An entire solution of a bistable parabolic equation on $\mathbb{R}$ with two colliding pulses

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Abstract. We consider semilinear parabolic equations of the form

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, t \in I, \tag{A}$$

where $I = (0, \infty)$ or $I = (-\infty, \infty)$. Solutions defined for all $(x, t) \in \mathbb{R}^2$ are referred to as entire solutions. Assuming that $f \in C^1(\mathbb{R})$ is of a bistable type with stable constant steady states 0 and $\gamma > 0$, we show the existence of an entire solution $U(x, t)$ of the following form. For $t \approx -\infty$, $U(\cdot, t)$ has two humps, or, pulses, one near ∞, the other near $-\infty$. As t increases, the humps move toward the origin $x = 0$, eventually "colliding" and forming a one-hump final shape of the solution. With respect to the locally uniform convergence, the solution $U(\cdot, t)$ is a heteroclinic orbit connecting the (stable) steady state 0 to the (unstable) ground state of the equation $u_{xx} + f(u) = 0$. We find the solution U as the limit of a sequence of threshold solutions of the Cauchy problem for equation (A).

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Contents

1 Introduction

We consider scalar parabolic equations of the form

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \in I. \tag{1.1}$$

Here, f is a C^1 function on $\mathbb{R}$ and $I \subset \mathbb{R}$ is an interval in $\mathbb{R}$. Usually, we take $I = (0, \infty)$ or $I = \mathbb{R}$. In the former case, we accompany (1.1) by an initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

with a suitable function u_0, say $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In the latter case, $I = \mathbb{R}$, solutions of (1.1) are referred to as entire solutions.

Entire solutions play a crucial role in the dynamics of solutions of the Cauchy problem (1.1), (1.2). For example, the ω-limits sets of bounded solutions of (1.1) and global attractors are comprised of entire solutions. For this reasons, entire solutions of reaction-diffusion equations have been widely studied (see, for example, [15, 21, 28] for overviews of results on attractors of evolution equations on bounded and unbounded domains). For equation (1.1) specifically, a variety of entire solutions have been found. These include, in addition to steady states, spatially periodic heteroclinic orbits between steady states (see [9, 10] and references therein), traveling waves and many types of "nonlinear superpositions" of traveling waves and other entire solutions (see [3, 4, 14, 16, 22, 23] and references therein).

In this paper, we exhibit a new type of entire solutions of (1.1). Our motivation to look for these entire solutions comes from the recent results in [25, 26]. In those papers, the second author considered the ω-limit sets of bounded solutions of (1.1) on $I = (0, \infty)$. For such a solution u, the ω-limit set is defined by

$$\omega(u) := \{\psi : u(\cdot, t_n) \to \psi \text{ for some sequence } t_n \to \infty\}, \tag{1.3}$$

where the convergence is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence). The goal of [25, 26] was to show examples of bounded non-quasiconvergent solutions, that is, bounded solutions u such that $\omega(u)$ does not consist entirely of steady states ($\omega(u)$ always contains at least one steady state [12, 13]). As noted above, $\omega(u)$ consists of entire solutions, and some of the examples of [25, 26] hint at the existence of rather curious entire solutions.

To be more specific, assume that f is an unbalanced bistable nonlinearity, that is, a function satisfying the following hypothesis:

(BU) $f \in C^1(\mathbb{R})$, and for some constants $\gamma > \beta > 0$ one has $f(0) = f(\beta) = f(\gamma) = 0$, $f'(0) < 0$, $f'(\beta) > 0$, $f'(\gamma) < 0$, $f < 0$ in $(0, \beta)$, $f > 0$ in (β, γ), and

$$\int_0^\gamma f(s)\, ds > 0. \tag{1.4}$$

It is well known that (BU) implies that the equation

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}, \tag{1.5}$$

has a solution v such that $\gamma > v > 0$ and $v \in C_0(\mathbb{R})$. Here and below $C_0(\mathbb{R})$ stands for the space of continuous functions on $\mathbb{R}$ converging to 0 at $x = \pm\infty$. We refer to any positive solution $v \in C_0(\mathbb{R})$ as a *ground state* of (1.5). The ground state is unique up to translations [2] and, if its point of maximum is placed at the origin, it is even in x and $v' < 0$ on $(0, \infty)$. We denote by ϕ the unique ground state with $\phi'(0) = 0$.

Two examples in [25, 26] show that for a suitable bounded solution u of (1.1) on $I = (0, \infty)$, the set $\omega(u)$ contains the ground state ϕ and the trivial steady state 0, and it does not contain any nonconstant steady state different from ϕ. As $t \to \infty$, such a solution $u(\cdot, t)$ must repeatedly visit small neighborhoods of ϕ, 0, and ϕ again. This is indicative of the existence of a "heteroclinic loop" between the steady states ϕ, 0, by which we mean a pair of heteroclinic entire solutions–one connecting ϕ to 0 and the one connecting 0 to ϕ. The existence of the former is well known and rather easy to establish: there is an entire solution $y(x, t)$ monotonically decreasing in t, such that $y(\cdot, t) \to \phi$ as $t \to -\infty$ and $y(\cdot, t) \to 0$ as $t \to \infty$, with the uniform convergence in both cases. The existence of a connection in the opposite direction, from 0 to ϕ, is not obvious at all and, in view of the asymptotic stability of 0, even seems to be impossible at the first glance. There is obviously no such connection with the uniform convergence at both ends. What one should be looking for, however, is a connection with the convergence in $L^\infty_{loc}(\mathbb{R})$, as that is the topology used in the definition of $\omega(u)$. In this paper, we show that such a connection indeed exists. Moreover, it takes a form of an entire solution U with an interesting spatial structure. For $t \approx -\infty$, $U(\cdot, t)$ has two humps, coming from spatial infinity, one from $-\infty$, the other one from $+\infty$. As t increases, the humps move toward the origin $x = 0$, eventually "colliding" and mixing up, after which just one hump forms as the solution approaches the ground state with $t \to \infty$ (see Figure 1). The presence of the moving humps, or, pulses, is perhaps the most interesting feature of this solution. It is well know that, unlike in reaction diffusion systems [17, 18], scalar equations (1.1) do not admit traveling pulses, that is, localized profiles moving with a constant nonzero speed. In accord with this, the humps in the solution $U(\cdot, t)$ do not move with constant speed; they slow down as $t \to -\infty$.

We state our main result formally as follows.

Theorem 1.1. *Suppose that f satisfies* (BU). *Then there exist an entire solution U of* (1.1) *with the following properties:*

- *(i) For each $t \in \mathbb{R}$ one has $0 < U(\cdot, t) < \gamma$, $U(\cdot, t) \in C_0(\mathbb{R})$, and $U(\cdot, t)$ is an even function.*
- *(ii) For each $t < 0$ the function $U(\cdot, t)$ has exactly three critical points, namely, two global maximum points and a local minimum point* 0. *For*

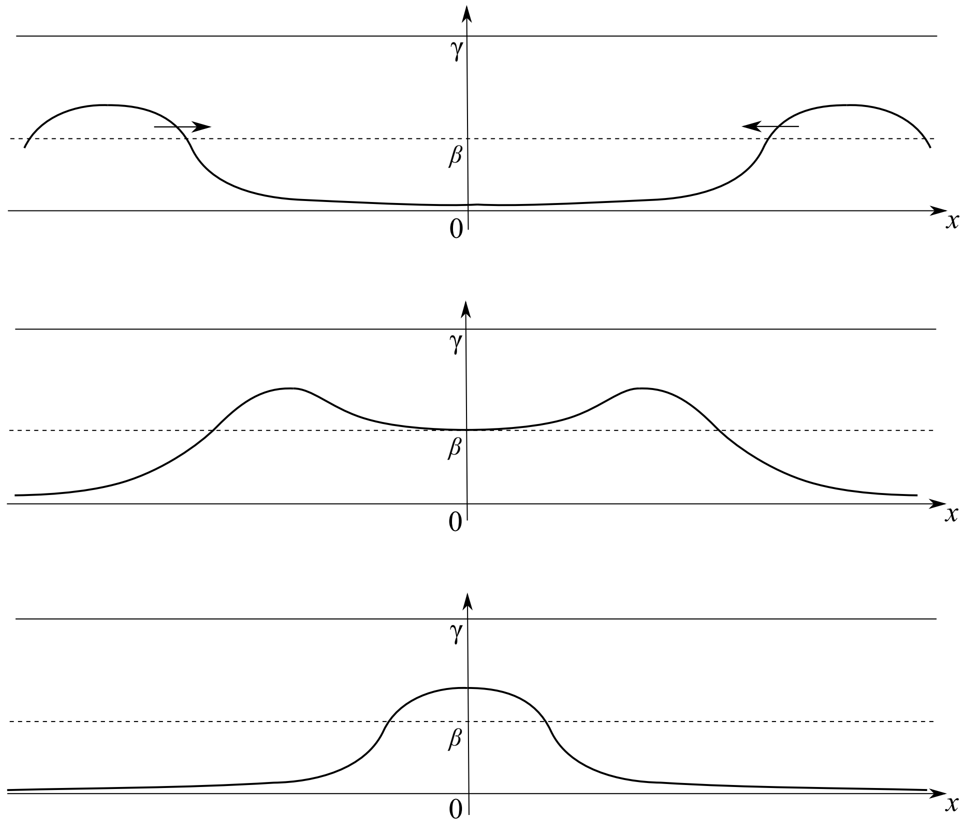

Figure 1: The graphs of the functions $U(\cdot,t)$ for $t \approx -\infty$, $t = 0$, and $t \approx \infty$ (top to bottom).

all sufficiently large $t > 0$, $x = 0$ is the unique critical point of $U(\cdot,t)$ and it is the global maximum point of $U(\cdot,t)$.

