, then W is contained in $\bar{\Omega}$. Note that a rectangle, or a rectangle with smoothed out corners, do not satisfy the geometric condition of [15].

In this paper we settle the symmetry problem for nonnegative solutions of (1.1), (1.2) for a general C^2 domain Ω:

Theorem 1.1. *Assume that $N \geq 2$ and Ω is a C^2 bounded domain in $\mathbb{R}^N$ which is convex in x_1 and symmetric about H_0. If $u \in C^2(\bar{\Omega})$ is a nonnegative solution of* (1.1), (1.2) *for some locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, then either $u \equiv 0$ (hence, necessarily, $f(0) = 0$) or else $u > 0$ and u has the symmetry and monotonicity properties* (1.3) *and* (1.4).

The symmetry of u, as stated in Theorem 1.1, follows from [19] once we know that u is strictly positive. We also remark that, by the Schauder theory, any classical solution of (1.1), (1.2) belongs to $C^2(\bar{\Omega})$ (and even to $C^{2+\alpha}(\bar{\Omega})$) if Ω is a $C^{2+\alpha}$ domain for some $\alpha \in (0, 1)$.

The outline of the proof of Theorem 1.1 is as follows. Assume that $u \not\equiv 0$. We first carry out a standard process of moving hyperplanes, moving the hyperplanes parallel to H_0 from the right, say. If the process can be continued all the way to the central position H_0, we obtain (1.4) as in [19] and this gives the positivity conclusion. Thus we assume that the process cannot be continued beyond a certain position $H_{\lambda_1} \neq H_0$. We then prove that there is a subdomain G of Ω, on which u solves an overdetermined problem: it simultaneously satisfies the homogeneous Dirichlet and Neumann boundary conditions on ∂G. In a key step of the proof, we show that G is of class C^2. This facilitates an application of the well-known symmetry result of [29] (see also [28]) which states that G is a necessarily a ball and u is radially symmetric about the center of G. From this and a unique continuation property, we infer that u must vanish on an open subset of Ω and then, by unique continuation again, it has to vanish everywhere in Ω, a contradiction.

Remark 1.2. (i) The positivity statement of Theorem 1.1 is nontrivial only if $f(0) < 0$. If $f(0) \geq 0$ one has $\Delta u + c(x)u \leq 0$ in Ω, where $c(x)$ is a bounded function ($c(x) = (f(u(x)) - f(0))/u(x)$ if $u(x) \neq 0$). Thus the strong maximum principle gives $u > 0$ in Ω, unless $u \equiv 0$.

(ii) Theorem 1.1 holds if (1.1) is replaced with the quasilinear equation

$$a(u, \nabla u)\Delta u + f(u, \nabla u) = 0, \quad x \in \Omega, \tag{1.5}$$

where $a(u, p)$, $f(u, p)$ are locally Lipschitz functions of $(u, p) \in \mathbb{R} \times \mathbb{R}^N$ which are radially symmetric in p, and a is positive. The proof is essentially the same as for (1.1), see Section 3.

We remark that the unique continuation has already appeared in proofs of some symmetry results, see [13, 24, 27]. The use of Serrin's symmetry result in the proof of positivity seems to be new.

There have been numerous extensions and generalizations of Serrin's result, see for example [1, 11, 28, 30] and references therein. We wish to mention, in particular, the recent work of [17] and its sequel [18], where partially overdetermined problems are considered. In such problems, a solution of (1.1) on a domain G is assumed to satisfy the Dirichlet boundary condition on the whole boundary ∂G and the Neumann boundary condition on a proper part S of ∂G. This is relevant for the symmetry problem at hand. If the domain Ω is not smooth, say it is only piecewise smooth, one can still show that if u is a nonnegative nonzero solution of (1.1), (1.2), then, unless u is strictly positive, it satisfies a partly overdetermined problem on some subdomain $G \subset \Omega$. It is not clear how to use this conclusion to derive a contradiction. As demonstrated by counterexamples in [18], for partly overdetermined problems the conclusion about the radial symmetry of G and u may not be valid. Thus the question whether a problem (1.1), (1.2) can have nontrivial nonnegative solutions with interior zeros on some nonsmooth domains remains open.

If f is continuous, but not Lipschitz, then nonnegative solutions of (1.1), (1.2) may not have properties (1.3), (1.4) even if Ω is a ball; see [5] for an example, also see [5, 13] for local symmetry results for continuous f. In case f is Lipschitz, but is allowed to depend on $x' = (x_2, \ldots, x_N)$, several examples of nonnegative nonzero solutions with interior zeros are available, see [27]. Such solutions do not have the monotonicity property (1.4), but, as shown in [27], they do have the symmetry property (1.3).

