Large-time behavior of solutions of parabolic equations on the real line with convergent initial data III: unstable limit at infinity

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Dedicated to Eiji Yanagida
on the occasion of his 65th birthday

Abstract

This is a continuation, and conclusion, of our study of bounded solutions u of the semilinear parabolic equation $u_t = u_{xx} + f(u)$ on the real line whose initial data $u_0 = u(\cdot, 0)$ have finite limits $\theta^\pm$ as $x \to \pm\infty$. We assume that f is a locally Lipschitz function on $\mathbb{R}$ satisfying minor nondegeneracy conditions. Our goal is to describe the asymptotic behavior of $u(x,t)$ as $t \to \infty$. In the first two parts of this series we mainly considered the cases where either $\theta^- \neq \theta^+$; or $\theta^\pm = \theta_0$ and $f(\theta_0) \neq 0$; or else $\theta^\pm = \theta_0$, $f(\theta_0) = 0$, and θ_0 is a stable equilibrium of the equation $\dot{\xi} = f(\xi)$. In all these cases we proved that the corresponding solution u is quasiconvergent—if bounded—which is to say that all limit profiles of $u(\cdot,t)$ as $t \to \infty$ are steady states. The limit profiles, or accumulation points, are taken in $L^\infty_{loc}(\mathbb{R})$. In the present paper, we take on the case that $\theta^\pm = \theta_0$, $f(\theta_0) = 0$, and θ_0 is an unstable equilibrium of the equation $\dot{\xi} = f(\xi)$. Our earlier quasiconvergence theorem in this case involved some restrictive technical conditions on the solution, which we now remove. Our sole condition on $u(\cdot,t)$ is that it is nonoscillatory (has only finitely many critical points) at some $t \geq 0$. Since it is known that oscillatory bounded solutions are not always quasiconvergent, our result is nearly optimal.

Key words: Parabolic equations on $\mathbb{R}$, quasiconvergence, entire solutions, chains, spatial trajectories, zero number

Contents

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1 Introduction and main results

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where f is a locally Lipschitz function on $\mathbb{R}$ and $u_0 \in C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We denote by $u(x,t,u_0)$, or simply $u(x,t)$ if there is no danger of confusion, the unique classical solution of (1.1), (1.2)—to ensure the uniqueness, we require classical solutions to satisfy $u(\cdot,t) \in L^\infty(\mathbb{R})$ as long as they are defined—and by $T(u_0) \in (0,+\infty]$ its maximal existence time. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = +\infty$, that is, the solution is global.

As in the previous two parts of this paper series, [17, 18], we examine solutions with initial data u_0 taken in the space

$$\mathcal{V} := \{v \in C_b(\mathbb{R}) :\ \text{the limits } v(-\infty), v(+\infty) \in \mathbb{R} \text{ exist}\}. \tag{1.3}$$

In other words, we assume that the initial datum u_0 has limits as $x \to \pm\infty$. It is well known that the space $\mathcal{V}$ is invariant for equation (1.1): if u_0 admits finite limits as $x \to \pm\infty$, then so does $u(\cdot, t, u_0)$ for any $t \in (0, T(u_0))$ (see Lemma 3.9 below; note that the limits may vary with t). So it is natural to consider $\mathcal{V}$ as a state space for equation (1.1). Our goal is to understand the large-time behavior of bounded solutions $u(\cdot,t) \in \mathcal{V}$ and, in particular, to clarify if, in any fixed bounded interval, the shape of $u(\cdot,t)$ at large times is determined by steady states of (1.1). To express this formally, we introduce the ω-limit set of a bounded solution u:

$$\omega(u) := \{\varphi \in L^\infty(\mathbb{R}),\ u(\cdot,t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}. \tag{1.4}$$

Here the convergence is in the topology of $L^\infty_{loc}(\mathbb{R})$, that is, the locally uniform convergence. By standard parabolic estimates, the trajectory $\{u(\cdot,t),\ t \geq 1\}$ of a bounded solution is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that $\omega(u)$ is nonempty, compact, and connected in $L^\infty_{loc}(\mathbb{R})$, and it attracts the solution in (the metric space) $L^\infty_{loc}(\mathbb{R})$:

$$\operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot,t), \omega(u)) \underset{t\to\infty}{\longrightarrow} 0.$$

If the $\omega-$limit set reduces to a single element φ, then u is *convergent:* $u(\cdot,t) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$ as $t \to \infty$. Necessarily, φ is a steady state of (1.1). If all functions $\varphi \in \omega(u)$ are steady states of (1.1), the solution u is said to be *quasiconvergent*. Convergence and quasiconvergence both express a relatively tame character of the solution in question. In both cases, $u_t(\cdot, t)$ approaches zero locally uniformly on $\mathbb{R}$ as $t \to \infty$. For this reason, it is difficult to numerically distinguish convergence from quasiconvergence (analytically, convergence is characterized by the existence of the improper Riemann integral of $u_t(x, t)$ on $[1, \infty)$ for each x).

For analogs of (1.1) on bounded intervals under Dirichlet, Neumann, Robin, or periodic boundary conditions, or sometimes even for (1.1) itself when solutions in suitable energy spaces are considered, quasiconvergence of solutions can be established by means of energy estimates (see, for example, [7]). However, the existence of the limits $u_0(\pm\infty)$ alone is not sufficient for quasiconvergence. As shown in [19, 21], bounded solutions in $\mathcal{V}$ which are not quasiconvergent do exist. (We emphasize here that the locally uniform convergence is taken in the definition of the ω-limit set and the corresponding notion of quasiconvergence; if the uniform convergence is taken instead, the existence of bounded solutions which are not quasiconvergent is rather trivial). Moreover, the existence of such solutions is not an exceptional phenomenon at all; it is guaranteed by a robust condition on f, namely the existence of a bistable interval. Note, however, that steady states are not completely irrelevant for the behavior of non-quasiconvergent solutions. A result of [11, 12] shows that the ω-limit set of any bounded solution of (1.1) contains at least one steady state. There are convergence and quasiconvergence results for various specific classes of solutions of (1.1), see [5, 6, 8, 14, 15, 16, 24, 25, 26]; overviews can be found in [22, 18]). More recent results include a convergence theorem of [4] for positive solutions of periodic versions of (1.1) with compact initial data and a description of the large-time behavior of entire solutions with localized past [13] (the latter paper deals with equations on $\mathbb{R}^N$ and its introduction also contains an overview of earlier results for multidimensional parabolic problems).

Our study of solutions in $\mathcal{V}$, which we conclude in this paper, yields a rather complete information on the quasiconvergence property of bounded solutions in this space. In our first result, the main theorem of [17], we proved that if the limits $\theta^{\pm} := u_0(\pm\infty)$ are distinct, then the solution u of (1.1), (1.2) is quasiconvergent, if bounded. In [18], we then showed that the same is true if $\theta^- = \theta^+ =: \theta_0$, and either $f(\theta_0) \neq 0$ or $f(\theta_0) = 0$ and θ_0 is a stable equilibrium of the equation $\dot{\xi} = f(\xi)$. In this result, we assumed the following nondegeneracy condition on the nonlinearity:

(ND) For each $\gamma \in f^{-1}\{0\}$, f is of class C^1 in a neighborhood of γ and $f'(\gamma) \neq 0$.

Hence, the stability of θ_0 simply means that $f'(\theta_0) < 0$.

In the remaining case, $\theta^{\pm} = \theta_0$, $f(\theta_0) = 0$ with θ_0 unstable ($f'(\theta_0) > 0$), the above quasiconvergence result is not valid without additional conditions on the solution; this is documented by the examples of [19, 21], as already mentioned above. It is intriguing, however, that all non-quasiconvergent solutions u found in these examples share a prominent feature: they are oscillatory in the sense that $u(\cdot, t)$ has infinitely many critical points at all times $t > 0$. This raises a natural question whether without the oscillations the solution is necessarily quasiconvergent, if bounded. More precisely, the question is whether the solution of (1.1), (1.2) is quasiconvergent, provided it is bounded and satisfies the following condition:

(NC) There is $t > 0$ such that $u(\cdot, t)$ has only finitely many critical points.

We remark that if (NC) holds for some t, then it holds for any larger t due to well known properties of the zero number of $u_x(\cdot, t)$ (see Section 3.1). In Remark 1.3 below, we mention some sufficient conditions for the validity of (NC) in terms of u_0.

In [18], we left open the question whether (NC) alone is sufficient for the quasiconvergence of u; we only proved the quasiconvergence assuming (NC) holds together with some additional and somewhat artificial conditions. The main theorem of the present paper gives a positive answer without any extra condition:

Theorem 1.1. *Assume that* (ND) *holds, and* $u_0 \in \mathcal{V}$ *has both its limits* $u_0(\pm\infty)$ *equal to some* $\theta_0 \in \mathbb{R}$ *with* $f(\theta_0) = 0 < f'(\theta_0)$*. Then, if the solution* u *of* (1.1), (1.2) *is bounded and satisfies* (NC)*, it is quasiconvergent:* $\omega(u)$ *consists of steady states of* (1.1).