(iii) $\lim_{t\to\infty} \|U(\cdot,t) - \phi\|_{L^\infty(\mathbb{R})} = 0$.

(iv) There is a C^1 function ζ on $(-\infty,0)$ such that $\zeta(t) \to \infty$, $\zeta'(t) \to 0$ as $t \to -\infty$, and

$$\lim_{t\to-\infty} \|U(\cdot,t) - \phi(\cdot - \zeta(t)) - \phi(\cdot + \zeta(t))\|_{L^\infty(\mathbb{R})} = 0. \tag{1.6}$$

According to statements (iii), (iv) of the theorem, for large negative t the function $U(\cdot,t)$ has two humps, each roughly with the shape of ϕ, whose positions drift slowly to $\pm\infty$ as $t \to -\infty$. For large positive t, $U(\cdot,t)$ has one hump, also with the shape of ϕ. Since $\phi \in C_0(\mathbb{R})$, relation (1.6) in particular implies that, as $t \to -\infty$, $\|U(\cdot,t)\| \to 0$ in $L^\infty_{loc}(\mathbb{R})$. Thus, in $L^\infty_{loc}(\mathbb{R})$, U is a heteroclinic connection between the steady states 0 and ϕ.

The rest of the paper is organized as follows. The proof of Theorem 1.1 is given in Section 4. It is based on results concerning threshold solutions, that is, solutions separating the domains of attraction of the two stable steady states 0 and γ; we recall these results in Section 3. In the preliminary Section 2, we summarize useful results concerning steady states of (1.1), the limit sets of time dependent solutions of (1.1), and the zero number of solutions of linear parabolic equations.

Although our main result concerns a solution with range in $[0,\gamma]$, it will be convenient to make the following assumption on the global shape of f:

(A) f' is bounded and there is $\delta_0 < 0$ such that $f(\delta_0) = 0$, $f > 0$ in $(\delta_0, 0)$, and $f' > 1$ in $(-\infty, \delta_0]$.

Clearly, we can achieve that (A) holds by a suitable modification of f away from the interval $[0,\gamma]$. This has no effect on the solutions with range in $[0,\gamma]$.

2 Preliminaries

2.1 Steady states

The equation for the steady states of (1.1) is

$$v_{xx} + f(v) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

In this subsection, we recall some elementary properties of solutions of this equation.

By a solution of (2.1) we always mean a maximally defined solution. In view of the Lipschitz continuity of f (see hypothesis (A)), the solutions are all global, that is, defined for all $x \in \mathbb{R}$. Given a solution v, we denote by $\tau(v)$ its trajectory, or, orbit:

$$\tau(v) = \{(v(x), v_x(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2. \tag{2.2}$$

More precisely, $\tau(v)$ is an orbit of the first-order system associated with (2.1):

$$v_x = w, \quad w_x = -f(v). \tag{2.3}$$

This is a Hamiltonian system with respect to the energy

$$H(v,w) := w^2/2 + F(v), \quad F(u) := \int_0^u f(s)\,ds.$$

Thus, the orbits of (2.3) are contained in the level sets of H. Note that these level sets are symmetric about the v axis. Whenever a connected component of a level set of H is compact and contains no equilibrium of (2.3), this connected component consists of a single closed orbit of (2.3) corresponding to a periodic solution of (2.1). The symmetry of H and the fact that $w = v_x$ imply that each nonconstant periodic solution v has precisely two critical points in each interval $[y, y+\rho)$, where ρ is the minimal period of v. Moreover, if 0 is a critical point of v, then v is even. This follows from the reversibility of (2.3) ($v(-x)$ is a solution with the same initial conditions).

Recall that ϕ is the ground state of (2.1) with $\phi'(0) = 0$. The orbit $\tau(\phi)$ is a homoclinic orbit to the equilibrium $(0,0)$. We set $\hat{\beta} := \phi(0)$. By (BU), $u = \beta$ is the unique critical point of F in $(0, \hat{\beta})$ and $F(0) = F(\hat{\beta}) > F(u)$ for $u \in (0, \hat{\beta})$. Therefore, elementary considerations using the level curves of the Hamiltonian H show that the region in $\mathbb{R}^2$ bounded by the homoclinic $\tau(\phi)$ and its limit equilibrium $(0,0)$ is filled by closed orbits and the equilibrium point $(\beta, 0)$, which is inside each of these closed orbits.

Hypotheses (A) further implies that F has a unique critical point, a minimum point, in $(-\infty, 0)$ and $F(u) \to \infty$ as $u \to -\infty$. It follows from this and (BU), again by elementary considerations using the Hamiltonian, that each orbit intersecting the segment $\{(\eta, 0) : \eta \in (\hat{\beta}, \gamma)\}$ is a closed orbit which also intersects the w-axis and is contained in the half-plane $\{(v,w) : v < \gamma)\}$ (cp. Figure 2).

Consider now the solution v^η of (2.1) with the initial conditions

$$v(0) = \eta, \quad v_x(0) = 0. \tag{2.4}$$

Note that if φ is a solution of (2.1) with $\eta := \max \varphi = \varphi(\xi)$ for some $\xi \in \mathbb{R}$, then $\varphi = v^\eta(\cdot - \xi)$. The above remarks can be summarized in terms of v^η as follows .

Lemma 2.1. *Assume the hypotheses* (BU) *and* (A). *For each* $\eta \in (\beta, \gamma)$, $\eta \neq \hat{\beta}$, *the solution* v^η *is even and periodic, and the following statements are valid:*

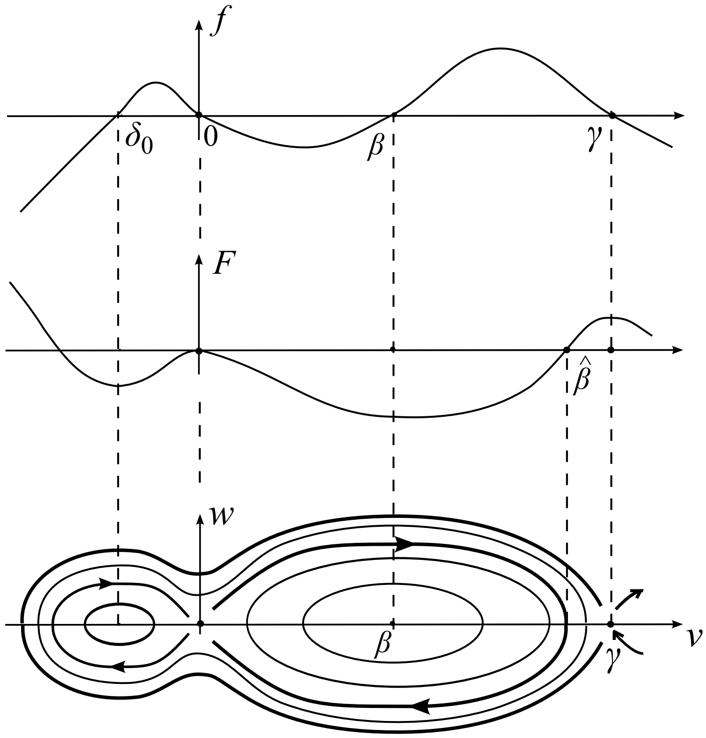

Figure 2: The graphs of the functions f and F, and the phase portrait of system (2.3).