Finally, we remark that while the proof of Theorem 1.1 applies to some quasilinear equations, such as (1.5), it does not apply to general nonlinear equations for which the reflectional symmetry of positive solutions has been established, such as the reflectionally symmetric fully nonlinear equations considered in [4, 23]. The reason is that our proof based on [29] requires the equation to be invariant under reflections in all hyperplanes and not just the hyperplanes perpendicular to e_1.

2 Notation and preliminaries on linear equations

In the rest of the paper we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^2 which is convex in x_1 and symmetric about the hyperplane H_0. We adopt the following notation.

For any $\lambda \in \mathbb{R}$ and any open set $G \subset \Omega$, we set

$$\begin{aligned} H_\lambda &:= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Sigma_\lambda^G &:= \{x \in G : x_1 > \lambda\}, \\ \Gamma_\lambda^G &:= H_\lambda \cap G, \\ \ell^G &:= \sup\{x_1 \in \mathbb{R} : (x_1, x') \in G \text{ for some } x' \in \mathbb{R}^{N-1}\}. \end{aligned} \tag{2.1}$$

When $G = \Omega$, we often omit the superscript Ω and simply write Σ_λ for Σ_λ^Ω, ℓ for ℓ^G, etc. By $\nu = (\nu_1, \dots, \nu_N)$ we denote the unit normal vector field on $\partial\Omega$ pointing out of Ω.

Let P_λ stand for the reflection in the hyperplane H_λ. Note that since Ω is convex in x_1 and symmetric in the hyperplane H_0, $P_\lambda(\Sigma_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$.

For any function z on $\bar{\Omega}$ and any $\lambda \in [0, \ell]$, we define z^λ and $V_\lambda z$ by

$$\begin{aligned} z^\lambda(x) &= z(P_\lambda x) = z(2\lambda - x_1, x'), \\ V_\lambda z(x) &= z^\lambda(x) - z(x) \quad (x = (x_1, x') \in \bar{\Sigma}_\lambda). \end{aligned} \tag{2.2}$$

Below we rely on the following standard observations concerning linearization of equation (1.1). If u is a solution of (1.1), or of the more general equation (1.5) then u^λ satisfies the same equation as u in Σ_λ. Hence, for any

$x \in \Sigma_\lambda$ we have (omitting the argument x of u, u^λ)

$$a(u,\nabla u)(\Delta(u^\lambda - u)) + (a(u^\lambda,\nabla u^\lambda) - a(u,\nabla u))\Delta u^\lambda \\ + f(u^\lambda,\nabla u^\lambda) - f(u,\nabla u) = 0.$$

Using Hadamard's formulas in the integral form (which is legitimate since a and f are Lipschitz on the range of $(u,\nabla u)$), one shows that the function $v = V_\lambda u$ solves on $U = \Sigma_\lambda$ a linear equation

$$a_0(x)\Delta v + b_i(x)v_{x_i} + c(x)v = 0, \quad x \in U, \tag{2.3}$$

where $a_0(x) = a(u(x),\nabla u(x))$ and the coefficients b_i, c depend on λ, but are bounded (uniformly in λ) in terms of the Lipschitz constants of a and f. Note that a_0 is bounded above and below by positive constants and is Lipschitz. In (2.3) and below, we use the summation convention (the summation over repeated indices); thus, in (2.3),

$$b_i(x)v_{x_i} = b_1(x)v_{x_1} + \ldots, b_N(x)v_{x_N}.$$

Since $u \geq 0$, the Dirichlet condition (1.2) gives

$$v(x) \geq 0 \quad (x \in \partial\Sigma_\lambda \setminus \Gamma_\lambda). \tag{2.4}$$

Of course, on the remaining part of $\partial\Sigma_\lambda$, Γ_λ, we have

$$v(x) = 0 \quad (x \in \Gamma_\lambda). \tag{2.5}$$

Another way to linearize (1.5) is via translations. Just like $V_\lambda u$, the function $w(x) = (u(x_1+\epsilon,x') - u(x))/\epsilon$, where $\epsilon \approx 0$, satisfies a linear equation(2.3) on $U = \Omega \cap (\Omega - \epsilon e_1)$ with coefficients bounded independently of ϵ. Using standard elliptic interior estimates and taking the limit as $\epsilon \to 0$, one shows that $v = u_{x_1}$ is a strong solution of (2.3) (see below for the definition of a strong solution).

We now recall some results on linear elliptic equations that we use in the proof of the symmetry result. We formulate them in slightly more general form than needed for this paper. Let $U \subset \Omega$ be a nonempty open set. Consider a linear equation

$$a_{ij}(x)v_{x_ix_j} + b_i(x)v_{x_i} + c(x)v = 0, \quad x \in U, \tag{2.6}$$

where

(L1) a_{ij}, b_i, c are measurable functions on U and there are positive constants α_0, β_0 such that

$$|a_{ij}(x)|,\ |b_i(x)|,\ |c(x)| < \beta_0 \quad (i,j = 1,\dots,N, x \in U),$$
$$a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in U).$$

For a unique continuation result, we shall need the leading coefficients to be more regular.