This theorem, combined with the results of [17, 18], gives the following corollary concerning general bonded solutions which are nonoscillatory in the spatial variable:

Corollary 1.2. *Assume that* (ND) *holds and let* u *be a bounded solution of* (1.1) *such that* (NC) *holds. Then* u *is quasiconvergent.*

Proof. Choose a large enough t_0 such that (NC) holds with $t = t_0$: $u(\cdot, t_0)$ has only finitely many critical points. Replacing the initial datum of the solution u by $u_0 := u(\cdot, t_0)$, we achieve that u_0 is monotone near $\pm\infty$; in particular, $u_0 \in \mathcal{V}$. If the limits $\theta^\pm := u_0(\pm\infty)$ are distinct, or are both equal to θ_0 where either $f(\theta_0) \neq 0$ or θ_0 is a stable equilibrium of $\dot{\xi} = f(\xi)$, we apply the results of [17] or [18], respectively. If the limits are both equal to an unstable equilibrium of $\dot{\xi} = f(\xi)$, we apply Theorem 1.1. We thus obtain the quasiconvergence conclusion in all cases. □

Remark 1.3. (i) We mention here some simple sufficient conditions, in terms of the initial data u_0, for the validity of the assumptions on the solution u in Theorem 1.1. A sufficient condition for the boundedness of the solution of (1.1), (1.2) is that u_0 takes values between two constants $\xi < \eta$ satisfying $f(\xi) > 0 > f(\eta)$. This follows from the comparison principle. A sufficient condition for (NC) is that u_0 has only finitely many critical points if it is of class C^1. This is a consequence of the monotonicity of the zero number of $u_x(\cdot, t)$. More generally, (NC) holds if there are constants $a < b$ such that the function u_0 is monotone and nonconstant on each of the intervals $(-\infty, a)$, (b, ∞). Indeed, if this holds, one shows easily, using the comparison principle (comparing u and its spatial shifts) that for small $t > 0$ the function is strictly monotone on each of the intervals $(-\infty, a - 1)$, $(b + 1, \infty)$; the strong comparison principle then shows that $u_x(x, t)$ has no zero in these intervals for small t. Consequently, by properties of the zero number of $u_x(\cdot, t)$ (cp. Section 3.1), $u(\cdot, t)$ has only a finite number of critical points for all $t > 0$.

(ii) As mentioned above, bounded solutions that do not satisfy (NC) are not quasiconvergent in general. In this sense, condition (NC) is optimal. However, some generalization are probably still possible. For example, one may ask if it is sufficient to assume that for some t there is ρ such that $u(\cdot, t)$ has no critical points in at least one of the intervals $(-\infty, \rho)$, (ρ, ∞). (Note that, as in (i), if $u(\cdot, t)$ has no critical points in the union of these intervals, then (NC) holds for larger times). Another question is whether condition (NC) can be replaced by the weaker requirement that $u(\cdot, t) - \theta_0$ has only finitely many zeros for some t. Our proof does not apply in these cases and we do not pursue these generalizations.

In the proof of Theorem 1.1, we build on the strategy and some technical results of [18]. The strategy consists in careful analysis of a certain type of entire solutions of (1.1). By an entire solution we mean a solution $U(x, t)$ of (1.1) defined for all $t \in \mathbb{R}$ (and $x \in \mathbb{R}$). It is well known that for any $\varphi \in \omega(u)$ there exists a unique entire solution $U(x, t)$ of (1.1) such that $U(\cdot, 0) = \varphi$, and this solution satisfies $U(\cdot, t) \in \omega(u)$ for all $t \in \mathbb{R}$. This is how entire solutions are relevant for our problem. The assumption $u_0 \in \mathcal{V}$ poses some restrictions on the structure of entire solutions that can possibly be contained in $\omega(u)$. Using these structural properties in combination with the chain recurrence property of $\omega(u)$, we were able to prove in [18], assuming that $\theta^{\pm} = \theta_0$ and $f(\theta_0) = 0 > f'(\theta_0)$, that all the entire solutions in $\omega(u)$ are necessarily steady states. To prove the same in the present case, $\theta^{\pm} = \theta_0$ and $f(\theta_0) = 0 < f'(\theta_0)$, assuming (NC), we need to consider a class of entire solution not covered by the analysis of [18] (see Section 2.2 below for more details on this). We prove a classification result for such entire solution (see Proposition 5.1), after which a general conclusion from [18] becomes applicable and we obtain our quasiconvergent result.

The rest of the paper is organized as follows. In the next section, we define the concepts of a chain of steady states of (1.1) and spatial trajectories of solutions of (1.1). We use these concepts to state a proposition which has Theorem 1.1 as a corollary. The proposition is then proved in Sections 4 and 5. In the preliminary Section 3, we recall several technical results from earlier papers, and discuss the basic properties of α and ω-limit sets and the zero number.

Below, it will be convenient to assume the following additional condition on the nonlinearity:

(MF) f is globally Lipschitz and there is $\kappa > 0$ such that for all s with $|s| > \kappa$ one has $f(s) = s/2$.

Since this condition concerns the behavior of $f(u)$ for large values of $|u|$, it can be assumed with no loss of generality. Indeed, our quasiconvergence theorem deals with an individual bounded solution, thus modifying f outside the range of this solution has no effect on the validity of the theorem.

Conditions (ND), (MF), are our *standing hypotheses on f*. With no loss of generality, shifting f if necessary, we will also assume that θ_0 in Theorem 1.1 is equal to zero. Thus, we henceforth also assume that

$$f(0) = 0, \quad f'(0) > 0. \tag{1.5}$$

2 Spatial trajectories and chains

As in [18], we employ a geometric technique involving spatial trajectories of solutions of (1.1). Our analysis consists mainly in the examination of how spatial trajectories of entire solutions of (1.1) are related to chains of the planar system corresponding to the equation for the steady states of (1.1):

$$u_{xx} + f(u) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

We define the concept of a chain in the next subsection, after recalling some basic properties of the planar trajectories of (2.1). Spatial trajectories of solutions of entire solutions of (1.1) are defined in Subsection 2.2. In that subsection, we state a result concerning entire solutions which implies Theorem 1.1.

2.1 Steady states of (1.1) and chains

Consider the planar system

$$u_x = v, \qquad v_x = -f(u), \tag{2.2}$$

associated with equation (2.1).

It is a Hamiltonian system with respect to the energy

$$H(u, v) = \frac{v^2}{2} + F(u), \tag{2.3}$$

where $F(u) = \int_0^u f(s)\, ds$. Thus, each orbit of (2.2) is contained in a level set of H. The level sets are symmetric with respect to the $u-$axis, and our extra hypothesis (MF) implies that they are all bounded. Therefore, all orbits of (2.2) are bounded and there are only four types of them: equilibria (all of which are on the $u-$axis), nonstationary periodic orbits (by which we mean orbits of nonstationary periodic solutions), homoclinic orbits, and heteroclinic orbits. Following a common terminology, we say that a solution φ of (2.1) is a *ground state at level* γ if the corresponding solution (φ, φ_x) of (2.2) is homoclinic to the equilibrium $(\gamma, 0)$; we say that φ is *a standing wave of* (1.1) *connecting* γ_- *and* γ_+ if (φ, φ_x) is a heteroclinic solution of (2.2) with limit equilibria $(\gamma_-, 0)$ and $(\gamma_+, 0)$.

Each nonstationary periodic orbit $\mathcal{O}$ is symmetric about the $u-$axis and for some $p < q$ one has

$$\begin{aligned} &\mathcal{O} \cap \{(u,0) : u \in \mathbb{R}\} = \{(p,0), (q,0)\}\,, \\ &\mathcal{O} \cap \{(u,v) : v > 0\} = \left\{ \left(u, \sqrt{2(F(p) - F(u))}\right) : u \in (p,q) \right\}. \end{aligned} \tag{2.4}$$

Let

$$\begin{aligned} \mathcal{E} &:= \{(a,0) : f(a) = 0\} \quad \text{(the set of all equilibria of (2.2))}, \\ \mathcal{P}_0 &:= \{(a,b) \in \mathbb{R}^2 : (a,b) \text{ lies on a nonstationary periodic orbit of (2.2)}\}, \\ \mathcal{P} &:= \mathcal{P}_0 \cup \mathcal{E} \quad \text{(the union of all periodic orbits of (2.2), including the equilibria)}. \end{aligned}$$

The following lemma is the same as [18, Lemma 2.1], which, except for the last two statements in (i), was originally proved in [15, Lemma 3.1]. It gives a description of the phase plane portrait of (2.2) without the nonstationary periodic orbits.

Lemma 2.1. *The following two statements are valid.*

(i) Let Σ be a connected component of $\mathbb{R}^2 \setminus \mathcal{P}_0$. Then Σ is a compact set contained in a level set of the Hamiltonian H and one has

$$\Sigma = \left\{ (u,v) \in \mathbb{R}^2 : u \in J,\ v = \pm\sqrt{2(c - F(u))} \right\}$$

where c is the value of H on Σ and $J = [p,q]$ for some $p, q \in \mathbb{R}$ with $p \le q$. Moreover, if $(u,0) \in \Sigma$ and $p < u < q$, then $(u,0)$ is an equilibrium. If $p < q$, the points $(p,0)$ and $(q,0)$ lie on homoclinic orbits. If $p = q$, then $\Sigma = \{(p,0)\}$, and p is an unstable equilibrium of the equation $\dot{\xi} = f(\xi)$.