(a) *If $\eta \in (\beta, \hat{\beta})$, then $0 < v^\eta < \hat{\beta}$ and the function $v^\eta - \beta$ has infinitely many zeros (all these zeros are necessarily simple by the uniqueness for initial value problems).*

(b) *If $\eta \in (\hat{\beta}, \gamma)$, then $v^\eta < \gamma$ and the functions v^η, $v^\eta - \beta$ have infinitely many zeros (all of them simple).*

(c) *If ρ^η denotes the minimal period of v^η, then v^η has precisely two critical points in each interval $[y, y + \rho^\eta)$, $y \in \mathbb{R}$.*

Finally, we note that the minimal period ρ^η depends continuously on η. This can be shown by the implicit function theorem, using the fact that $v^\eta_{xx}(\rho^\eta) = -f(\eta) \neq 0$. Moreover, as $\eta \to \beta$, ρ^η approaches $2\pi/\sqrt{f'(\beta)}$, that is, the minimal period of the nontrivial solutions of the linearization

$$v_{xx} + f'(\beta)v = 0.$$

Of course, $\rho^\eta \to \infty$ as $\eta \to \hat{\beta}$ or $\eta \to \gamma$. Therefore, the following statement is valid:

Lemma 2.2. *With ρ_η as in Lemma 2.1(c), one has*

$$\inf_{\eta \in (\beta, \hat{\beta}) \cup (\hat{\beta}, \gamma)} \rho^\eta > 0. \tag{2.5}$$

2.2 Solutions of (1.1) and their limit sets

Recall that an *entire solution* of (1.1) refers to a solution defined for all $t \in I := \mathbb{R}$. When dealing with the Cauchy problem (1.1), (1.2), we always take $I := (0, \infty)$ and $u_0 \in \mathcal{B}$, where $\mathcal{B}$ stands for the set of measurable functions on $\mathbb{R}$ such that $0 \leq u_0 \leq \gamma$; usually, we take u_0 piecewise continuous. We denote by $u(x, t, u_0)$ the unique maximally defined solution of (1.1), (1.2). This solution is global (that is, defined for all $t > 0$) and bounded: by the comparison principle one has $0 \leq u(\cdot, \cdot, u_0) \leq \gamma$. The map $u_0 \to u(\cdot, t, u_0) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is locally Lipschitz continuous, uniformly with respect to t in any compact interval $[0, T]$. Moreover, if u_0 is uniformly continuous, then $\|u(\cdot, t, u_0) - u_0\|_{L^\infty(\mathbb{R})} \to 0$ as $t \searrow 0$, and if, in addition, the derivatives $u_0^{(k)}$, $k = 1, 2$, are bounded and uniformly continuous, then

$$\lim_{t \searrow 0} \|\partial_x^k u(\cdot, t, u_0) - u_0^{(k)}\|_{L^\infty(\mathbb{R})} = 0 \quad (k = 0, 1, 2). \tag{2.6}$$

All these results can be found in [19]. We recall the following additional continuity properties of the solutions.

Lemma 2.3. *Given $T > \tau > 0$ and $p \in [1, \infty]$, there is a constant $L(\tau, T, p)$ such that if u_0, $\tilde{u}_0 \in \mathcal{B}$, then for each $t \in [\tau, T]$ one has*

$$\begin{aligned} \|u(\cdot, t, u_0) - u(\cdot, t, \tilde{u}_0)\|_{L^\infty(\mathbb{R})}, \ \|u_x(\cdot, t, u_0) - u_x(\cdot, t, \tilde{u}_0)\|_{L^\infty(\mathbb{R})} \\ \leq L(\tau, T, p)\|u_0 - \tilde{u}_0\|_{L^p(\mathbb{R})}. \end{aligned}$$

The estimate for $v := u(\cdot, t, u_0) - u(\cdot, t, \tilde{u}_0)$ is a standard $L^p - L^\infty$ estimate for the linear equation satisfied by v (see equation (2.12) below and note that the coefficient c is bounded independently of u_0, $\tilde{u}_0$, since the solutions stay between 0 and γ). The estimate for the derivatives then follows, enlarging $L(\tau, T, p)$ if necessary, from parabolic regularity estimates.

The following lemma gives a continuity with respect to the topology of $L^\infty_{loc}(\mathbb{R})$. Note that $\mathcal{B}$ with the topology induced from $L^\infty_{loc}(\mathbb{R})$ is a metric space with the metric given by the weighted sup-norm

$$\|v\|_w \equiv \sup_{x\in\mathbb{R}} w(x)|v(x)|, \tag{2.7}$$

where $w(x) := 1/(1+|x|^2)$.

Lemma 2.4. *Given any finite $T>0$ there is a constant $L(T)$ such that for any u_0, $\tilde{u}_0 \in \mathcal{B}$, one has*

$$\|u(\cdot,t,u_0) - u(\cdot,t,\tilde{u}_0)\|_w \le L(T)\|u_0 - \tilde{u}_0\|_w \quad (t\in[0,T]).$$

This continuity result is proved easily by considering the linear parabolic equation satisfied by $v(x,t) := w(x)(u(x,t,u_0) - u(x,t,\tilde{u}_0))$, see [8, Lemma 6.2].

Next we recall invariance properties of the limit sets of solutions of (1.1). If u is global bounded solution, then, in addition to the ω-limit set defined in (1.3), we define the following larger set

$$\Omega(u) := \{\psi : u(\cdot + x_n, t_n) \to \psi \text{ for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \tag{2.8}$$

Here, too, the convergence is in $L^\infty_{loc}(\mathbb{R})$. Clearly, $\omega(u) \subset \Omega(u)$, but the opposite inclusion is not true in general.

Standard parabolic regularity estimates imply that the derivatives u_t, u_x, u_{xx} are bounded on $\mathbb{R}\times[1,\infty)$ and they are globally α-Hölder there for each $\alpha \in (0,1)$. Therefore, if $\{(x_n,t_n)\}$ is a sequence in $\mathbb{R}\times(0,\infty)$ such that $t_n\to\infty$, then it has a subsequence (still denoted by $\{(x_n,t_n)\}$) such that $u(x_n+\cdot,t_n)$ converges in $L^\infty_{loc}(\mathbb{R})$ to some function ψ, obviously an element of $\Omega(u)$. As is well known, with each such ψ, there is an entire solution U of (1.1) such that $U(\cdot,0)=\psi$ and $U(\cdot,t)\in\Omega(u)$ for each $t\in\mathbb{R}$. Specifically, one finds U as follows. Consider the sequence of functions $(x,t)\mapsto u(x+x_n,t_n+t)$, $(x,t)\in\mathbb{R}\times(-t_n,\infty)$, $n=1,2,\dots$. The Hölder estimates on u_t, u_x, u_{xx} allow us to pass to a subsequence of this sequence such that

$$D^{2,1}u(x_n+\cdot,t_n+\cdot) \to D^{2,1}U, \tag{2.9}$$

uniformly on each compact set in $\mathbb{R}^2$, where $U(x,t)$ is an entire solution with the indicated properties and the symbol $D^{2,1}u$ stands for (u,u_x,u_{xx},u_t).

We now introduce analogous, although not so commonly used, limit sets for entire solutions. If u is a bounded entire solution u of (1.1), its α-limit set and A-limit set are defined as follows:

$$\alpha(u) := \{\psi : u(\cdot, t_n) \to \psi \text{ for some sequence } t_n \to -\infty\}, \tag{2.10}$$

$$A(u) := \{\psi : u(\cdot + x_n, t_n) \to \psi \text{ for some sequences } t_n \to -\infty \text{ and } x_n \in \mathbb{R}\}. \tag{2.11}$$

The convergence is in $L^\infty_{loc}(\mathbb{R})$ in both cases. By analogous arguments as above one shows the following invariance property:

Lemma 2.5. *Let u be a bounded entire solution of* (1.1)*. Given any sequence $\{(x_n, t_n)\}$ in $\mathbb{R} \times (0, \infty)$ with $t_n \to -\infty$, one can pass to a subsequence to obtain*

$$D^{2,1}u(x_n + \cdot, t_n + \cdot) \to D^{2,1}U \text{ in } L^\infty_{loc}(\mathbb{R}^2),$$

where U is an entire solution of (1.1) *such that $U(\cdot, t) \in A(u)$ for each $t \in \mathbb{R}$.*

We remark that in the above result, it is sufficient to assume that, rather than being an entire solution, u is an *ancient solution*, that is, a solution defined for all $t < 0$. However, in the present paper this distinction is rather meaningless. We deal with solutions whose range is contained in $[0, \gamma]$ and any ancient solution with this property extends to an entire solution (with the range in $[0, \gamma]$).

2.3 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + c(x, t)v, \quad x \in \mathbb{R}, \ t \in (s, T), \tag{2.12}$$

where $-\infty \le s < T \le \infty$ and c is a bounded function. Note that if u, $\tilde{u}$ are bounded solutions of (1.1) on $I = (s, T)$, then their difference $v = u - \tilde{u}$ is a solution of (2.12), with

$$c(x, t) = \int_0^1 f'(\tilde{u}(x, t) + s(\tilde{u}(x, t) - u(x, t)))\, ds$$

Also, $v = u_x$ is a solution of (2.12) with $c(x, t) = f'(\tilde{u}(x, t))$.

For an interval $J = (a, b)$, with $-\infty \le a < b \le \infty$, we denote by $z_J(v(\cdot, t))$ the number, possibly infinite, of all zeros $x \in J$ of the function $x \to v(x, t)$. If $J = \mathbb{R}$, we usually omit the subscript $\mathbb{R}$:

$$z(v(\cdot, t)) := z_{\mathbb{R}}(v(\cdot, t)).$$

The following intersection-comparison principle holds (see [1, 5]).

Lemma 2.6. *Let v be a nontrivial solution of* (2.12) *and $J = (a, b)$, where $-\infty \le a < b \le \infty$. Assume that the following conditions are satisfied:*

(c1) *if $b < \infty$, then $v(b, t) \neq 0$ for all $t \in (s, T)$,*

(c2) *if $a > -\infty$, then $v(a, t) \neq 0$ for all $t \in (s, T)$.*

Then the following statements hold true:

(i) *For each $t \in (s, T)$, all zeros of $v(\cdot, t)$ are isolated. In particular, if $a > -\infty$ and $b < \infty$, then $z_J(v(\cdot, t)) < \infty$ for all $t \in (s, T)$.*

(ii) *$t \mapsto z_J(v(\cdot, t))$ is a monotone nonincreasing function on (s, T) with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s, T)$, the function $v(\cdot, t_0)$ has a multiple zero in J and $z_J(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in (s, T)$ with $t_1 < t_0 < t_2$ one has*

$$z_J(v(\cdot, t_1)) > z_J(v(\cdot, t_2)). \tag{2.13}$$

If (2.13) holds, we say that $z_J(v(\cdot, t))$ *drops in the interval* (t_1, t_2) and if it holds for each interval (t_1, t_2) containing t_0, we say that $z_J(v(\cdot, t))$ *drops at* t_0.