(L2) The functions a_{ij}, $i,j = 1,\dots,N$ are Lipschitz on U.

By a solution of (2.6) we mean a strong solution, that is, a function $v \in W^{2,N}_{loc}(U)$ such that (2.6) is satisfied almost everywhere in U. In the following proposition, $|U|$ stands for the Lebesgue measure on $\mathbb{R}^N$.

Proposition 2.1. *Assume that (L1) holds and let $v \in W^{2,N}_{loc}(U)$ be a solution of* (2.6)*.*

(i) If $v \geq 0$ in U and U is connected then either $v \equiv 0$ or $v > 0$ in U.

(ii) Assume that $v \in C^1(\bar{B})$, where B is a ball in U, and $x_0 \in \partial U \cap \bar{B}$. If $v > 0$ in B and $v(x_0) = 0$, then $\partial v/\partial \eta < 0$ at x_0, where η is a normal vector to ∂B at x_0 pointing out of B.

(iii) Assume that $v \in C(\bar{U})$. There is $\delta_0 > 0$ depending only on N, α_0, β_0 such that the relation $v \geq 0$ on ∂U implies $v \geq 0$ in $\bar{U}$, provided one of the following two conditions is satisfied

(a) $|U| < \delta_0$,

(b) $U \subset \{x \in \mathbb{R}^N : m - \delta_0 \leq x \cdot e \leq m\}$ for some unit vector $e \in \mathbb{R}^N$ and some $m \in \mathbb{R}$.

(iv) Assume that U is a connected component of Σ^G_λ for some bounded C^2 domain G and some $\lambda \in \mathbb{R}$, and the normal vector to ∂G at a point $x^0 \in \partial G \cap \bar{U} \cap H_\lambda$ is contained in the hyperplane H_λ. Assume further that $v \in C^2(\bar{U})$, $v \not\equiv 0$, $v \geq 0$ in U, and $v(x^0) = 0$. Let η be any direction at x^0 pointing inside U and not tangent to ∂G. Then the following alternative holds at x^0:

$$\text{either } \frac{\partial v}{\partial \eta} > 0 \text{ or else } \frac{\partial^2 v}{\partial \eta^2} > 0.$$

Statements (i), (ii) are the standard strong maximum principle and Hopf boundary lemma for nonnegative solutions. Statement (iii) is the maximum principle for small or narrow domains (see [4, 6]). Statement (iv) is a corner point lemma proved in [29]. The version of the lemma given in [29, Lemma 2] is different in that it requires $c \equiv 0$, but, on the other hand, it allows v to be a supersolution. One derives (iv) from this result by the transformation $\tilde{v} = e^{kx_1}v$ with k sufficiently large (cf. p. 316 in [29]). Note that no sign condition on the coefficient c is needed in Proposition 2.1.

Proposition 2.2. *Assume that (L1), (L2) hold and U is connected. Let $v \in W^{2,N}_{loc}(U)$ be a solution of (2.6). If $v \equiv 0$ in a nonempty open subset of U, then $v \equiv 0$ in U.*

This is a weak unique continuation theorem. The proof can be found in [21, Section 17.2], for example.

3 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1 and then indicate simple modifications that extend the proof to the more general equations as in Remark 1.2.

Throughout this section we assume that the hypotheses of Theorem 1.1 are satisfied. We use the notation introduced in Section 2. Also, we use without notice the fact that for $\lambda \in [0, \ell)$ the function $v = V_\lambda u$ satisfies a linear equation (2.6) on $U = \Sigma_\lambda$, with coefficients satisfying (L1), (L2), and that it satisfies (2.4), (2.5).

In preparation for the process of moving hyperplanes, we prove the following three lemmas.

Lemma 3.1. *Given $\lambda \in [0, \ell)$ let D be a connected component of Σ_λ such that $V_\lambda u \geq 0$ in D. Then either $V_\lambda u \equiv 0$ on D or $V_\lambda u(x) > 0$ for each $x \in D$. In the latter case one has*

$$\partial_{x_1} u(x) < 0 \quad (x \in \Gamma_\lambda \cap \partial D). \tag{3.1}$$

Proof. This follows directly from statements (i), (ii) of Proposition 2.1 and the fact that $\partial_{x_1} u(x) = -\partial_{x_1}(V_\lambda u(x))/2$ on Γ_λ. □

Lemma 3.2. *If $u_{x_1} \equiv 0$ in a nonempty open subset of Ω, then $u \equiv 0$ in Ω.*