(ii) Each connected component of the set $\mathbb{R}^2 \setminus \mathcal{P}$ consists of a single orbit of (2.2)*, either a homoclinic orbit or a heteroclinic orbit.*

We define a *chain* as any connected component of the set $\mathbb{R}^2 \setminus \mathcal{P}_0$. Each chain consists of equilibria, homoclinic orbits, and, possibly, heteroclinic orbits of (2.2). We say that a chain is *trivial* if it consists of a single equilibrium. By a *loop* we mean a set $\Lambda \subset \mathbb{R}^2$ which is either the union of a homoclinic orbit and its limit equilibrium or the union of two heteroclinic orbits, one reflection of the other around the u axis, and their common limit equilibria. Obviously, every loop Λ is contained in a chain and it can be viewed as a Jordan curve in $\mathbb{R}^2$. We denote by $\mathcal{I}(\Lambda)$ the interior of Λ (the bounded connected component of $\mathbb{R}^2 \setminus \Lambda$). Similarly we define $\mathcal{I}(\mathcal{O})$ when $\mathcal{O}$ is a nonstationary periodic orbit of (2.2). If Σ is a chain, $\mathcal{I}(\Sigma)$ denotes the union of the interiors of the loops contained in Σ. We also define $\overline{\mathcal{I}}(\Sigma) = \mathcal{I}(\Sigma) \cup \Sigma$. The set $\overline{\mathcal{I}}(\Sigma)$ is closed and equal to the closure of $\mathcal{I}(\Sigma)$, except when Σ consists of a single point, in which case $\overline{\mathcal{I}}(\Sigma) = \Sigma$. For a nonstationary periodic orbit $\mathcal{O}$ of (2.2), $\overline{\mathcal{I}}(\mathcal{O})$ denotes the closure of $\mathcal{I}(\mathcal{O})$.

The following lemma introduces the inner chain and the outer loop associated with a connected component of $\mathcal{P}_0$ (see Figure 1). The lemma is identical with [18, Lemma 2.2].

Lemma 2.2. *Let Π be any connected component of $\mathcal{P}_0$. The following statements hold true.*

(i) The set Π is open.

(ii) There exists a unique chain Σ_{in} such that for all periodic orbits $\mathcal{O} \subset \Pi$ one has

$$\overline{\mathcal{I}}\,(\Sigma_{in}) \subset \mathcal{I}(\mathcal{O}) \text{ and } \mathcal{I}(\mathcal{O}) \setminus \overline{\mathcal{I}}(\Sigma_{in}) \subset \Pi.$$

(iii) If Π is bounded, there exists a unique loop Λ_{out} such that for all periodic orbits $\mathcal{O} \subset \Pi$ one has

$$\overline{\mathcal{I}}(\mathcal{O}) \subset \mathcal{I}(\Lambda_{out}), \text{ and } \mathcal{I}(\Lambda_{out}) \setminus \overline{\mathcal{I}}(\mathcal{O}) \subset \Pi.$$

(iv) There is a zero β of f such that $f'(\beta) > 0$ and $(\beta, 0) \in \mathcal{I}(\mathcal{O})$, for all periodic orbits $\mathcal{O} \subset \Pi$.

(v) If $\mathcal{O}_1, \mathcal{O}_2$ are two distinct periodic orbits contained in Π, then either $\mathcal{O}_1 \subset \mathcal{I}\,(\mathcal{O}_2)$ or $\mathcal{O}_2 \subset \mathcal{I}\,(\mathcal{O}_1)$ (thus, Π is totally ordered by this relation).

We refer to Σ_{in} and Λ_{out} as the *inner chain and outer loop* associated with Π. If the correspondence to Π is to be explicitly indicated, we denote them by $\Sigma_{in}(\Pi)$ and $\Lambda_{out}(\Pi)$, respectively.

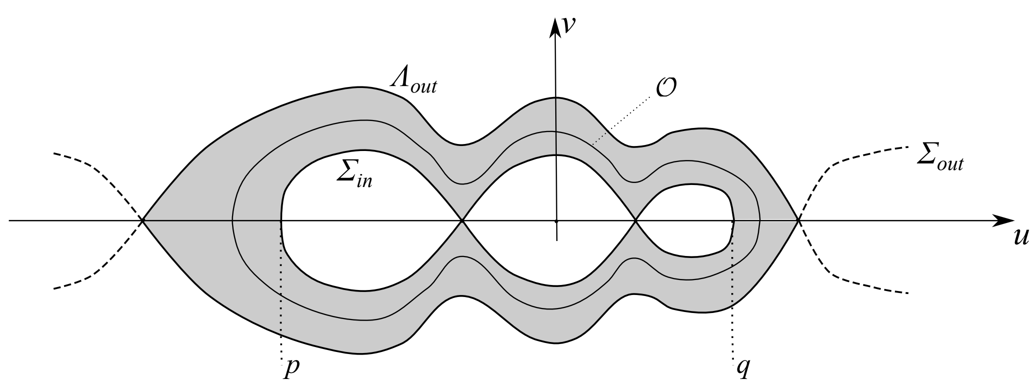

Figure 1: The inner chain and outer loop associated with a connected component Π of $\mathcal{P}_0$: Λ_{out} and Σ_{in} form the boundary of Π. The outer loop can be a heteroclinic loop (as in this figure) or a homoclinic loop, and it is part of a chain Σ_{out}. The points p and q are as in Lemma 2.1 for $\Sigma = \Sigma_{in}$.

Below, the connected component of $\mathcal{P}_0$ whose closure contains $(0, 0)$ will play a prominent role. We denote it by Π_0. Note that Π_0 is well defined, for $f'(0) > 0$ implies that $(0, 0)$ is a center for (2.2), which is to say that it has a neighborhood foliated by periodic orbits.

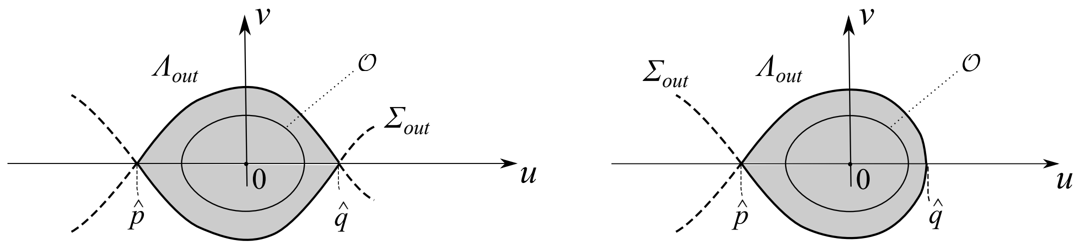

Figure 2: The shaded region indicates the connected component Π_0 containing the point $(0, 0)$. The corresponding outer loop Λ_{out} is a heteroclinic loop in the figure on the left and a homoclinic loop in the figure of the right. Points $\hat{p}$ and $\hat{q}$ indicate the intersections of Λ_{out} with the u-axis. The inner chain is trivial: $\Sigma_{in} = \{(0, 0)\}$.

2.2 A key result on spatial trajectories of entire solutions

In this subsection, we introduce spatial trajectories of entire solutions of (1.1). As we explain, Theorem 1.1 follows from a result on entire solutions stated in Proposition 2.3 below.

For any $\varphi \in C^1(\mathbb{R})$, we define

$$\tau(\varphi) := \{(\varphi(x), \varphi_x(x)) : x \in \mathbb{R}\} \tag{2.5}$$

and refer to this set as the *spatial trajectory (or orbit)* of φ. Note that if φ is a steady state of (1.1), then $\tau(\varphi)$ is the usual trajectory of the solution (φ, φ_x) of the planar system (2.2). If U is an entire solution of (1.1); we refer to the collection $\tau(U(\cdot, t))$, $t \in \mathbb{R}$, as the spatial trajectories of U.

If $Y \subset C^1(\mathbb{R})$, $\tau(Y) \subset \mathbb{R}^2$ is the union of the spatial trajectories of the functions in Y:

$$\tau(Y) := \{(\varphi(x), \varphi_x(x)) : x \in \mathbb{R},\, \varphi \in Y\}\,. \tag{2.6}$$

Assume now that $u_0 \in C_b(\mathbb{R})$, $u_0(\pm\infty) = 0$ (recalling that relations (1.5) are assumed to hold), and the solution u of (1.1), (1.2) is bounded. For a description of $\omega(u)$, some results relating the spatial trajectories of entire solutions of (1.1) to the chains of (2.2) are crucial. If one can prove that, for an entire solution U, the spatial trajectories $\tau(U(\cdot, t))$, $t \in \mathbb{R}$, are all contained in a chain, then a unique-continuation type result shows that U is a steady state of (1.1) (see Lemma 3.4 below). Thus, the quasiconvergence of u can be proved by showing that $\tau(\omega(u))$ is contained in a chain. We now explain a key idea of how this can be accomplished.