Remark 2.7. It is clear that if the assumptions of Lemma 2.6 are satisfied and for some $s_0 \in (s, T)$ one has $q := z_J(v(\cdot, s_0)) < \infty$, then $z_J(v(\cdot, t))$ cannot drop more q times in (s_0, T). In other words, the set

$$\{t \in (s_0, T) : v(\cdot, t) \text{ has a multiple zero in } J\}$$

has at most q elements. Also, if $z_J(v(\cdot, t))$ is constant on (s_0, T), then $v(\cdot, t)$ has only simple zeros in J for each $t \in (s_0, T)$.

We next state a persistence property of multiple zeros in solutions of (2.12). The following lemma is a reformulation of [6, Lemma 2.6].

Lemma 2.8. *Assume that w is a nontrivial solution of* (2.12) *such that for some $s_0 \in (s, T)$ the function $w(\cdot, s_0)$ has a multiple zero at some x_0: $w(x_0, s_0) = w_x(x_0, s_0) = 0$. Assume further that for some $\delta > 0$, v_n is a sequence in $C^1([x_0 - \delta, x_0 + \delta] \times [s_0 - \delta, s_0 + \delta])$ which converges in this space to w. Then for all sufficiently large n the function $v_n(\cdot, t)$ has a multiple zero in $(x_0 - \delta, x_0 + \delta)$ for some $t \in (s_0 - \delta, s_0 + \delta)$.*

3 Threshold solutions

Similarly as in [26], our construction of a solution of (1.1) with an interesting behavior uses properties of threshold solutions. For some $\mu_0 > 0$, we consider a family of functions ψ^μ, $\mu \in [0, \mu_0]$, in $\mathcal{B}$ with the following properties:

(a1) Either for each $\mu \in [0, \mu_0]$ one has $\psi^\mu \in C_0(\mathbb{R})$, or for each each $\mu \in [0, \mu_0]$ the function ψ^μ has compact support (in the latter case, ψ^μ is not required to be continuous).

(a2) For some $p \in [1, \infty]$, the function $\mu \to \psi^\mu : [0, \mu_0] \to L^p(\mathbb{R})$ is continuous and monotone increasing in the sense that if $\mu < \nu$, then $\psi_\mu \leq \psi_\nu$ everywhere, with the strict inequality on a nonempty open set.

Lemma 3.1. *For each $\theta \in (\beta, \gamma)$ the following statements are valid.*

(i) *There exists $\ell = \ell(\theta)$ such that if $u_0 \in \mathcal{B}$ and $u_0 \geq \theta$ on an interval of length ℓ, then $u(\cdot, t, u_0) \to \gamma$ in $L^\infty_{loc}(\mathbb{R})$.*

(ii) *With ℓ as in* (i)*, let ψ^μ, $\mu \in [0, \mu_0]$, be a family of functions in $\mathcal{B}$ satisfying* (a1), (a2) *and the following two conditions:*

 (a3) *$\psi_{\mu_0} \geq \theta$ on an interval of length ℓ,*

 (a4) *$\lim_{t\to\infty} u(\cdot, t, \psi_0) = 0$ in $L^\infty(\mathbb{R})$.*

 Then there exists a unique $\mu^* \in (0, \mu_0)$ with the following properties:

 (t1) *If $u_0 = \psi^\mu$ with $\mu \in (0, \mu^*)$, then $\lim_{t\to\infty} u(\cdot, t, u_0) = 0$ in $L^\infty(\mathbb{R})$.*

 (t2) *If $u_0 = \psi^\mu$ with $\mu \in (\mu^*, \mu_0]$, then $\lim_{t\to\infty} u(\cdot, t, u_0) = \gamma$ in $L^\infty_{loc}(\mathbb{R})$.*

 (t3) *If $u_0 = \psi_{\mu^*}$, then $\lim_{t\to\infty} u(\cdot, t, u_0) \to v$ in $L^\infty(\mathbb{R})$ for some ground state v of* (1.5).

We refer to μ^* as the threshold value (relative to the family ψ^μ, $\mu \in [0, \mu_0]$), to the solution in (t3) as the *threshold solution*, and to the solutions in (t1) and (t2) as *subthreshold solutions* and *superthreshold solutions*, respectively.

Statement (i) of Lemma 3.1 is due to [11] (see also [6, Lemma 4.2], [7, Lemma 2.4], [8, Lemma 6.3], or [24, Lemma 3.5]). Statement (ii) for families of functions with compact supports is proved in [6] (an earlier result for specific families was proved in [29]). The result for $\psi^\mu \in C_0(\mathbb{R})$ is proved in [20]. We remark that the result for functions ψ^μ with compact support follows from the result of [20], even though ψ^μ are not required to be continuous. This can be shown by considering the functions $u(\cdot, \delta, \psi^\mu)$ which are contained in $C_0(\mathbb{R})$ for each $\delta > 0$.

Related results can also be found in [26], where families with $\psi^\mu(\pm\infty) > 0$ are considered.

4 Proof of Theorem 1.1

We find the entire solution as in the statement of the theorem as the limit of a sequence of threshold solutions for suitably constructed families.

4.1 A sequence of threshold solutions

Fix some $\theta_0 \in (\beta, \gamma)$, and let $\ell := \ell(\theta_0)$ (see Lemma 3.1(i)). For each $n \in \mathbb{N}$ and $\mu \in [0, \theta_0]$, we define an even function $\tilde{\psi}_n^\mu$ on $\mathbb{R}$ as follows (see Figure 3):

$$\tilde{\psi}_n^\mu(x) = \begin{cases} 0 & (0 \le x < n), \\ \mu & (n \le x \le n + \ell), \\ 0 & (x > n + \ell), \\ \psi_n^\mu(-x) & (x < 0). \end{cases} \tag{4.1}$$

Clearly, the family $\tilde{\psi}_n^\mu$, $\mu \in [0, \theta_0]$, satisfies the hypotheses of Lemma 3.1. We denote by μ_n the threshold value for this family and set

$$\psi_n := \tilde{\psi}_n^{\mu_n} \quad (n = 1, 2, \dots). \tag{4.2}$$

In the following lemma, we establish several key properties of the threshold solutions $u(\cdot, \cdot, \psi_n)$, $n \in \mathbb{N}$. Recall that ϕ is the ground state of (2.1) with $\phi'(0) = 0$ and $\hat{\beta} = \phi(0)$.

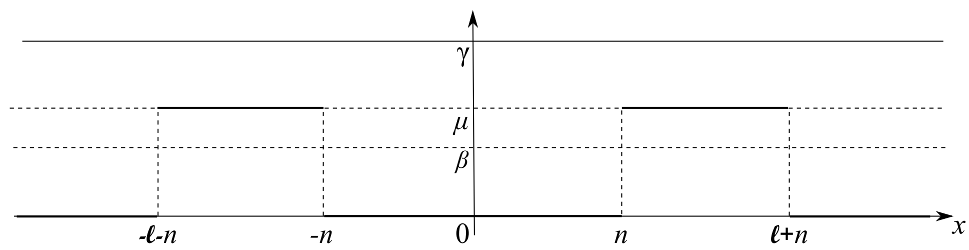

Figure 3: The graph of $\tilde{\psi}_n^{\mu}$.

Lemma 4.1. *For each $n \in \mathbb{N}$, the following statements are valid:*

(i) $\lim_{t\to\infty} u(\cdot, t, \psi_n) = \phi$ *in* $L^\infty(\mathbb{R})$.

(ii) There is a unique $\tau_n \in (0, \infty)$ *such that* $u(0, \tau_n, \psi_n) = \beta$.

(iii) With τ_n *as in (ii), one has*

$$z(u(\cdot, t, \psi_n) - \beta) = \begin{cases} 4 & (t \in (0, \tau_n)), \\ 2 & (t \in (\tau_n, \infty)). \end{cases} \tag{4.3}$$

(iv) One has $z(u_x(\cdot, t, \psi_n)) \leq 3$ *for all* $t > 0$, *and* $z(u_x(\cdot, t, \psi_n)) = 3$ *for all* $t \in (0, \tau_n)$.

(v) There is $q \in \mathbb{N}$, *independent of* n, *such that if* φ *is a periodic solution of* (2.1) *with* $\max \varphi \in (\beta, \gamma)$ *and* J *is a connected component of the set* $\{x \in \mathbb{R} : \varphi(x) > 0\}$ *(possibly,* $J = \mathbb{R}$*), then*

$$z_J(u(\cdot, t, \psi_n) - \varphi) \leq q \quad (t > 0). \tag{4.4}$$

Proof. Statement (i) follows directly from the fact that $u(\cdot, \cdot, \psi_n)$ is an even threshold solution. This also implies that $\mu_n > \beta$ (otherwise, $\psi_n \leq \beta$ and then $u(\cdot, \cdot, \psi_n) \leq \beta$, by the comparison principle).