Proof. As noted in Section 2, u_{x_1} solves a linear equation (2.3) on Ω. If $u_{x_1} \equiv 0$ in a nonempty open subset of Ω, then, by Proposition 2.2, $u_{x_1} \equiv 0$ in Ω. Hence u is constant in x_1 on Ω and the boundary condition forces $u \equiv 0$. $\square$

Lemma 3.3. *Assume that $u \not\equiv 0$ and let $\lambda \in (0, \ell]$. If $V_\lambda u(x) > 0$ for all $x \in \Sigma_\lambda$, then for each $\tilde\lambda \in [0, \lambda]$ sufficiently close to λ one has $V_{\tilde\lambda} u > 0$ in $\Sigma_{\tilde\lambda}$.*

Note that the assumption $V_\lambda u(x) > 0$ for all $x \in \Sigma_\lambda$ is trivially satisfied for $\lambda = \ell$, as $\Sigma_\lambda = \emptyset$.

Proof of Lemma 3.3. We first show that $V_{\tilde\lambda} u \geq 0$ in $\Sigma_{\tilde\lambda}$ for each $\tilde\lambda \approx \lambda$. For that we apply the maximum principle on small domains in a similar way as in [4]. Fix $\delta = \delta_0$ such that Proposition 2.1(iii) applies to equations (2.3) satisfied by $v = V_\lambda u$ for any $\lambda \in [0, \ell)$. Since δ is determined by the ellipticity constant of (2.3) and a bound on the coefficients, it is independent of λ.

Choose a compact set $K \subset \Sigma_\lambda$ such that $|\Sigma_\lambda \setminus K| < \delta$. Since $V_\lambda u > 0$ on $\Sigma_\lambda \supset K$, for each $\tilde\lambda \in [0, \lambda]$ sufficiently close to λ one has $V_{\tilde\lambda} u > 0$ on K and $|\Sigma_{\tilde\lambda} \setminus \Sigma_\lambda| < \delta - |\Sigma_\lambda \setminus K|$. Say this is true for all $\tilde\lambda \in [\lambda - \epsilon, \lambda]$, where ϵ is a small positive constant. Then for any $\tilde\lambda \in [\lambda - \epsilon, \lambda]$, $|\Sigma_{\tilde\lambda} \setminus K| < \delta$. Also, using (2.4), (2.5) with $\tilde\lambda$ replacing λ, together with the condition $V_{\tilde\lambda} u > 0$ on K, we obtain $V_{\tilde\lambda} u \geq 0$ on the boundary of $\Sigma_{\tilde\lambda} \setminus K$. Proposition 2.1(iii) then implies that $V_{\tilde\lambda} u$ is nonnegative in $\Sigma_{\tilde\lambda} \setminus K$, hence in $\Sigma_{\tilde\lambda}$.

Before taking on the strict positivity of $V_{\tilde\lambda} u$, we note that the relations $V_{\tilde\lambda} u \geq 0$ in $\Sigma_{\tilde\lambda}$, $V_{\tilde\lambda} u = 0$ on $\Gamma_{\tilde\lambda}$ imply that

$$u_{x_1}(x) = -\partial_{x_1}(V_{\tilde\lambda} u(x))/2 \leq 0 \quad (x \in \Gamma_{\tilde\lambda},\ \tilde\lambda \in [\lambda - \epsilon, \lambda]).$$

Hence

$$u_{x_1} \leq 0 \text{ in } \Sigma_{\lambda-\epsilon} \setminus \Sigma_\lambda. \tag{3.2}$$

Now we show that if $\tilde\lambda \in (\lambda - \epsilon, \lambda]$, then $V_{\tilde\lambda} u > 0$ in any connected component $D_{\tilde\lambda}$ of $\Sigma_{\tilde\lambda}$. This follows immediately from Lemma 3.1 if $D_{\tilde\lambda} \cap K \neq \emptyset$, as $V_{\tilde\lambda} u > 0$ in K. We need a different argument in the case $D_{\tilde\lambda} \cap K = \emptyset$ (which we cannot avoid if $D_{\tilde\lambda} \cap \Sigma_\lambda = \emptyset$). By Lemma 3.1, we only need to rule out the possibility $V_{\tilde\lambda} u \equiv 0$ in $D_{\tilde\lambda}$. Assume it holds. As a consequence we obtain that if $u_{x_1} < 0$ at some point $x \in D_{\tilde\lambda}$ then $u_{x_1} > 0$ at the point $P_{\tilde\lambda} x$. Since $P_{\tilde\lambda} x \in \Sigma_{\lambda-\epsilon} \setminus \Sigma_\lambda$ if $x \in D_{\tilde\lambda}$ is close enough to $H_{\tilde\lambda}$, (3.2) implies that $u_{x_1}(x) < 0$ cannot hold at any such point x. Hence, $u_{x_1} \equiv 0$ at all points