Let us first scrutinize the possibility that for some entire solution U with $U(\cdot, t) \in \omega(u)$ a spatial trajectory $\tau(U(\cdot, t_0))$ is not contained in any chain for some $t_0 \in \mathbb{R}$. It was proved in [18, Proposition 3.2] that then none of the trajectories $\tau(U(\cdot, t))$, $t \in \mathbb{R}$, can intersect any chain. This clearly implies that there is a connected component Π of $\mathcal{P}_0$ such that

$$\bigcup_{t\in\mathbb{R}} \tau\left(U(\cdot, t)\right) \subset \Pi. \tag{2.7}$$

The connected component Π has to be bounded as also shown in [18].

Trying to rule (2.7) out, we look for a contradiction. We consider the ω and α-limit sets of the entire solution U, denoted by $\omega(U)$, $\alpha(U)$, respectively; $\omega(U)$ is defined as in (1.4) and the definition of $\alpha(U)$ is analogous, with $t_n \to \infty$ replaced by $t_n \to -\infty$. Take the inner chain $\Sigma_{in}(\Pi)$ and the outer loop $\Lambda_{out}(\Pi)$ associated with Π, as in Lemma 2.2. As in [18, Section 6], a contradiction is obtained if the following relations can be derived from (2.7):

$$\tau\left(\alpha(U)\right) \subset \Sigma_{in}(\Pi), \qquad \tau\left(\omega(U)\right) \subset \Lambda_{out}(\Pi). \tag{2.8}$$

The reason why (2.8) leads to a contradiction can intuitively be explained as follows. Relations (2.8) show that there is a specific "direction" of the flow of (1.1) in $\omega(u)$: $U(\cdot,t)$ always goes *from the inner chain to the outer loop* as t increases from $-\infty$ to ∞. However, the existence of such a flow direction is inconsistent with well-known chain-recurrence properties of the ω-limit sets and thus the contradiction (see [18, Section 6] for details).

Under the assumptions (ND), (MF), and (1.5), it has been proved in [18] that relations (2.8) do follow from (2.7) for any connected component Π of $\mathcal{P}_0$, with the notable exception of $\Pi = \Pi_0$. Recall that Π_0 is the connected component whose closure contains $(0,0)$; in this case, $\Sigma_{in}(\Pi_0)$ is the trivial chain $\{(0,0)\}$. As noted in [18, Remark 6.3], the lack of (2.8) in the case $\Pi = \Pi_0$ was the only reason why we could not give a general quasiconvergence theorem in the case $u_0(\pm\infty) = 0$ and $f(0) = 0 < f'(0)$. In the present paper, we provide the proof of (2.8) in the case $\Pi = \Pi_0$ and thereby prove Theorem 1.1.

For reference, we state here the result which implies Theorem 1.1, as explained above.

Proposition 2.3. *Assuming* (ND), (MF), *and* (1.5), *let* $u_0 \in \mathcal{V}$ *be a function satisfying* $u_0(\pm\infty) = 0$ *such that the solution* u *of* (1.1), (1.2) *is bounded and* (NC) *holds. Let* U *be an entire solution of* (1.1) *such that* $U(\cdot,t) \in \omega(u)$ *for all* $t \in \mathbb{R}$. *If* (2.7) *holds with* $\Pi = \Pi_0$, *then*

$$\alpha(U) = \{0\}, \qquad \tau\left(\omega(U)\right) \subset \Lambda_{out}(\Pi_0). \tag{2.9}$$

Of course, $\alpha(U) = \{0\}$ is equivalent to $\tau\left(\alpha(U)\right) = \{(0,0)\}$, so (2.8) and (2.9) are the same statements when $\Pi = \Pi_0$ (and $\Sigma_{in}(\Pi_0) = \{(0,0)\}$).

The case $\Pi = \Pi_0$ differs from the case when $\Sigma_{in}(\Pi)$ is a nontrivial chain in several aspects. One key difference is that we need to take into account the possibility that the spatial limits $U(\pm\infty,t)$ of the entire solution U depend on t (this can be ruled out easily if $\Sigma_{in}(\Pi)$ is a nontrivial chain, see [18, Lemma 3.9]). Even when $U(\pm\infty,t)$ are independent of t, the case when one or both of them is equal to 0, an unstable equilibrium of $\dot{\xi} = f(\xi)$, is not encountered in the case $\Pi \neq \Pi_0$ (the limits $U(\pm\infty,t)$ are always equal to a stable equilibrium of $\dot{\xi} = f(\xi)$ if $\Sigma_{in}(\Pi)$ is a nontrivial chain). On the other hand, assumption (NC) has some consequences on the structure of relevant entire solutions (see Section 4), which we exploit in the proof of (2.8).

3 Preliminaries

In this section, we first recall basic properties of the zero-number functional and various limit sets of bounded solutions of (1.1) and then state some results from earlier paper that will be referred to in the proof of Proposition 2.3.

3.1 Zero number

Consider a linear parabolic equation

$$v_t = v_{xx} + c(x,t)v, \qquad x \in \mathbb{R},\ t \in (s,T)\,, \tag{3.1}$$

where $-\infty \leq s < T \leq \infty$ and c is a bounded measurable function. Note that whenever u, $\bar{u}$ are bounded solutions of (1.1), their difference $v = u - \bar{u}$ satisfies (3.1) with a suitable function c. Similarly, $v = u_x$ and $v = u_t$ are solutions of such a linear equation.

For an interval $I = (a,b)$, with $-\infty \leq a < b \leq \infty$, we denote by $z_I(v(\cdot,t))$ the number, possibly infinite, of zeros $x \in I$ (counted without their multiplicities) of the function $x \mapsto v(x,t)$.

If $I = \mathbb{R}$ we usually omit the subscript $\mathbb{R}$:

$$z(v(\cdot, t)) := z_{\mathbb{R}}(v(\cdot, t)).$$

The following intersection-comparison principle holds (see [1, 3]).

Lemma 3.1. *Let v be a nontrivial solution of* (3.1) *and $I = (a, b)$, with $-\infty \le a < b \le \infty$. Assume that the following conditions are satisfied:*

- *if $b < \infty$, then $v(b, t) \neq 0$ for all $t \in (s, T)$,*
- *if $a > -\infty$, then $v(a, t) \neq 0$ for all $t \in (s, T)$.*

Then the following statements hold true.

(i) For each $t \in (s, T)$, all zeros of $v(\cdot, t)$ are isolated. In particular, if I is bounded, then $z_I(v(\cdot, t)) < \infty$ for all $t \in (s, T)$.

(ii) The function $t \mapsto z_I(v(\cdot, t))$ is monotone nonincreasing on (s, T) with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.

(iii) If for some $t_0 \in (s, T)$ the function $v(\cdot, t_0)$ has a multiple zero in I and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in (s, T)$ with $t_1 < t_0 < t_2$, one has

$$z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_0)) \ge z_I(v(\cdot, t_2)). \tag{3.2}$$

If (3.2) holds, we say that $z_I(v(\cdot, t))$ drops at t_0.

We will also use a version of Lemma 3.1 for time-dependent intervals; it is derived easily from Lemma 3.1 (cp. [2, Section 2]).

Lemma 3.2. *Let v be a nontrivial solution of* (3.1) *and $I(t) = (a(t), b(t))$, where $-\infty \le a(t) < b(t) \le \infty$ for $t \in (s, T)$. Assume that the following conditions are satisfied:*

(c1) *Either $b \equiv \infty$ or b is a (finite) continuous function on (s,T). In the latter case, $v(b(t), t) \neq 0$ for all $t \in (s, T)$.*

(c2) *Either $a \equiv -\infty$ or a is a continuous function on (s,T). In the latter case, $v(a(t), t) \neq 0$ for all $t \in (s, T)$.*

Then statements (i), (ii) of Lemma 3.1 are valid with I, a, b replaced by $I(t)$, $a(t)$, $b(t)$, respectively; and statement (iii) of Lemma 3.1 is valid with all occurrences of $z_I(v(\cdot, t_j))$, $j = 0, 1, 2$, replaced by $z_{I(t_j)}(v(\cdot, t_j))$, $j = 0, 1, 2$, respectively.

The next lemma is a robustness result of [5].

Lemma 3.3. *Let $w_n(x, t)$ be a sequence of functions converging to $w(x, t)$ in $C^1(I \times (s, T))$ where I is an open interval. Assume that $w(x, t)$ solves a linear equation* (3.1)*, $w \not\equiv 0$, and $w(\cdot, t)$ has a multiple zero $x_0 \in I$ for some $t_0 \in (s, T)$. Then there exist sequences $x_n \to x_0$, $t_n \to t_0$ such that for all sufficiently large n the function $w_n(\cdot, t_n)$ has a multiple zero at x_n.*

3.2 Limit sets and entire solutions

The $\omega-$limit set of a bounded solution u of (1.1) is defined as in (1.4), with the convergence in $L^\infty_{loc}(\mathbb{R})$. As already noted above, it is a nonempty, compact, and connected set in $L^\infty_{loc}(\mathbb{R})$. It is also well known that $\omega(u)$ has the following invariance property: for any $\varphi \in \omega(u)$, there is an entire solution $U(x,t)$ of (1.1) such that

$$U(\cdot,0)=\varphi, \qquad U(\cdot,t)\in\omega(u) \quad (t\in\mathbb{R}). \tag{3.3}$$

In fact, if a sequence $t_n \to \infty$ is such that $u(\cdot,t_n)\to\varphi$ in $L^\infty_{loc}(\mathbb{R})$, then, for a subsequence, we have $u(\cdot,t_n+\cdot)\to U$ in $C^1_{loc}(\mathbb{R}^2)$, where U is an entire solution of (1.1) satisfying (3.3). This follows by compactness arguments based on parabolic estimates (see [18, Section 3.2] for more details.) Note that the entire solution satisfying $U(\cdot,0)=\varphi$ is uniquely determined by φ; this follows from the uniqueness and backward uniqueness for the Cauchy problem (1.1), (1.2).