Next, we claim that for all $t > 0$, $t \approx 0$, the following relations hold:

$$u(0, t, \psi_n) < \beta, \tag{4.5}$$

$$z(u(\cdot, t, \psi_n) - \beta) = 4, \tag{4.6}$$

$$z(u_x(\cdot, t, \psi_n)) = 3. \tag{4.7}$$

To prove the claim, we first appeal to the boundedness of the solution $u(\cdot, \cdot, u_0)$ and the variation of constants formula

$$u(\cdot, t, u_0) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} f(u(\cdot, s, u_0))\, ds, \tag{4.8}$$

where $e^{\Delta t}$, $t \geq 0$, stands for the heat semigroup on $L^\infty(\mathbb{R})$ (see [19]). Using the integral representation of the heat semigroup, we obtain from (4.8) that for all sufficiently small $t > 0$ relation (4.5) as well as the following relations are valid:

$$u(\pm(n + \frac{\ell}{2}), t, \psi_n) > \beta. \tag{4.9}$$

Since also $u(\cdot, t, \psi_n) \in C_0(\mathbb{R})$, we see that for $t > 0$, $t \approx 0$,

$$z(u(\cdot, t, \psi_n) - \beta) \geq 4, \tag{4.10}$$

$$z(u_x(\cdot, t, \psi_n)) \geq 3. \tag{4.11}$$

We show that these inequalities cannot be strict. Indeed, suppose, for example, that $z(u(\cdot, t_0, \psi_n) - \beta) > 4$ for some $t_0 > 0$. Since $u(\cdot, t, \psi_n) \in C_0(\mathbb{R})$ for each $t > 0$, Lemma 2.6 implies that $z(u(\cdot, t, \psi_n) - \beta)$ is finite. Hence, by Lemma 2.6(ii) and Remark 2.7, making $t_0 > 0$ smaller we can assume that $u(\cdot, t_0, \psi_n) - \beta$ has more than 4 simple zeros. Using the definition of ψ_n, one easily finds a sequence of smooth, even functions $w_k \in \mathcal{B}$ with compact support such that $\lim_{k\to\infty}(w_k - \psi_n) = 0$ in $L^2(\mathbb{R})$ and $w_k - \beta$ has exactly 4 zeros, all of them simple. Then, $z(u(\cdot, t, w_k) - \beta) \leq 4$ for all $t > 0$ (this follows from the monotonicity of the zero number and the continuity at $t = 0$, see (2.6)). On the other hand, by Lemma 2.3,

$$\lim_{k\to\infty} \|u(\cdot, t_0, w_k) - u(\cdot, t_0, \psi_n)\|_{L^\infty(\mathbb{R})} = 0,$$

and this implies that $z(u(\cdot, t_0, w_k) - w_k) > 4$ if k is sufficiently large. This contradiction shows that (4.6) holds.

By a similar approximation argument (taking w_k such that w_k' has exactly 3 zeros, all of them simple) one shows that the inequality in (4.11) cannot be strict, hence (4.7) holds for $t \approx 0$.

From (4.7) and the monotonicity of the zero number (Lemma 2.6), we immediately obtain the first relation in statement (iv). To prove the second relation in (iv) and statement (iii), we argue as follows. By (4.5) and statement (i), there is $\tau_n > 0$ such that $u(0, \tau_n, \psi_n) = \beta$. Since $u(\cdot, \tau_n, \psi_n)$ is even, this means that $x = 0$ is a multiple zero of $u(\cdot, t, \psi_n)$. Hence $z(u(\cdot, t, \psi_n) - \beta)$ drops at τ_n. Therefore, since $u(\cdot, t, \psi_n) \in C_0(\mathbb{R})$ and it is an even threshold solution, (4.3) must hold. This proves statement (iii). Also, (4.3) implies that for $t \in (0, \tau_n)$, $z(u_x(\cdot, t, \psi_n)) \geq 3$, and this completes the proof of statement (iv).

Since $z(u(\cdot, t, \psi_n) - \beta)$ is constant in (τ_n, ∞) and in $(0, \tau_n)$, the function $u(\cdot, t, \psi_n) - \beta$ has only simple zeros for $t \neq \tau_n$ (cp. Remark 2.7), in particular, $u(0, t, \psi_n) \neq \beta$. This completes the proof of statement (ii).

It remains to prove statement (v). Let φ be a periodic solution of (2.1) with $\eta := \max \varphi \in (\beta, \gamma)$. Then, clearly, $\eta \neq \hat{\beta}$ and $\varphi = v^\eta(\cdot - \xi)$ for some $\xi \in \mathbb{R}$ (see Section 2.1).

Assume first that $\eta \in (\beta, \hat{\beta})$, so that $\varphi > 0$. Then, since $u(\cdot, t, \psi_n) \in C_0(\mathbb{R})$ for each $t > 0$, Lemma 2.6 implies that $z(u(\cdot, t, \psi_n) - \varphi)$ is finite. It is sufficient to show that there is q independent of n and a sequence of smooth, even functions $w_k \in \mathcal{B}$ with compact support such that $\lim_{k\to\infty}(w_k - \psi_n) = 0$ in $L^2(\mathbb{R})$ and $w_k - \varphi$ has at most q zeros, all of them simple. An approximation argument similar to the one used above in the proof of (4.6) then shows that (4.4) holds with $J = \mathbb{R}$. First off, recall that the minimal period ρ^η of $\varphi = v^\eta(\cdot - \xi)$ is bounded from below by a positive constant d and that φ has precisely two critical points in each interval of the form $[y, y + \rho^\eta)$ (see Lemmas 2.1, 2.2). Therefore, there is a number $\tilde{q} \in \mathbb{N}$, depending only on the quotient ℓ/d (and not on φ) such that

$$z_{[n,n+\ell]}(\mu - \varphi) \leq \tilde{q} \quad (\mu \in (0, \gamma)).$$

Using this property of $\tilde{q}$ and the definition of ψ_n, one easily finds a sequence of functions $w_k \in \mathcal{B}$ with the desired properties. For example, take for each k a smooth even function w_k such that $0 \leq w_k \leq \psi_n$, $w_k \equiv \psi_n$ in $[0, n) \cup (n + \ell/(4k), n + \ell - \ell/(4k)) \cup (n + \ell, \infty)$. If w_k has suitable monotone transitions in the intervals $(n, n + \ell/(4k))$, $(n + \ell - \ell/(4k), n + \ell)$, then

$$z_{[n,n+\ell]}(w_k - \varphi) \leq \tilde{q} + 2,$$

and, consequently, since $\varphi > 0 = w_k$ in $\mathbb{R} \setminus ([n, n + \ell] \cup [-n - \ell, -n]$,

$$z(w_k - \varphi) \leq q := 2\tilde{q} + 4.$$

Let now $\eta \in (\beta, \hat{\beta})$ so that φ changes sign. Let $J = (a, b)$ be any connected component of the set $\{x \in \mathbb{R} : \varphi(x) > 0\}$. Then $\varphi(a) = \varphi(b) = 0$ and since

$u(\cdot, t, \psi_n) > 0$ for each $t > 0$, Lemma 2.6 applies to $u(\cdot, t, \psi_n) - \varphi$. This time, φ has precisely one critical point in $[a, b]$, namely, $(a+b)/2$ and similar considerations as above show that (4.4) holds for a suitable q independent of n and φ.

This completes the proof of statement (v). □

Lemma 4.2. *There is a number $m \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and any periodic solution φ of* (2.1) *with* $\max \varphi \in (\beta, \gamma)$*, the set*

$$\mathcal{M} := \{t > 0 : u(\cdot, t, \psi_n) - \varphi \text{ has a multiple zero}\}$$

has at most m elements.

Proof. For positive φ, this statement holds with $m = q$, where q is as in Lemma 4.1(v). This follows directly from (4.4) and Remark (2.7). If φ changes sign, then (4.4) and Remark (2.7) give the following conclusion. If J is any connected component of $\mathcal{N} := \{x \in \mathbb{R} : \varphi(x) > 0\}$, then the set

$$\mathcal{M}_J := \{t > 0 : u(\cdot, t, \psi_n) - \varphi \text{ has a multiple zero in } J\}$$

has at most q elements. Note that there can be at most 4 connected components J of $\mathcal{N}$ with the property that

$$\bar{J} \cap \{-n, -n-\ell, n, n+\ell\} \neq \emptyset. \tag{4.12}$$

For any other component J, we show in a moment (see Lemma 4.3 below) that $u(\cdot, t, \psi_n) - \varphi$ has only simple zeros in J for all $t > 0$. Of course, in the complement of the set $\mathcal{N}$ one has $u(\cdot, t, \psi_n) > 0 > \varphi$, hence, $u(\cdot, t, \psi_n) - \varphi$ has no zeros there at all. Therefore, all multiple zeros of $u(\cdot, t, \psi_n) - \varphi$, for any t, are confined in the components J which satisfy (4.12). Using this and Lemma 4.1(v), we conclude that the set $\mathcal{M}$ has at most $m := 4q$ elements. □

The following property was used in the proof of Lemma 4.2.