$x \in D_{\tilde\lambda}$ near $H_{\tilde\lambda}$. Lemma 3.2 then implies that $u \equiv 0$ in Ω a contradiction. This contradictions rules out $V_{\tilde\lambda} u \equiv 0$ in $D_{\tilde\lambda}$ and the proof is complete. $\square$

We proceed to the proof of Theorem 1.1. Assume that $u \not\equiv 0$ is a nonnegative solution of (1.1), (1.2). Our task is to prove that $u > 0$ in Ω. Define

$$\lambda_1 := \inf \{\mu \in (0,\ell) : V_\lambda u(x) > 0 \text{ for all } x \in \Sigma_\lambda \text{ and } \lambda \in [\mu,\ell)\}. \tag{3.3}$$

The fact that λ_1 is well defined (and $\lambda_1 < \ell$), that is, that the set in (3.3) is nonempty, follows directly from Lemma 3.3, where we take $\lambda = \ell$. Now, if $\lambda_1 = 0$, then Lemma 3.1 implies $u_{x_1} < 0$ in Σ_0, hence necessarily $u > 0$ in $\bar\Sigma_0$. Also $V_0 u \geq 0$, by the continuity of u, and this implies that u is positive in $-\Sigma_0$ as well. Thus $u > 0$ in Ω and Theorem 1.1 is proved if $\lambda_1 = 0$.

The crux of the proof now consists in ruling out the case $\lambda_1 > 0$. This is done in several steps, using the claims below.

Claim 3.4. *If $\lambda_1 > 0$, then there is a connected component D of Σ_{λ_1} such that $V_{\lambda_1} u \equiv 0$ in D.*

Proof. By the continuity of u, $V_{\lambda_1} u \geq 0$ in Σ_{λ_1}. At the same time, it is impossible for $V_{\lambda_1} u$ to be positive in Σ_{λ_1}, for Lemma 3.3 would immediately give a contradiction to the definition of λ_1. Hence, by Lemma 3.1, there is a connected component $D \neq \emptyset$ of Σ_{λ_1} such that $V_{\lambda_1} u \equiv 0$ on D. $\square$

If $\lambda_1 > 0$ and D is as in Claim 3.4, we set

$$G := \operatorname{int}(\bar D \cup P_{\lambda_1}(\bar D)) = D \cup P_{\lambda_1}(D) \cup (\Gamma_{\lambda_1} \cap \bar D). \tag{3.4}$$

Then G is a subdomain of Ω which is convex in x_1 and symmetric about H_{λ_1}: $P_{\lambda_1}(G) = G$ (see Figure 1). Obviously,

$$\begin{aligned} \partial G &= (\partial D \cap \partial\Omega) \cup P_{\lambda_1}(\partial D \cap \partial\Omega) \\ &= (\partial D \setminus H_{\lambda_1}) \cup (P_{\lambda_1}(\partial D) \setminus H_{\lambda_1}) \cup (\partial G \cap H_{\lambda_1}) \end{aligned} \tag{3.5}$$

and

$$\Sigma_\lambda^G = \Sigma_\lambda^D = \Sigma_\lambda \cap D \quad (\lambda \in [\lambda_1,\ell)). \tag{3.6}$$

Claim 3.5. *If $\lambda_1 > 0$, D is as in Claim 3.4, and G is as in* (3.4)*, then*

$$V_{\lambda_1} u \equiv 0 \quad (x \in G), \qquad u_{x_1} < 0 \quad (x \in \Sigma_{\lambda_1}^G), \tag{3.7}$$

$$u > 0 \quad (x \in G), \qquad u = 0 \quad (x \in \partial G). \tag{3.8}$$

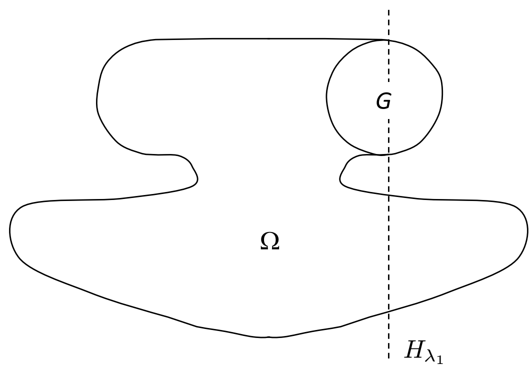

Figure 1: Domain Ω and its subdomain G

Proof. The identity in (3.7) is obvious from the definition of G. The inequality in (3.7) is a direct consequence of the definition of λ_1 and Lemma 3.1. Relations (3.8) follow from (3.7) and the Dirichlet boundary condition (1.2). □

Our next goal is to show that G is a C^2 domain and $\nabla u = 0$ on ∂G. Recall that $\nu = (\nu_1, \dots, \nu_N)$ is the outward unit normal vector field on $\partial\Omega$. Since $\bar{\Sigma}_{\lambda_1} \cap \partial G \subset \partial\Omega$, ν is defined on this set.