The above considerations also imply that $\omega(u)$ is unaffected if the convergence in (1.4) is taken in $C^1_{loc}(\mathbb{R})$, rather than in $L^\infty_{loc}(\mathbb{R})$, and therefore $\omega(u)$ is connected in $C^1_{loc}(\mathbb{R})$ as well. Hence, the set

$$\tau\left(\omega(u)\right)=\{(\varphi(x),\varphi_x(x)):\varphi\in\omega(u),x\in\mathbb{R}\}=\bigcup_{\varphi\in\omega(u)}\tau(\varphi)$$

is connected in $\mathbb{R}^2$. (Here, $\tau(\varphi)$ is as in (2.5).) Also, obviously, $\tau(\varphi)$ is connected in $\mathbb{R}^2$ for all $\varphi\in\omega(u)$.

If U is a bounded entire solution of (1.1), we define its $\alpha-$limit set by

$$\alpha(U):=\{\varphi\in C_b(\mathbb{R}):U(\cdot,t_n)\to\varphi \text{ for some sequence } t_n\to-\infty\}\,. \tag{3.4}$$

Here, again, the convergence is in $L^\infty_{loc}(\mathbb{R})$, but due to parabolic regularity, it can be taken in $C^1_{loc}(\mathbb{R})$ with no effect on $\alpha(U)$. The α-limit set has similar properties as the $\omega-$limit set: it is nonempty, compact and connected in $L^\infty_{loc}(\mathbb{R})$ as well as in $C^1_{loc}(\mathbb{R})$, and for any $\varphi\in\alpha(U)$ there is an entire solution $\tilde U$ such that $\tilde U(\cdot,0)=\varphi$ and $\tilde U(\cdot,t)\in\alpha(U)$ for all $t\in\mathbb{R}$. The connectivity property of $\alpha(U)$ implies that the set

$$\tau\left(\alpha(U)\right)=\{(\varphi(x),\varphi_x(x)):\varphi\in\alpha(U),x\in\mathbb{R}\}=\bigcup_{\varphi\in\alpha(U)}\tau(\varphi)$$

is connected in $\mathbb{R}^2$.

For a bounded entire solution U of (1.1), we define generalized notions of α and ω-limit sets as follows:

$$\Omega(U):=\{\varphi\in C_b(\mathbb{R}):U(\cdot+x_n,t_n)\to\varphi \text{ for some sequences } x_n\in\mathbb{R},\ t_n\to\infty\}\,, \tag{3.5}$$

$$A(U):=\{\varphi\in C_b(\mathbb{R}):U(\cdot+x_n,t_n)\to\varphi \text{ for some sequences } x_n\in\mathbb{R},\ t_n\to-\infty\}\,. \tag{3.6}$$

The convergence can be taken in $L^\infty_{loc}(\mathbb{R})$ or $C^1_{loc}(\mathbb{R})$ without altering the sets $\Omega(U)$, $A(U)$. Both these sets are nonempty, compact and connected in $C^1_{loc}(\mathbb{R})$, and they have a similar invariance property as $\omega(U)$, $\alpha(U)$. Also, by their definitions, the sets $\Omega(U)$, $A(U)$ are translation invariant as well. Further, the definitions and parabolic regularity imply that the sets

$$\tau\left(A(U)\right)=\bigcup_{\varphi\in A(U)}\tau(\varphi), \quad \tau\left(\Omega(U)\right)=\bigcup_{\varphi\in\Omega(U)}\tau(\varphi)$$

are connected and compact in $\mathbb{R}^2$. We remark that the sets $\tau(\omega(u))$, $\tau(\alpha(u))$ are both connected (as noted above), but they are not necessarily compact in $\mathbb{R}^2$.

3.3 Further technical results

Throughout this subsection, we assume that u_0 is in $C_b(\mathbb{R})$ (not necessarily in $\mathcal{V}$), u is the solution of (1.1), (1.2) and it is bounded.

In view of the invariance property of $\omega(u)$ (see (3.3)), the following lemma gives a criterion for an element $\varphi \in \omega(u)$ to be a steady state. This unique-continuation type result is proved in a more general form in [24, Lemma 6.10].

Lemma 3.4. *Let $\varphi := U(\cdot, 0)$, where U is a solution of* (1.1) *defined on a time interval $(-\delta, \delta)$ with $\delta > 0$ (this holds in particular if $\varphi \in \omega(u)$). If $\tau(\varphi) \subset \Sigma$ for some chain Σ, then φ is a steady state of* (1.1).

As already noted above, it is proved in [11] (see also [12]) that the ω-limit set of any bounded solution of (1.1) contains a steady state. We will use this result for entire solutions:

Theorem 3.5. *If U is a bounded entire solution of* (1.1)*, then each of the sets $\omega(U)$ and $\alpha(U)$ contains a steady state of* (1.1).

The result concerning the α-limit set follows from the result of the ω-limit set via compactness and invariance properties of $\alpha(U)$: taking an entire solution $\tilde{U}$ with $\tilde{U}(\cdot, t) \in \alpha(U)$ for all t, we have $\omega(\tilde{U}) \subset \alpha(U)$ and $\omega(\tilde{U})$ contains a steady state.

The following lemma is essentially the same as [18, Lemma 2.11] (see also [2, Proof of Proposition 2.1]). The only difference is that in [18, Lemma 2.11], θ is a constant whereas here we allow $\theta = \theta(t)$ to continuously depend on t. This makes just a notational difference in the proof given in [18, Lemma 2.11].

Lemma 3.6. *Let U be a solution of* (1.1) *on $\mathbb{R} \times J$, where $J \subset \mathbb{R}$ is an open time interval, and let $\theta(t)$ be a continuous real function on J. Assume that for each $t \in J$ the function $U(\cdot, t) - \theta(t)$ has at least one zero and*

$$\xi(t) := \sup\{x : U(x, t) = \theta(t)\}$$

is finite and depends continuously on $t \in J$. Then, for any $t_0, t_1 \in J$ satisfying the relations $t_1 > t_0$ and $\xi(t_1) < \xi(t_0)$, the function $U_x(\cdot, t_1)$ is of constant sign on the interval $(\xi(t_1), \xi(t_0)]$. If $J = (-\infty, b)$ for some $-\infty < b \leq \infty$ and $\limsup_{t \to -\infty} \xi(t) = \infty$, then U_x is of constant sign on $(\xi(t), \infty)$, for all $t \in J$.

Analogous statements hold for $\xi(t) = \inf\{x : U(x, t) = \theta(t)\}$.

We next state a quasiconvergence result from [17] (cp. [18, Theorem 2.12]). For any $\lambda \in \mathbb{R}$, consider the function $V_\lambda u$ defined by

$$V_\lambda u(x, t) = u(2\lambda - x, t) - u(x, t), \quad x \in \mathbb{R},\, t \geq 0. \tag{3.7}$$

Theorem 3.7. *Assume that $u_0 \in \mathcal{V}$ and one of the following conditions holds:*

(i) $u_0(-\infty) \neq u_0(\infty)$,

(ii) there is $t > 0$ such that for all $\lambda \in \mathbb{R}$, one has $z(V_\lambda u(\cdot, t)) < \infty$.

Then, u is quasiconvergent. Moreover, $\omega(u)$ does not contain any nonconstant periodic function.

The following result is the same as [18, Lemma 2.13]. It is a variant of the Squeezing Lemma from [23].

Lemma 3.8. *Let U be a bounded entire solution of* (1.1) *such that if $\beta \in f^{-1}\{0\}$ is an unstable equilibrium of the equation $\dot{\xi} = f(\xi)$, then*

$$z\left(U(\cdot,t) - \beta\right) \leq N \quad (t \in \mathbb{R}) \tag{3.8}$$

for some $N < \infty$. Let K be any one of the following subsets of $\mathbb{R}^2$:

$$\bigcup_{t\in\mathbb{R}} \tau\left(U(\cdot,t)\right), \quad \tau\left(\omega(U)\right), \quad \tau\left(\Omega(U)\right), \quad \tau\left(\alpha(U)\right), \quad \tau\left(A(U)\right).$$

Assume that $\mathcal{O}$ is a nonstationary periodic orbit of (2.2) *such that one of the following inclusions holds:*

$$\text{(i)} \quad K \subset \mathcal{I}(\mathcal{O}), \qquad \text{(ii)} \quad K \subset \mathbb{R}^2 \setminus \overline{\mathcal{I}}(\mathcal{O}).$$

Let Π be the connected component of $\mathcal{P}_0$ containing $\mathcal{O}$. If (i) holds, then $K \subset \overline{\mathcal{I}}\left(\Sigma_{in}(\Pi)\right)$; and if (ii) holds, then $K \subset \mathbb{R}^2 \setminus \mathcal{I}\left(\Lambda_{out}(\Pi)\right)$ (in particular, Π is necessarily bounded in this case).