Lemma 4.3. *Let φ be a sign-changing periodic solution of* (2.1) *with* $\max \varphi \in (\beta, \gamma)$ *and let $J = (a, b)$ be a connected component of the set* $\{x \in \mathbb{R} : \varphi(x) > 0\}$ *such that* (4.12) *does not hold, that is,* $\bar{J} \cap \{-n, -n-\ell, n, n+\ell\} = \emptyset$*. Then for each $t > 0$ the function $u(\cdot, t, \psi_n) - \varphi$ has precisely 2 zeros in J both of them simple.*

Proof. First, we rule out the possibility that $u(\cdot, t_0, \psi_n) - \varphi \geq 0$ in (a, b) for some $t_0 > 0$. Suppose that it holds. Then $u(\cdot, t_0, \psi_n) \geq \varphi^*$, where φ^* is a continuous function defined by

$$\varphi^*(x) = \begin{cases} \varphi(x), & \text{if } x \in [a,b], \\ 0, & \text{if } x \in \mathbb{R} \setminus [a,b]. \end{cases}$$

By the comparison principle,

$$u(\cdot, t - t_0, \psi_n) \geq u(\cdot, t, \varphi^*) \quad (t \geq t_0). \tag{4.13}$$

Since both φ and 0 are steady states of (1.1), it is not difficult to show (see [20, Proof of Lemma 4.2] for details) that $u(\cdot, t, \phi^*)$ is increasing in t and, as $t \to \infty$, it converges in $L^\infty(\mathbb{R})$ to γ (which is the smallest constant steady state of (1.1) above φ). This and (4.13) give a contradiction to the fact that $u(\cdot, t, \psi_n)$ is a threshold solution.

Thus we have ruled out the inequality $u(\cdot, t, \psi_n) - \varphi \geq 0$ in (a, b) for any $t > 0$. Since $u(\cdot, t, \psi_n) > 0$ and $\varphi(a) = \varphi(b) = 0$, we have

$$z_{(a,b)}(u(\cdot, t, \psi_n) - \varphi) \geq 2 \quad (t > 0). \tag{4.14}$$

We next show that the equality holds here, hence the zeros of $u(\cdot, t, \psi_n) - \varphi$ are both simple by Remark 2.7.

With all points $-n$, $-n - \ell$, n, $n + \ell$ out of $\bar{J}$, the function ψ_n is constant on $\bar{J}$. It is therefore easy to show, taking smooth approximations of ψ_n as in the proof of Lemma 4.1 that

$$z_{(a,b)}(u(\cdot, t, \psi_n) - \varphi) \leq 2$$

for all sufficiently small $t > 0$. By the monotonicity of the zero number, this inequality remains valid for all $t > 0$, hence, we have the equality in (4.14). $\square$

We prove one more property of the sequence $\{u(\cdot, \cdot, \psi_n)\}$:

Lemma 4.4. *For any $T > 0$ and $M > 0$, one has*

$$\lim_{n \to \infty} \|u(\cdot, \cdot, \psi_n)\|_{L^\infty((-M,M) \times (0,T))} = 0. \tag{4.15}$$

In particular, if τ_n, $n = 1, 2, \ldots$, are as in Lemma 4.1, then $\tau_n \to \infty$ as $n \to \infty$.

Proof. Clearly, $\psi_n \to 0$ in $L^\infty_{loc}(\mathbb{R})$. Therefore, relation (4.15) follows directly from Lemma 2.4, where we take $\tilde{u}_0 \equiv 0$. $\square$

4.2 The limit entire solution

Using the notation from the previous subsection, consider the sequence of functions

$$u(x, t + \tau_n, \psi_n), \quad x \in \mathbb{R},\ t > -\tau_n.$$

They are solutions of (1.1) taking values in $(0, \gamma)$. Hence, using parabolic estimates similarly as in Section 2.2, we find an increasing sequence $\{n_k\}$ and an entire solution U of (1.1) such that

$$u(\cdot, \cdot + \tau_{n_k}, \psi_{n_k}) \to U \text{ as } k \to \infty, \tag{4.16}$$

where the convergence is in $C^{2,1}_{loc}(\mathbb{R}^2)$.

Our goal is to prove that the statement of Theorem 1.1 is satisfied by this entire solution. We start by establishing some basic properties of U.

Lemma 4.5. *The entire solution $U(x,t)$ takes values in $(0, \gamma)$, is even in x, and has the following properties:*

(i) $U(0, 0) = \beta$ *and* $t(U(0,t) - \beta) > 0$ *for* $t \neq 0$.

(ii) For each $t \in \mathbb{R}$ *one has*

$$z(U(\cdot, t) - \beta) \le 4, \quad z(U_x(\cdot, t)) \le 3. \tag{4.17}$$

(iii) There is $q \in \mathbb{N}$ *such that if* φ *is any periodic solution of* (2.1) *with* $0 < \varphi < \gamma$, *then*

$$z(U(\cdot, t) - \varphi) \le q \quad (t \in \mathbb{R}). \tag{4.18}$$

(iv) There is $m \in \mathbb{N}$ *such that if* φ *is any periodic solution of* (2.1) *with* $\max \varphi \in (\beta, \gamma)$, *then the set*

$$\{t \in \mathbb{R} : U(\cdot, t) - \varphi \text{ has a multiple zero}\} \tag{4.19}$$

has at most m *elements.*

Proof. Obviously, U inherits the following properties of the functions $u(\cdot, \cdot + \tau_{n_k}, \psi_{n_k})$ (cp. Lemma 4.1): $U(x,t)$ is even in x,

$$0 \le U \le \gamma, \quad U(0,0) = \beta, \quad t(U(0,t) - \beta) \ge 0 \ (t \in \mathbb{R}). \tag{4.20}$$

The relation $U(0,0) = \beta$ implies that $U \not\equiv 0$, $U \not\equiv \gamma$. Since U is an entire solution, the strong comparison principle implies that the first two relations in (4.20) are strict, that is, U takes values in $(0,\gamma)$.

We now show that for any $t \in \mathbb{R}$, $U(\cdot,t) \not\equiv \beta$. Indeed, suppose that $U(\cdot,t) \equiv \beta$ for some t. Fix a periodic solution $\varphi = v^\eta$ with $\eta \in (\beta,\hat{\beta})$ (cp. Lemma 2.1). Then $0 < \varphi < \gamma$ and, by (4.16) and Lemma 2.1(a),

$$z(u(\cdot,t+\tau_{n_k},\psi_{n_k}) - \varphi) \to \infty$$

as $k \to \infty$. This is a contradiction to Lemma 4.1(v) (note that $J = \mathbb{R}$ in that statement, as $\varphi > 0$).

Next, we claim that for each $t \neq 0$ the function $U(\cdot,t) - \beta$ has only simple zeros. Suppose this is not true: for some $t_0 \neq 0$ the function $U(\cdot,t_0) - \beta$ has a multiple zero. Then, an application of Lemma 2.8 shows that for sufficiently large k the function $u(\cdot,t_1+\tau_{n_k},\psi_{n_k}) - \beta$ has a multiple zero for some $t_1 \approx t_0$ (in particular, $t_1 \neq 0$). This is impossible by (4.3) and Remark 2.7. Thus, our claim is true.

By very similar arguments, using Lemma 4.1(iv), one shows that for each $t < 0$ the function $U_x(\cdot,t)$ has only simple zeros.

These simplicity properties, in conjunction with (4.16) and Lemma 4.1(iii)-(iv), imply that relations (4.17) hold for all $t < 0$, hence for all $t \in \mathbb{R}$ by the monotonicity of the zero number. This proves statement (ii).

The simplicity of the zeros of $U(\cdot,t) - \beta$ and the evenness of U imply that $U(\cdot,t) \neq \beta$ for $t \neq 0$. Therefore, Lemma 4.1(i) and (4.16) imply that statement (i) holds.

To prove statements (iii) and (iv), we first note that for any nonconstant periodic solution of (2.1) with $\max\varphi \in (\beta,\gamma)$ one has $U \not\equiv \varphi$. This is obvious from $U > 0$ if φ changes sign; for $0 < \varphi < \gamma$ it follows from (4.17) and Lemma 2.1(a). Now, if the function $U(\cdot,t_0) - \varphi$ has a multiple zero for some t_0, then, by Lemma 2.8, for all sufficiently large k the function $u(\cdot,t+\tau_{n_k},\psi_{n_k}) - \varphi$ has a multiple zero for some $t \approx t_0$. This and Lemma 4.2 clearly imply that statement (iv) holds.

Finally, we prove that statement (iii) holds with q as in Lemma 4.1(v). Suppose $z(U(\cdot,t_0) - \varphi) > q$ for some periodic solution φ with $0 < \varphi < \gamma$ and for some $t_0 \in \mathbb{R}$. Using the monotonicity of the zero number, we then find $t \leq t_0$ such that $U(\cdot,t) - \varphi$ has more than q simple zeros. Consequently, for all sufficiently large k the function $u(\cdot,t+\tau_{n_k},\psi_{n_k}) - \varphi$ has more than q zeros, contradicting Lemma 4.1(v).