Claim 3.6. *If $\lambda_1 > 0$, D is as in Claim 3.4, and G is as in* (3.4), *then*

(i) $\nu_1(x) > 0 \quad (x \in (\bar{\Sigma}_{\lambda_1} \setminus H_{\lambda_1}) \cap \partial G)$,

(ii) $\nu_1(x) = 0 \quad (x \in H_{\lambda_1} \cap \partial G)$,

(iii) $\nabla u = 0$ *on* ∂G.

Proof. Define

$$\mu := \inf\{\lambda \in [\lambda_1, \ell^G) : \nu_1(x) > 0 \ \text{ for each } x \in \bar{\Sigma}_\lambda \cap \partial G\}. \tag{3.9}$$

Since Ω is of class C^2 and $\partial G \cap \bar{\Sigma}_{\lambda_1} \subset \partial\Omega$, each $\lambda \in [\lambda_1, \ell^G)$ sufficiently close to ℓ^G belongs to the set in (3.9). Hence $\mu < \ell$. We now prove that

$$\nabla u = 0 \text{ on } K := P_{\lambda_1}\big((\bar{\Sigma}_\mu \setminus H_\mu) \cap \partial G\big). \tag{3.10}$$

As $\Sigma_\mu = \cup_{\lambda>\mu}\Sigma_\lambda$, the definition of μ implies that $\nu_1(x) > 0$ for each $x \in (\bar{\Sigma}_\mu \setminus H_\mu) \cap \partial G$. This and the convexity of Ω in x_1 imply that $K \subset \Omega$. Also, since $\mu \geq \lambda_1$ and $P_{\lambda_1}(G) = G$,

$$K \subset P_{\lambda_1}(\bar{\Sigma}_{\lambda_1} \cap \partial G) \subset \partial G. \tag{3.11}$$

These properties yield (3.10), as $u \geq 0$ in Ω and $u = 0$ on ∂G (see (3.8)).

Using (3.10) and the symmetry relation in (3.7), we obtain

$$\nabla u = 0 \text{ on } (\bar{\Sigma}_\mu \setminus H_\mu) \cap \partial G. \tag{3.12}$$

We next show that $\mu = \lambda_1$. This will prove statement (i) and, in view of (3.10), (3.12), and the continuity of ∇u, also statement (iii). Assume that $\mu > \lambda_1$. The definition of μ and the fact that $\partial\Omega \in C^2$ imply that there is $x^0 \in \bar{\Sigma}_\mu \cap \partial G$ such that $\nu_1(x^0) = 0$, that is, $\nu(x_0)$ is contained in H_μ. This means that the direction e_1 of the x_1 axis is tangent to $\partial\Omega$ at x^0. This and (3.12) imply

$$u_{x_j x_1}(x^0) = 0 \quad (j = 1, \ldots, N). \tag{3.13}$$

Now consider the function $v = V_\mu u$. It satisfies a linear equation (2.6) in $\bar{\Sigma}_\mu^G$ and, by the relation $\mu > \lambda_1$ and the definition of λ_1, $v > 0$ in Σ_μ^G. Clearly, for $i, j = 2, \ldots, N$, $v_{x_i} = V_\mu(u_{x_i})$ and $v_{x_i x_j} = V_\mu(u_{x_i x_j})$. These relations imply that $v = v_{x_i} = v_{x_i x_j} = 0$ on Γ_μ^G and in particular at $x^0 \in \operatorname{cl}\Gamma_\mu^G$:

$$v(x^0) = v_{x_i}(x^0) = v_{x_i x_j}(x^0) = 0 \quad (i, j = 2, \ldots, N).$$

Further, $v_{x_1} = -2u_{x_1}$ on Γ_μ^G and, similarly, $v_{x_j x_1} = -2u_{x_j x_1}$ on Γ_μ^G. Therefore, by (3.12) and (3.13),

$$v_{x_1}(x^0) = v_{x_j x_1}(x^0) = 0 \quad (j = 2, \ldots, N).$$

Finally, since $x^0 \in \operatorname{cl}\Gamma_\mu^G$, we have $v_{x_1 x_1}(x^0) = u_{x_1 x_1}(x^0) - u_{x_1 x_1}(x^0) = 0$. We have thus verified that all derivatives of v up to the second order vanish at x^0. This is clearly a contradiction to Proposition 2.1(iv). This contradiction shows that $\mu = \lambda_1$, hence statements (i) and (iii) of Claim 3.6 are proved.