Finally, we recall the following well known result concerning the solutions in $\mathcal{V}$ (the proof can be found in [27, Theorem 5.5.2], for example).

Lemma 3.9. *Assume that $u_0 \in \mathcal{V}$. Then the limits*

$$\theta_-(t) := \lim_{x\to-\infty} u(x,t), \qquad \theta_+(t) := \lim_{x\to\infty} u(x,t) \tag{3.9}$$

exist for all $t > 0$ and are solutions of the following initial-value problems:

$$\dot{\theta}_\pm = f(\theta_\pm), \qquad \theta_\pm(0) = u_0(\pm\infty). \tag{3.10}$$

4 Entire solutions in $\omega(u)$

Throughout this section, we assume—in addition to the standing hypotheses (ND), (MF), and (1.5)— that $u_0 \in \mathcal{V}$, $u_0(\pm\infty) = 0$, and the solution of (1.1), (1.2) is bounded and satisfies (NC). We reserve the symbol $u(x,t)$ for this fixed solution.

In the following lemma, we derive some consequences of the assumption (NC) concerning entire solutions $U(\cdot,t)$ in $\omega(u)$.

Lemma 4.1. *Let U be an entire solution of* (1.1) *such that $U(\cdot,t) \in \omega(u)$ for all $t \in \mathbb{R}$. Then either $U_x \equiv 0$ or U has the following properties:*

- *(i) Each of the functions $U(\cdot,t)$, $U_x(\cdot,t)$ has only finitely many zeros, all of them simple, and the number of these zeros is bounded by a constant independent of t.*
- *(ii) For each $t \in \mathbb{R}$, the following limits exist:*

$$\Theta_-(t) := \lim_{x\to-\infty} U(x,t), \quad \Theta_+(t) := \lim_{x\to\infty} U(x,t). \tag{4.1}$$

- *(iii) For each $t \in \mathbb{R}$, the spatial trajectory $\tau(U(\cdot,t))$ is a simple curve in $\mathbb{R}^2$, that is, it has no self-intersections.*
- *(iv) If* (2.7) *holds with $\Pi = \Pi_0$, then, for any $t \in \mathbb{R}$, the function $U(\cdot,t)$ has no positive local minima and no negative local maxima.*

Proof. Assume that $U_x \not\equiv 0$.

We know (cp. Section 3.2) that for some sequence $t_n \to \infty$ we have

$$u(\cdot, \cdot + t_n) \to U, \quad u_x(\cdot, \cdot + t_n) \to U_x \tag{4.2}$$

with the convergence in $L^\infty_{loc}(\mathbb{R}^2)$ in both cases. Moreover, since f is Lipschitz, the function u_x is bounded in $C^{1+\alpha}(\mathbb{R} \times [1, +\infty))$ for some $\alpha \in (0, 1)$. Therefore, possibly after $\{t_n\}$ is replaced by a subsequence, the second convergence in (4.2) takes place in $C^1_{loc}(\mathbb{R}^2)$ as well.

We now prove that all zeros of $U_x(\cdot, t)$ are simple. Suppose for a contradiction that x_0 is multiple zero of $U_x(\cdot, t_0)$ for some t_0. It then follows from (4.2) and Lemma 3.3, that there is a sequence $\tau_n \to 0$ such that $u_x(\cdot, \cdot + t_n + \tau_n)$ has a multiple zero. Thus, by Lemma 3.1, the zero number $z(u_x(\cdot, t))$ drops at $t = t_n + \tau_n$. Since $t_n + \tau_n \to \infty$, the monotonicity of the zero number gives $z(u_x(\cdot, t)) = \infty$ for all $t > 0$, in contradiction to (NC). The contradiction proves that the zeros of $U_x(\cdot, t)$ are indeed simple for any $t \in \mathbb{R}$.

From (4.2) and (NC), it now follows that $z(U_x(\cdot, t))$ is bounded by a constant independent of t. Consequently, by the mean value theorem, $z(U(\cdot, t))$ is also bounded by a constant independent of t. The proof of statement (i) is complete.

Statement (i) implies that for each t the (bounded) function $U(\cdot, t)$ is monotone near $\pm\infty$. This gives (ii).

To prove statement (iii), we go by contradiction. Suppose that for some $t_0 \in \mathbb{R}$ the curve $\tau(U(\cdot, t_0))$ is not simple: there are $x_0, \eta \in \mathbb{R}$ with $\eta \neq 0$, such that

$$(U(x_0, t_0), U_x(x_0, t_0)) = (U(x_0 + \eta, t_0), U_x(x_0 + \eta, t_0)). \tag{4.3}$$

Consider the function $v(x, t) := U(x + \eta, t) - U(x, t)$. It is a solution of a linear parabolic equation (3.1), and (4.3) means that x_0 is a multiple zero of $v(\cdot, t_0)$. By (4.2) and Lemma 3.3, there is a sequence $t_n \to \infty$ such that the function $u(\cdot + \eta, t_n) - u(\cdot, t_n)$ has a multiple zero (near x_0). Therefore, the same arguments as the ones used above for the function u_x show that for all $t > 0$ one has $z(u(\cdot + \eta, t) - u(\cdot, t)) = \infty$. By Lemma 3.1, all zeros of $u(\cdot + \eta, t) - u(\cdot, t)$ are isolated for $t > 0$. Therefore, given any $t > 0$, there is a sequence x_n with $|x_n| \to \infty$ such that $u(x_n + \eta, t) - u(x_n, t) = 0$. Now, for each n, the function $u(\cdot, t)$ has a critical point between x_n and $x_n + \eta$, so it has infinitely many critical points. This contradiction to condition (NC) proves statement (iii).

For statement (iv), we refer the reader to [18, Lemma 3.11(ii)]. □

5 Entire solutions with spatial trajectories in Π_0

In this section, we investigate entire solutions U of (1.1) such that

(c0)
$$\bigcup_{t \in \mathbb{R}} \tau\left(U(\cdot, t)\right) \subset \Pi_0. \tag{5.1}$$

Moreover, we will assume that if $U_x \not\equiv 0$, then U satisfies the following conditions:

(ci) There is an integer m such that

$$z(U(\cdot, t)),\ z(U_x(\cdot, t)) \leq m \quad (t \in \mathbb{R}) \tag{5.2}$$

and all zeros of $U(\cdot, t)$ and $U_x(\cdot, t)$ are simple.

(cii) The following limits exist

$$\Theta_-(t) := \lim_{x\to-\infty} U(x,t), \quad \Theta_+(t) := \lim_{x\to\infty} U(x,t). \tag{5.3}$$

(ciii) For each $t \in \mathbb{R}$, the spatial trajectory $\tau(U(\cdot,t))$ is a simple curve in $\mathbb{R}^2$ (it has no self-intersections).

(civ) For any $t \in \mathbb{R}$, the function $U(\cdot,t)$ has no positive local minima and no negative local maxima.

As shown in Lemma 4.1, if the entire solution U satisfying (c0) comes from an ω-limit set: $U(\cdot,t) \in \omega(u)$, where u is as in Section 4, then either $U_x \equiv 0$ or (ci)–(civ) hold. Therefore the following proposition, which is the main result of this section, implies Proposition 2.3 (and thereby Theorem 1.1).

Proposition 5.1. *Let U be an entire solution satisfying* (c0) *such that either $U_x \equiv 0$ or conditions* (ci)–(civ) *hold. Then*

$$\alpha(U) = \{0\}, \qquad \tau\left(\omega(U)\right) \subset \Lambda_{out}(\Pi_0). \tag{5.4}$$

Note that, by Lemma 3.4, the second inclusion means that $\omega(U)$ consists of steady states of (1.1) whose trajectories are contained in $\Lambda_{out}(\Pi_0)$. Thus the proposition says that any entire solution U with the indicated properties is a connection, in $L^\infty_{loc}(\mathbb{R})$, from 0 to a set of steady states with trajectories in the outer loop. In the process of proof of the proposition, we will make this conclusion more precise in some cases.

Although our main purpose of investigating entire solutions U satisfying the assumptions of Proposition 5.1 is to prove Theorem 1.1 concerning the bounded solution u of (1.1), below we make no further reference to the solution u and just examine the entire solutions satisfying conditions (ci)–(civ). Thus, Proposition 5.1 can also be viewed as a statement concerning a class of entire solutions with some additional properties, regardless of whether they belong to an ω-limit set or not. Such a classification result for entire solutions may be of independent interest.

Let us first of all dispose of the trivial case $U_x \equiv 0$. In this case, U, being independent of x, is a solution of $\dot{\xi} = f(\xi)$. Condition (c0) implies that U is nonconstant and it connects the unstable equilibrium 0 to an equilibrium ζ with $(\zeta, 0) \in \Lambda_{out}$. Thus in this case, (5.4) is proved.