The proof of Lemma 4.5 is now complete. □

Corollary 4.6. *(i) If $\theta \in (\beta, \gamma)$ and $t \in \mathbb{R}$, then the length of any interval in the set*

$$\{x \in \mathbb{R} : U(x,t) > \theta\} \tag{4.21}$$

is not greater $\ell(\theta)$, where $\ell(\theta)$ is as in Lemma 3.1(i).

(ii) Given any $\epsilon \in (0, \hat{\beta}/2)$, there is $d := d(\epsilon) > 0$ such that for each $t \in \mathbb{R}$ the length of any interval in the set

$$\{x \in \mathbb{R} : \epsilon < U(x,t) < \hat{\beta} - \epsilon\} \tag{4.22}$$

is not greater than d.

Proof. Statement (i) follows from Lemma 3.1. Indeed, if the set in (4.21) contained an interval of length greater than $\ell(\theta)$, then for sufficiently large k the same would be true of the set

$$\{x \in \mathbb{R} : u(x, t + \tau_{n_k}, \psi_{n_k}) > \theta\}.$$

However, this is impossible since $u(\cdot, \cdot, \psi_{n_k})$ is a threshold solution.

To prove statement (ii), fix any $\epsilon > 0$. There is a periodic solution φ of (2.1) such that $0 < \min \varphi < \epsilon$ and $\hat{\beta} - \epsilon < \max \varphi < \hat{\beta}$ (see Section 2.1). Let ρ be the minimal period of φ and let q be as in Lemma 4.5(ii). Then the statement of the corollary holds with $d := (q+1)\rho$. Indeed, if the set in (4.22) contained an interval of length greater than d, then, clearly, $U(\cdot, t) - \varphi$ would have at least $q + 1$ zeros, which is not possible by Lemma 4.5(ii). $\square$

Corollary 4.7. *For each $t \in \mathbb{R}$, one has $U(\cdot, t) \in C_0(\mathbb{R})$.*

Proof. For each $t \in \mathbb{R}$, Lemma 4.5(ii) implies that $U_x(x,t) \neq 0$ for all x with sufficiently large $|x|$. Therefore, the limits

$$\xi^{\pm}(t) = \lim_{x \to \pm\infty} U(x,t)$$

exist for each $t \in \mathbb{R}$. Corollary 4.6(i) clearly rules out the possibility that $\xi^-(t) > \beta$ or $\xi^+(t) > \beta$ for some t, and Corollary 4.6(ii) rules out the possibility that $\xi^-(t) \in (0, \beta]$ or $\xi^+(t) \in (0, \beta]$ for some t. Therefore $\xi^{\pm} \equiv 0$, which shows that $U(\cdot, t) \in C_0(\mathbb{R})$ for each $t \in \mathbb{R}$. $\square$

Corollary 4.8. *One has*

$$z(U(\cdot,t)-\beta) = \begin{cases} 4 & (t<0), \\ 2 & (t>0). \end{cases} \tag{4.23}$$

and

$$z(U_x(\cdot,t)) = 3 \quad (t<0). \tag{4.24}$$

Proof. For any $t > 0$, the relations $U(0,t) > \beta$ and $U(\cdot,t) \in C_0(\mathbb{R})$ (see Lemmas 4.5(i) and 4.7) give $z(U(\cdot,t)-\beta) \geq 2$. We also know, by Lemma 4.5(i) and the evenness of $U(\cdot,t)$, that $z(U(\cdot,t))$ drops at $t=0$. This, Lemma 4.5(ii), and the evenness of $U(\cdot,t)$ imply (4.23).

The evenness of $U(\cdot,t)$ and (4.23) imply that $z(U_x(\cdot,t)) \geq 3$ for $t<0$. This and Lemma 4.5(ii) give (4.24). □

Lemma 4.9. *One has*

$$\lim_{t\to\infty} \|U(\cdot,t)-\phi\|_{L^\infty(\mathbb{R})} = 0. \tag{4.25}$$

Proof. Fix any $t_0 \in \mathbb{R}$. We define a family of functions ψ^μ, $\mu \in [0,\mu_0]$, in $\mathcal{B} \cap C_0(\mathbb{R})$ as follows:

$$\psi^\mu(x) = \begin{cases} \mu U(x,t_0) & (\mu \in [0,1]), \\ \min\{\mu U(x,t_0),\gamma\} & (\mu \in [1,\mu_0]). \end{cases}$$

Since $U(\cdot,t_0) \in C_0(\mathbb{R})$ and $U(\cdot,t_0) > 0$, it is clear that if $\mu_0 > 1$ is sufficiently large, then this family satisfies the hypotheses of Lemma 3.1(ii). Let $\mu^* \in (0,\mu_0)$ be the threshold value for this family, as in Lemma 3.1(ii). Relation (4.25) can now be equivalently stated as $\mu^* = 1$. We prove this by contradiction. If $1 > \mu^*$, so that $U(\cdot,t)$ is a superthreshold solution, then $U(\cdot,t) \to \gamma$ in $L^\infty_{loc}(\mathbb{R})$ as $t\to\infty$. This is impossible by Corollary 4.6(i). Suppose now that $1 < \mu^*$, that is, $U(\cdot,t)$ is a subthreshold solution. Then $U(\cdot,t) \to 0$ in $L^\infty(\mathbb{R})$ as $t \to \infty$. In particular, there is $t_1 > 0$ such that $U(\cdot,t_1) < \beta$, contradicting (4.23). These contradictions show that $\mu^* = 1$, proving that (4.25) holds. □

By (4.24) and the evenness, for each $t<0$, $U(\cdot,t)$ has exactly three critical points $0, \pm\zeta(t)$, where $\zeta(t) > 0$. By Remark 2.7, the zeros of $U_x(\cdot,t)$ are simple for $t<0$, hence, by the implicit function theorem, ζ is a C^1 function on $(-\infty,0)$. As $U(\cdot,t) \in C_0(\mathbb{R})$, $\pm\zeta(t)$ are the global maximizers of $U(\cdot,t)$.

Our next step in the proof of Theorem 1.1 is the following statement.

Lemma 4.10. *With $\zeta(t)$ as above, one has*

$$\lim_{t\to\infty} U(\cdot + \zeta(t), t) = \phi \;\; in \; L^\infty_{loc}(\mathbb{R}). \tag{4.26}$$

Proof. It is clearly sufficient to prove that each sequence $t_j \to -\infty$ can be replaced by a subsequence such that

$$\lim_{j\to\infty} U(\cdot + \zeta(t_j), t_j) = \phi \;\text{ in } L^\infty_{loc}(\mathbb{R}). \tag{4.27}$$

For that aim, take an arbitrary sequence $t_j \to -\infty$. Passing to a subsequence, we may assume that

$$U(\cdot + \zeta(t_j), \cdot + t_j) \to V \;\text{ in } C^{2,1}_{loc}(\mathbb{R}^2), \tag{4.28}$$

where V is another entire solution of (1.1) with

$$\psi := V(\cdot, 0) = \lim_{j\to\infty} U(\cdot + \zeta(t_j), t_j) \in A(U)$$

(see Lemma 2.5). We need to prove that $\psi \equiv \phi$.

Observe that since $U(\zeta(t), t)$ is the global maximum of $U(\cdot, t)$ and

$$z(U(\cdot, t) - \beta) \geq 2,$$

we have $\beta < U(\zeta(t), t)$. Hence,

$$\beta \leq \psi(0) \leq \gamma, \quad \psi'(0) = 0. \tag{4.29}$$

From Lemma 4.5, we have $0 < U < \gamma$, which gives $0 \leq V \leq \gamma$. In fact, $0 < V < \gamma$ by the strong comparison principle, since $V \not\equiv 0$ by (4.29) and $V \not\equiv \gamma$ as shown in the next paragraph.

We next show that ψ cannot be identical to any periodic steady state of (1.1). This is trivial for sign-changing periodic solutions. The possibilities $\psi \equiv \gamma$, $\psi \equiv \beta$ are ruled out by Corollary 4.6. Finally, if φ is a periodic solution with $0 < \varphi < \gamma$, then the identity $\psi \equiv \varphi$ and Lemma 2.1 would imply that $z(U(\cdot, t_j) - \beta) \to \infty$ as $j \to \infty$. This is impossible by Lemma 4.5(ii). Thus, $\psi \not\equiv \varphi$.

Although, at this stage we do not know if ψ is a steady state of (1.1) or not, it is still useful to introduce its "spatial trajectory:"

$$\{(\psi(x), \psi_x(x)) : x \in \mathbb{R}\}$$

(cp. (2.2)). We claim that

$$\tau(\psi) \subset \tau(\phi). \tag{4.30}$$

Suppose this is not true. Then there is a periodic steady state φ of (1.1), such that $\max \varphi \in (0, \gamma)$ and

$$\tau(\psi) \cap \tau(\varphi) \neq \emptyset. \tag{4.31}$$

Indeed, if $\psi(0) \neq \hat{\beta} = \phi(0)$, we can simply take $\varphi = v^\eta$ with $\eta = \psi(0)$ (cp. Lemma 2.1) so that the sets $\tau(\psi)$ and $\tau(\varphi)$ share the point $(\eta, 0)$. Assume that $\psi(0) = \hat{\beta}$, which means that $(\psi(0), \psi'(0)) \in \tau(\phi)$. Since ψ is a C^1 function, if (4.30) does not hold, then $\tau(\psi)$ must intersect one of the trajectories $\tau(v^\eta)$ for $\eta \approx \hat{\beta}$.