To prove (ii), pick any $x^0 \in H_{\lambda_1} \cap \partial G$. We have $\nu_1(x^0) \geq 0$ by the definition of μ and the continuity of ν. Assume that $\nu_1(x^0) > 0$. By statement (iii) and (3.8),

$$u = 0,\ \nabla u = 0 \quad \text{on} \quad \bar{\Sigma}_{\lambda_1} \cap \partial G. \tag{3.14}$$

Therefore, similarly to (3.13),

$$u_{\tau_1\tau_2}(x^0) = u_{\nu\tau_1}(x^0) = 0, \tag{3.15}$$

where $\nu = \nu(x^0)$ and τ_1, τ_2 are any tangent directions to $\partial\Omega$ at x^0.

We now show that also $u_{x_1x_1}(x^0) = 0$. By (3.14), $u_{x_1} = 0$ on $\bar{\Sigma}_{\lambda_1} \cap \partial G$ and the symmetry relation in (3.7) implies $u_{x_1} = 0$ on $\Gamma^G_{\lambda_1}$. Since $x^0 \in H_{\lambda_1} \cap \partial G$ and G is convex in x_1 and symmetric about H_{λ_1}, there is a sequence $x^k = (\lambda_1, y^k) \in \Gamma^G_{\lambda_1}$ such that $x^k \to x^0$ as $k \to \infty$, that is, $y^k \to y^0$, where $y^0 \in \mathbb{R}^{N-1}$ is such that $x^0 = (\lambda_1, y^0)$. For each k there is $x^k_1 > \lambda_1$ such that $(x^k_1, y^k) \in \bar{\Sigma}_{\lambda_1} \cap \partial G$ and

$$(x_1, y^k) \in \Sigma^G_{\lambda_1} \quad (x_1 \in (\lambda_1, x^k_1)).$$

The relations $u_{x_1}(\lambda_1, y^k) = u_{x_1}(x^k_1, y^k) = 0$ then imply that there exists $\mu_k \in [\lambda_1, x^k_1]$ such that $u_{x_1x_1}(\mu_k, y^k) = 0$. Passing to a subsequence, we may assume that $\mu_k \to \mu_0 \in [\lambda_1, \ell^G]$, so that $(\mu_k, y^k) \to (\mu_0, y^0) =: \tilde{x}^0 \in \bar{\Sigma}^G_{\lambda_1}$. Now, since $x^0 = (\lambda_1, y^0) \in \partial G \cap H_{\lambda_1}$ and G is convex in x_1, we necessarily have $\tilde{x}^0 \in \partial G$ and, if $\tilde{x}^0 \neq x^0$, then the whole line segment joining $\tilde{x}^0$ and x^0 is contained in ∂G. However, the latter would imply that e_1 is a tangent vector to $\partial\Omega$ at $\tilde{x}^0 = (\mu_0, y^0)$, hence $\nu_1(\tilde{x}^0) = 0$. This would contradict the definition of $\mu = \lambda_1$ (as $\mu_0 > \lambda_1$ if $\tilde{x}^0 \neq x^0$). Thus necessarily $\tilde{x}^0 = x^0$ and from $u_{x_1x_1}(\mu_k, y^k) = 0$ and the fact that $u \in C^2(\bar{\Omega})$ we conclude that $u_{x_1x_1}(x^0) = 0$.

Since we are assuming that $\nu_1(x^0) > 0$, relations (3.15) and $u_{x_1x_1}(x^0) = 0$ imply $D^2u(x^0) = 0$. Hence

$$0 = \Delta u(x^0) = -f(u(x^0)) = f(0). \tag{3.16}$$

However, if $f(0) = 0$, then the nontrivial nonnegative solution u of (1.1) cannot have a zero in Ω (see Remark 1.2) and we have a contradiction to (3.8). This contradiction rules out the relation $\nu_1(x^0) > 0$ for any $x^0 \in \partial G \cap H_{\lambda_1}$. Statement (ii) of Claim 3.6 is proved. □

Claim 3.7. *If $\lambda_1 > 0$, D is as in Claim 3.4, and G is as in* (3.4)*, then the domain G is of class C^2.*

Proof. Since D is a connected component of $\Sigma^\Omega_{\lambda_1}$ and Ω is of class C^2, the sets $\partial D \setminus H_{\lambda_1}$ and $P_{\lambda_1}(\partial D) \setminus H_{\lambda_1}$ are C^2 portions of the boundary of G, see (3.5). We only need to consider the boundary of G near H_{λ_1}. Pick any