In the remainder of this section, we assume that U is an entire solution of (1.1) satisfying conditions (c0)–(civ) and $U_x \not\equiv 0$. We continue assuming the standing hypotheses (ND), (MF), and (1.5) on f, and simplify the notation letting

$$\Lambda_{out} := \Lambda_{out}(\Pi_0).$$

There are two possibilities in regard to the structure of Λ_{out} (cp. Figure 2 in Section 2):

(A1) Λ_{out} is a *homoclinic loop,* that is, it is the union of a homoclinic orbit of (2.2) and its limit equilibrium, or, in other words,

$$\Lambda_{out} = \{(\gamma, 0)\} \bigcup \tau(\Phi), \tag{5.5}$$

where $f(\gamma) = 0$ and Φ is a ground state of (2.1) at level γ. We choose Φ so that $\Phi'(0) = 0$, that is, the only critical point of Φ is $x = 0$.

(A2) Λ_{out} is a *heteroclinic loop,* that is, it is the union of two heteroclinic orbits of (2.2) and their limit equilibria $(\gamma_\pm, 0)$. In other words,

$$\Lambda_{out} = \{(\gamma_-, 0), (\gamma_+, 0)\} \bigcup \tau(\Phi^+) \bigcup \tau(\Phi^-), \tag{5.6}$$

with $\gamma_- < \gamma_+$, $f(\gamma_\pm) = 0$, and $\Phi^\pm$ are standing waves of (2.1) connecting γ_- and γ_+, one increasing the other one decreasing. We adopt the convention that $\Phi^+_x > 0$ and $\Phi^-_x < 0$.

To have a unified notation, we set

$$\begin{aligned} \hat{p} &:= \inf\{a \in \mathbb{R} : (a, 0) \in \Pi_0\} = \inf\{a \in \mathbb{R} : (a, 0) \in \Lambda_{out}\}, \\ \hat{q} &:= \sup\{a \in \mathbb{R} : (a, 0) \in \Pi_0\} = \sup\{a \in \mathbb{R} : (a, 0) \in \Lambda_{out}\}. \end{aligned} \tag{5.7}$$

Thus, $\{\hat{p}, \hat{q}\} = \{\gamma, \Phi(0)\}$ if (A1) holds; and $\hat{p} = \gamma_-$, $\hat{q} = \gamma_+$ if (A2) holds.

Note that if $\bar{\gamma}$ is any of the constants γ, $\gamma_\pm$ in (A1), (A2), then $f'(\bar{\gamma}) < 0$. Indeed, $f'(\bar{\gamma}) = 0$ is not allowed by (ND), and $f'(\bar{\gamma}) > 0$ would imply that $(\bar{\gamma}, 0)$ is a center for (2.2) and thus cannot be the limit equilibrium for any homoclinic or heteroclinic orbit.

As for the limits (5.3), since they are solutions of the ordinary differential equation $\dot{\xi} = f(\xi)$, if $\Theta(t)$ stands for $\Theta_-(t)$ or $\Theta_+(t)$, then there are two possibilities: either $\Theta(t) =: \Theta$ is independent of t and $f(\Theta) = 0$, or else it is a strictly monotone solution. Due to (5.1), in the former case $(\Theta, 0)$ equals $(0, 0)$ or it is an element of Λ_{out}, and in the latter case $\Theta(t) \neq 0$ for all t and $\Theta(-\infty) = 0$, $(\Theta(\infty), 0) \in \Lambda_{out}$. We distinguish the following three cases:

(T1) $\Theta_\pm(t) \neq 0$ for all t.

(T2) $\Theta_\pm \equiv 0$.

(T3) $\Theta_+ \equiv 0$ and $\Theta_-(t) \neq 0$ for all t; or $\Theta_- \equiv 0$ and $\Theta_+(t) \neq 0$ for all t.

We treat these cases in separate subsections, proving (5.4) (and sometimes more) in each of them. The next subsection contains some general lemmas that apply to all three cases.

5.1 Some general lemmas

The following two lemmas show basic relations of $U(\cdot, t)$ to $\{(0, 0)\}$ and Λ_{out}. They are special cases (with $\Sigma_{in} = \{(0, 0)\}$) of [18, Lemmas 4.7 and 4.8]. Remember that we are assuming that that U satisfies conditions $U_x \not\equiv 0$ and (c0)–(civ) (in particular U is not a steady state of (1.1)).

Lemma 5.2. *Let K be any one of the sets $\{(0, 0)\}$, Λ_{out}. Then the following statements are valid.*

(i) If (x_n, t_n), $n = 1, 2, \ldots$, is a sequence in $\mathbb{R}^2$ such that

$$\operatorname{dist}((U(x_n, t_n), U_x(x_n, t_n)), K) \to 0, \tag{5.8}$$

then, possibly after passing to a subsequence, one has $U(\cdot + x_n, \cdot + t_n) \to \varphi$ in $C^1_{loc}(\mathbb{R}^2)$, where φ is a steady state of (1.1) with $\tau(\varphi) \subset K$.

(ii) There exists a sequence (x_n, t_n), $n = 1, 2, \ldots$ as in (i) with the additional property that $|t_n| \to \infty$. Consequently, there exists a steady state of (1.1) with $\tau(\varphi) \subset K$ and

$$\varphi \in A(U) \cup \Omega(U). \tag{5.9}$$

In the previous lemma, $\Omega(U)$, $A(U)$ are the generalized limit sets of U, as defined in (3.5), (3.6). Statement (i) of the lemma delineates a way spatial trajectories of U can get arbitrarily close to one of the sets $\{(0,0)\}$, Λ_{out}, and statement (ii) shows that the spatial trajectories of U cannot stay away from from both $\{(0,0)\}$ and Λ_{out} as $|t| \to \infty$. In the next lemma, we consider the case when the spatial trajectories stay away from one of the points $(\hat{p}, 0), (\hat{q}, 0) \in \Lambda_{out}$ (see (5.7) for the definition of $\hat{p}$, $\hat{q}$).

Lemma 5.3. *The following statements are valid.*

(i) If $U \leq \hat{q} - \vartheta$ for some $\vartheta > 0$, then $\omega(U) = \{\hat{p}\}$ (so, necessarily, $f(\hat{p}) = 0$) and $\alpha(U) = \{0\}$. Similarly, if $U \geq \hat{p} + \vartheta$ for some $\vartheta > 0$, then $\omega(U) = \{\hat{q}\}$ (so $f(\hat{q}) = 0$) and $\alpha(U) = \{0\}$.

(ii) If for some $t_0 \in \mathbb{R}$ and $\vartheta > 0$ one has $U(\cdot, t) \leq \hat{q} - \vartheta$ for all $t < t_0$, then $\alpha(U) = \{0\}$. If for some $t_0 \in \mathbb{R}$ and $\vartheta > 0$ one has $U(\cdot, t) \geq \hat{p} + \vartheta$ for all $t < t_0$, then $\alpha(U) = \{0\}$.

We are making an intentional duplicity in this lemma by including the conclusion $\alpha(U) = \{0\}$ in statement (i), although this conclusion is also contained in statement (ii) (which has weaker assumptions). This will allow us to make more straightforward references to one of these statements.

The next lemma gives other sufficient conditions for $\alpha(U) = \{0\}$. It may be useful to reiterate at this point that $\alpha(U)$ and $\omega(U)$ are considered with respect to the locally uniform convergence.

Lemma 5.4. *If one of the following conditions (a1)–(a4) is satisfied, then $\alpha(U) = \{0\}$.*

(a1) $\Theta_+ \not\equiv \hat{p}$ (so $\Theta_+ > \hat{p}$) and there is $m \in \mathbb{R}$ such that

$$\limsup_{x \in [m,\infty),\, t \to -\infty} U(x,t) \ \left(= \limsup_{t \to -\infty} \sup_{x \in [m,\infty),} U(x,t)\right) \leq 0. \tag{5.10}$$

(a2) $\Theta_+ \not\equiv \hat{q}$ and there is $m \in \mathbb{R}$ such that $\liminf_{x \in [m,\infty),\, t \to -\infty} U(x,t) \geq 0$.

(a3) $\Theta_- \not\equiv \hat{p}$ and and there is $m \in \mathbb{R}$ such that $\limsup_{x \in (-\infty,m),\, t \to -\infty} U(x,t) \leq 0$.

(a4) $\Theta_- \not\equiv \hat{q}$ and and there is $m \in \mathbb{R}$ such that $\limsup_{x \in (-\infty,m),\, t \to -\infty} U(x,t) \geq 0$.

Proof. We prove the conclusion assuming (a1) holds, all the other cases are analogous. Note that (5.10) in particular implies that

$$\varphi(x) \leq 0 \quad (x \in (m, \infty),\ \varphi \in \alpha(U)). \tag{5.11}$$

To start with, we claim that no nonzero steady state of (1.1) can belong to $\alpha(U)$. To show this, it is sufficient—because of (c0)—to exclude from $\alpha(U)$ all steady states φ with $\tau(\varphi) \subset \Lambda_{out}$ (note that no nonconstant periodic steady state can be contained in $\alpha(U)$ by (ci)). Relation (5.11) excludes the shifts of the increasing standing wave Φ_+ (if Λ_{out} a heteroclinic loop as in (A2)), the constant $\hat{q}$ (if $f(\hat{q}) = 0$), and the shift of the ground state Φ in the case (A1) with $\gamma = \Phi(\pm\infty) = \hat{q}$. It remains to exclude the constant $\hat{p}$, the shifts of the ground state Φ in the case (A1) with $\gamma = \hat{p}$, and the shifts of the decreasing standing wave Φ_- in the case (A2). The proofs for all these use very similar comparisons arguments involving periodic steady states, so we give the details for just one of them, say for the ground state Φ when $\gamma = \hat{p}$ (and $\Phi(0) = \hat{q}$).