Relation (4.31) means that for some points $x_0, x_1 \in \mathbb{R}$ we have $\psi(x_0) = \varphi(x_1)$, $\psi'(x_0) = \varphi'(x_1)$. Replacing φ by a translation, we may assume that $x_1 = x_0$, thus $\psi - \varphi$ has a multiple zero at x_0. Consider now the difference $w := V - \varphi$, which is a solution of a linear equation (2.12) on $\mathbb{R}^2$. It is nontrivial (as shown above, $V(\cdot, 0) = \psi \not\equiv \varphi$) and $w(\cdot, 0) = \psi - \varphi$ has a multiple zero at x_0. We intend to apply Lemma 2.8 to this solution w. Let $\rho > 0$ be the minimal period of φ. For $j = 1, 2, \ldots$, we can write

$$\zeta(t_j) = p_j \rho + r_j,$$

with $p_j \in \mathbb{N} \cup \{0\}$, $r_j \in [0, \rho)$. Replacing $\{t_j\}$ by a subsequence, we may assume that $r_j \to r_0 \in [0, \rho]$. Then, using (4.28), we obtain

$$U(\cdot + p_j\rho + r_0, \cdot + t_j) - \varphi \to w \text{ in } C^1_{loc}(\mathbb{R}^2). \tag{4.32}$$

Therefore, by Lemma 2.8, for all sufficiently large j, the function $U(\cdot + p_j\rho + r_0, \tilde{t}_j) - \varphi$ has a multiple zero in $(x_0 - 1, x_0 + 1)$ for some $\tilde{t}_j \in (t_j - 1, t_j + 1)$. But then the function

$$U(\cdot, \tilde{t}_j) - \varphi(\cdot - p_j\rho - r_0) = U(\cdot, \tilde{t}_j) - \varphi(\cdot - r_0)$$

has a multiple zero and, since $\tilde{t}_j \to -\infty$, we have a contradiction to Lemma 4.5(iv). Thus, (4.30) is true, as claimed.

To prove that $\psi \equiv \phi$, we now follow unique continuation arguments from [27]. Relation (4.30) and the evenness of ψ, ϕ imply that for each $x > 0$ there is a unique $\varsigma(x) \geq 0$ such that

$$\psi(x) = \phi(\varsigma(x)), \quad \psi'(x) = \phi'(\varsigma(x)). \tag{4.33}$$

(the uniqueness is due to $\phi' < 0$ on $(0, \infty)$). Since ψ is nonconstant, there is a (nonempty) open interval J such that $\varsigma(x) > 0$ for all $x \in J$. It then follows from the uniqueness and the implicit function theorem that $\varsigma \in C^1$ on J. Differentiating the first identity in (4.33) and comparing to the second one, we obtain that $\varsigma' \equiv 1$. Thus, on J we have $\psi \equiv \phi(\cdot + \theta)$ for some $\theta \in \mathbb{R}$. Consider now the function $W := V - \phi(\cdot + \theta)$. It is an entire solution of a linear equation (2.12) and $W(\cdot, 0) \equiv \psi - \phi(\cdot + \theta)$ vanishes on J. By Lemma 2.6, this is possible only if $W \equiv 0$. In particular $\psi \equiv \phi(\cdot + \theta)$ on $\mathbb{R}$. By the evenness, $\theta = 0$ and $\psi \equiv \phi$. This completes the proof. □

Corollary 4.11. *One has $\zeta(t) \to \infty$ as $t \to -\infty$.*

Proof. By the evenness of $U(\cdot, t)$, in addition to (4.26) we have

$$\lim_{t \to \infty} U(\cdot - \zeta(t), t) = \phi \text{ in } L^\infty_{loc}(\mathbb{R}). \tag{4.34}$$

Since ϕ is even and $\phi' < 0$, if $\zeta(t_j)$ stayed bounded for a sequence $t_j \to -\infty$, then (4.26), (4.34) would mean that $U(\cdot, t_j)$ is not single-valued near $x = 0$ if j is sufficiently large. Thus $\zeta(t) \to \infty$ as $t \to -\infty$. □

Completion of the proof of Theorem (1.1). Statement (i) of the theorem follows from Lemma 4.5 and Corollary 4.7; statement (ii) is a direct consequence of (4.24) and Lemma 4.5; and statement (iii) is the same as Lemma 4.9.

It remains to prove statement (iv). First, we prove that (1.6) holds if $\zeta(t)$ is as in above: for $t < 0$, $\zeta(t)$ is the unique critical point of $U(\cdot, t)$ in $(0, \infty)$.

Given any $\epsilon > 0$, choose $b > 0$ so large that

$$\phi(x) < \epsilon/4 \quad (x \in \mathbb{R} \setminus (-b, b)). \tag{4.35}$$

Lemma 4.10 implies that there is $T > 0$ such that

$$|U(x, t) - \phi(x - \zeta(t))| < \epsilon/4 \quad (x \in [\zeta(t) - b, \zeta(t) + b],\ t < -T). \tag{4.36}$$

Since $U_x(x, t) > 0$ in $[0, \zeta(t))$ and $U_x(x, t) < 0$ in $(\zeta(t), \infty)$, from (4.35), (4.36), we obtain

$$0 < U(x, t) < U(\zeta(t) - b, t) < \epsilon/2 \quad (x \in [0, \zeta(t) - b),\ t < -T), \tag{4.37}$$
$$0 < U(x, t) < U(\zeta(t) + b, t) < \epsilon/2 \quad (x \in [\zeta(t) + b, \infty),\ t < -T). \tag{4.38}$$

Using estimates (4.35)-(4.38), one shows easily that

$$|U(x, t) - \phi(x - \zeta(t)) - \phi(x + \zeta(t))| < \epsilon \quad (x \geq 0,\ t < -T).$$

Since $U(\cdot, t)$ and ϕ are even, this proves that (1.6) holds.

Obviously, (1.6) remains valid if ζ is replaced by any C^1 function $\tilde{\zeta}$ on $(-\infty, 0)$ satisfying $\zeta(t) - \tilde{\zeta}(t) \to 0$ as $t \to -\infty$. We show that $\tilde{\zeta}$ with this property can be chosen in such a way that $\tilde{\zeta}'(t) \to 0$ as $t \to -\infty$. Since $\tilde{\zeta}(t)$ approaches ∞ together with $\zeta(t)$ as $t \to -\infty$ (cp. Corollary 4.11), the proof of statement (iv) will be completed by this step.

We define $\tilde{\zeta}$ as follows. For $t < 0$, let $\vartheta(t)$ be the largest of the four zeros of the function $U(\cdot, t) - \beta$ (cp. Corollary 4.8). Since the zero is simple, ϑ is a C^1 function. From Lemma 4.10 it follows that, as $t \to -\infty$, $\vartheta(t) - \zeta(t) \to x_0$, where x_0 is the unique point in $(0, \infty)$ with $\phi(x_0) = \beta$. We set $\tilde{\zeta}(t) := \vartheta(t) - x_0$. This is obviously a function with the desired property $\zeta(t) - \tilde{\zeta}(t) \to 0$ as $t \to -\infty$. We conclude the proof by showing that $\tilde{\zeta}'(t) = \vartheta'(t) \to 0$ as $t \to -\infty$.

We use an argument from [27]. Recall that any sequence $t_n \to -\infty$ can be replaced by a subsequence such that $U(\cdot + \vartheta(t_n), \cdot + t_n)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to an entire solution V of equation (1.1) (see Section 2.2). By (4.34) and the relation $U(\vartheta(t), t) = \beta$, we have $V(\cdot, 0) = \phi(\cdot + x_0)$. Since $\phi(\cdot + x_0)$ is a steady state, by the uniqueness and backward uniqueness for (1.1) we have $V \equiv \phi(\cdot + x_0)$. Thus, the convergence in $C^1_{loc}(\mathbb{R}^2)$ yields

$$\begin{aligned}(U(\cdot + \vartheta(t_n), \cdot + t_n), U_x(\cdot + \vartheta(t_n), \cdot + t_n), U_t(\cdot + \vartheta(t_n), \cdot + t_n))\\ \to (\phi(\cdot + x_0), \phi'(\cdot + x_0), 0).\end{aligned}$$

Since this is true for any sequence $t_n \to -\infty$, the convergence takes place with t_n replaced by t, with $t \to -\infty$. In particular, taking $x = 0$, we obtain

$$\big(U(\vartheta(t), t), U_x(\vartheta(t), t), U_t(\vartheta(t), t)\big) \to (\beta, \phi'(x_0), 0), \tag{4.39}$$

as $t \to -\infty$. Differentiating the identity $U(\vartheta(t), t) = \beta$, we obtain

$$U_x(\vartheta(t), t)\vartheta'(t) + U_t(\vartheta(t), t) = 0.$$

Since $\phi'(x_0) < 0$, from (4.39) we conclude that $\vartheta'(t) \to 0$ as $t \to \infty$. The proof is now complete.

□

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