$x^0 \in \partial G \cap H_{\lambda_1}$. By Claim 3.6(ii), $\nu(x^0)$ is perpendicular to the x_1-axis. Using a rotation keeping the x_1-axis fixed, we may assume that $\nu(x^0) = (0, \dots, 0, 1)$, that is, the normal vector has the direction of the x_N-axis at x^0. Since Ω is of class C^2, there is a ball B in $\mathbb{R}^N$ centered at x^0 such that

$$\nu_N(x) > 0 \quad (x \in B \cap \partial\Omega) \tag{3.17}$$

and

$$B \cap \partial\Omega = \{(\tilde{x}, \phi(\tilde{x})) : \tilde{x} \in W\},$$

where $\tilde{x} = (x_1, \dots x_{N-1})$, W is an open set in $\mathbb{R}^{N-1}$ containing the point $\tilde{x}^0 = (x_1^0, \dots, x_{N-1}^0)$, and ϕ is a C^2 function with $\phi(\tilde{x}^0) = x_N^0$. Then, by (3.5),

$$\begin{aligned} B_1 \cap \partial G &= \{(\tilde{x}, \psi(\tilde{x})) : \tilde{x} \in W_1\}, \\ B_2 \cap G &\subset \{(\tilde{x}, x_N) : \tilde{x} \in W_1,\ x_N < \psi(\tilde{x})\}, \end{aligned} \tag{3.18}$$

where B_1, B_2 are possibly smaller balls centered at x^0 with $B_2 \subset B_1 \subset B$, $W_1 \subset W$ is an open set containing $\tilde{x}^0$, and

$$\psi(\tilde{x}) = \psi(x_1, \dots x_{N-1}) = \begin{cases} \phi(x_1, \dots x_{N-1}) & \text{if } \tilde{x} \in W_1 \text{ and } x_1 \geq \lambda_1, \\ \phi(2\lambda_1 - x_1, \dots x_{N-1}) & \text{if } \tilde{x} \in W_1 \text{ and } x_1 < \lambda_1. \end{cases} \tag{3.19}$$

Now, for any $\tilde{x} = (x_1, \dots x_{N-1}) \in W_1$ with $x_1 = \lambda_1$, one has $x = (\tilde{x}, \phi(\tilde{x})) \in \partial G \cap H_{\lambda_1}$, hence $\nu_1(x) = 0$. Therefore $\phi_{x_1}(\tilde{x}) = 0$ at any such $\tilde{x}$ (indeed, $(1, 0, \dots, 0, \phi_{x_1}(\tilde{x}))$ is a tangent vector to $\partial\Omega$ at x and and $\nu_N(x) > 0$ by (3.17)). This and (3.19) readily imply that ψ is a C^2 function. This proves that the domain G is of class C^2. □

Completion of the proof of Theorem 1.1. We show that the assumption $\lambda_1 > 0$ leads to a contradiction. Assume it holds. Let D be as in Claim 3.4 and G as in (3.4). By Claims 3.5-3.7, $G \subset \Omega$ is a C^2 domain and $u \in C^2(\bar{G})$ is a positive solution of the following overdetermined problem on G:

$$\begin{aligned} \Delta u + f(u) = 0, &\quad x \in G, \\ u = 0, &\quad x \in \partial G, \\ \nabla u = 0, &\quad x \in \partial G. \end{aligned} \tag{3.20}$$

By Theorem 2 of [29], G is necessarily a ball and $u\big|_G$ is radially symmetric around the center of G. Let z denote the center of G and let Q be any rotation of $\mathbb{R}^N$ around z. Then the function $u(Qx)$ solves (1.1) on $Q^{-1}(\Omega)$, hence $v(x) = u(Qx) - u(x)$ solves a linear equation (2.6) on $\Omega \cap Q^{-1}(\Omega)$. Since $v \equiv 0$ in G, Proposition 2.2 implies that $v \equiv 0$ on the connected component of $\Omega \cap Q^{-1}(\Omega)$ containing G. Since this conclusion holds for any Q, u is radially symmetric around z in a neighborhood of $\bar{G}$ in $\bar{\Omega}$. Take any ball B centered at z with radius slightly larger than the radius of G. Then ∂B intersects $\partial\Omega$ and hence the radial symmetry and Dirichlet boundary condition imply that $u = 0$ on a connected component of $\partial B \cap \Omega$. Taking all such balls B, we obtain that $u \equiv 0$ (and hence $u_{x_1} \equiv 0$) on a nonempty open subset of Ω. Therefore, by Lemma 3.2, $u \equiv 0$ in Ω, a contradiction. This contradiction rules out the possibility $\lambda_1 > 0$ and hence Theorem 1.1 is proved. □

If, in place of equation (1.1), one considers the quasilinear equation (1.5), the above arguments go through with the following modifications.

Relations (3.16) have to be replaced with

$$0 = a(0,0)\Delta u(x^0) = -f(u(x^0), \nabla u(x^0)) = f(0,0).$$

Just like in the above proof, the condition $f(0,0) = 0$ and the strong maximum principle imply that a nontrivial nonnegative solution u of (1.5), (1.2) cannot have a zero in Ω.

In the overdetermined problem (3.20), the equation has to be replaced with (1.5), but the paper [29] covers such quasilinear problem as well.

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