Assume for a contradiction that a shift $\tilde{\Phi}$ of Φ is contained in $\alpha(U)$. Hence, there is sequence $t_n \to -\infty$ such that $U(\cdot, t_n) \to \tilde{\Phi}$ in $L^\infty_{loc}(\mathbb{R})$. Let x_0 stand for the larger of the two zeros of $\tilde{\Phi}$.

Clearly, (5.11) implies that $x_0 \le m$ and $\tilde{\Phi} < 0$ on (m, ∞). Pick any $\epsilon \in (0, \hat{q})$. Then, by (5.10), there is $t_0 \in \mathbb{R}$ such that

$$U(x,t) < \epsilon \quad (x \ge m,\ t \le t_0). \tag{5.12}$$

For any $\nu \in [\epsilon, \hat{q})$, let ψ be any periodic solution of (2.1) with $\tau(\psi) \subset \Pi_0$ and $\psi(m) = \psi(m+\rho) = \nu$, $\rho > 0$ being the minimal period of ψ. Then, $\psi(m) = \psi(m+\rho) > 0 \ge \tilde{\Phi}(m) > \tilde{\Phi}(m+\rho)$; and, in fact, $\tilde{\Phi} < \psi$ on $(m, m+\rho)$ (otherwise, a right shift of the graph of ψ would be touching the graph of $\tilde{\Phi}$, which is impossible for two distinct solutions of (2.1)). Consequently, if n is large enough, we have $t_n < t_0 - 1$ and

$$U(x, t_n) < \psi(x) \quad (x \in (m, m+\rho)).$$

Since, by (5.12), we also have

$$U(m,t) < \psi(m) \;\text{ and }\; U(m+\rho, t) < \psi(m+\rho) \quad (t < t_0),$$

applying the comparison principle on the domain $(m, m+\rho) \times (t_n, t_0)$, we obtain

$$U(x, t_0) < \psi(x) \quad (x \in (m, m+\rho)).$$

This is true for all periodic solutions ψ with the indicated properties. We can choose a sequence of such periodic solutions converging locally uniformly to another shift $\bar{\Phi} := \Phi(\cdot - m)$ of the ground state Φ (by continuity with respect to the initial data, the periods ρ of these periodic solutions go to infinity). This implies that $U(x, t_0) \le \bar{\Phi}(x)$ for all $x > m$. In particular, $\Theta_+(t_0) = U(\infty, t_0) = \hat{p}$, in contradiction to the assumption on $\Theta_+(t_0)$. Our claim is proved.

We now prove that $\alpha(U) = \{0\}$. Take any $\varphi \in \alpha(U)$. Let $\tilde{U}$ be the entire solution of (1.1) with $\tilde{U}(\cdot, 0) = \varphi$ and $\tilde{U}(\cdot, t) \in \alpha(U)$ for all $t \in \mathbb{R}$. By (5.11), $\tilde{U}(\cdot, t) \le 0$ in (m, ∞) for all $t \in \mathbb{R}$. We go by contradiction: if $\varphi \not\equiv 0$, the strong comparison principle implies that $\tilde{U}(\cdot, t) < 0$ on (m, ∞) for all $t > 0$. Now, assumption (1.5) and the Hamiltonian structure of system (2.2) imply that $(0,0)$ is a center for (2.2): a neighborhood of $(0,0)$ is foliated by periodic orbits. We can thus choose a sequence ψ_n of nonconstant periodic solutions of (2.1), with their minimal periods bounded from above by a constant $\rho_0 > 0$, such that $\max|\psi_n| \to 0$. Pick any $x_0 > m$ and denote

$$s := \max\{\tilde{U}(x, 1) : x \in [x_0, x_0 + \rho_0]\} < 0.$$

If n is sufficiently large, then a suitable shift ψ of ψ_n satisfies

$$\psi(x_0) = \psi(x_1) = 0 \text{ and } s < \psi(x) < 0 \quad (x \in (x_0, x_1)),$$

for some $x_1 \in (x_0, x_0 + \rho_0)$. Then $\tilde{U}(\cdot, 1) \le \psi$ in (x_0, x_1) and $\tilde{U}(x_j, t) \le 0 = \psi(x_j)$, $j = 0, 1$, for all $t \ge 1$. Therefore, the comparison principle gives $\tilde{U}(\cdot, t) \le \psi$ on (x_0, x_1) for all $t > 1$. This and Theorem 3.5 imply that $\omega(\tilde{U})$, which is a subset of $\alpha(U)$ by compactness of $\alpha(U)$, contains a nonzero steady state, in contradiction to the above claim. We have thus shown that $\varphi \not\equiv 0$ is impossible, therefore $\alpha(U) = \{0\}$. □

5.2 Case (T1): $\Theta_\pm(t) \neq 0$ for all t

In this subsection, we assume that the entire solution U, fixed as above, satisfies the relations $\Theta_\pm(t) \neq 0$ for all t.

We first show that $\tau(\Omega(U)) \subset \Lambda_{out}$ (note that this is stronger than the needed conclusion $\tau(\omega(U)) \subset \Lambda_{out}$). In some cases, we even establish the existence of a limit

$$\phi = \lim_{t \to \infty} U(\cdot, t) \text{ in } L^\infty(\mathbb{R}). \tag{5.13}$$

Lemma 5.5. *The following statements are valid.*

(i) If Θ_-, Θ_+ have opposite signs, then necessarily Λ_{out} is a heteroclinic loop as in (A2)*, and the limit ϕ in* (5.13) *exists and is equal to a standing wave – a shift of Φ^+ or Φ^-.*

(ii) If the signs of Θ_-, Θ_+ are equal and Λ_{out} is a heteroclinic loop as in (A2)*, then the limit ϕ in* (5.13) *exists and is equal to one of the constants γ_-, γ_+.*

(iii) If the signs of Θ_-, Θ_+ are equal and Λ_{out} is a homoclinic loop as in (A1)*, then the following statements hold:*

(a) If the functions Θ_-, Θ_+ are constant (so they are both identical to the constant γ), then the limit ϕ in (5.13) *exists and is equal to the constant γ or a shift of the ground state Φ.*

(b) If one (or both) of the functions functions Θ_-, Θ_+ is nonconstant, then $\tau(\Omega(U)) \subset \Lambda_{out}$.

In particular, in all cases, we have $\tau(\omega(U)) \subset \Lambda_{out}$.

Remark 5.6. The convergence conclusion as in statement (iii)(a) is likely valid in (iii)(b) as well. However, a result on threshold solutions from [15] that we are using in the proof of (iii)(a) does not seem to be available in general when Θ_-, Θ_+ are nonconstant.

Proof of Lemma 5.5. Recall that the functions Θ_-, Θ_+ are solutions of the equation $\dot{\xi} = f(\xi)$. If they have opposite signs, as assumed in statement (i), then (whether they are constants or not) their limits

$$\theta_+ = \lim_{t\to\infty} \Theta_+(t), \quad \theta_- = \lim_{t\to\infty} \Theta_-(t) \tag{5.14}$$

are nonzero constants, both stable equilibria of $\dot{\xi} = f(\xi)$. Clearly, they have opposite signs, hence, in view of (c0), necessarily $\{\theta_-, \theta_+\} = \{\gamma_-, \gamma_+\}$, with $\gamma_\pm$ as in (A2). Thus U is a front-like solution in the sense that U takes values between γ_-, γ_+, and one of its spatial limits $U(\pm\infty, t)$ is in the interval $(\gamma_-, 0)$ while the other one is in $(0, \gamma_+)$. Since $f'(\gamma_\pm) < 0$, the convergence result in (i) is contained in [9, Theorem 3.1].

Under the assumptions of (ii), the limits θ_-, θ_+ of $\Theta_-(t)$, $\Theta_+(t)$ as $t \to \infty$ are equal to the same constant, either γ_- or γ_+. In this case, the convergence result stated in (ii) is also known and can easily be derived from [9, Theorem 3.1] using a comparison function (see [19, Proof of Lemma 3.4], for example).

Let now Λ_{out} be a homoclinic loop as in (A1). For definiteness, we assume $\Phi(0) > \gamma$ (the ground state Φ is above γ). The case $\Phi(0) < \gamma$ is analogous. Thus, in the notation introduced in (5.7), $\hat{p} = \gamma$ and $\hat{q} = \Phi(0)$.

Assume first that $\Theta_- \equiv \Theta_+ \equiv \gamma$. Since $(\gamma, \hat{q}]$ is the range of the ground state Φ, $F < F(\gamma)$ in $(\gamma, \hat{q}]$ (here, $F(u) = \int_0^u f(s)\,ds$ is as in (2.3)). This is the setup of [15, Theorem 2.5] whose conclusion, translated to the present notation, is the same as the conclusion in (iii)(a).

It remains to prove statement (iii)(b). To that end, we first claim that $(0,0) \notin \tau(\Omega(U))$. Observe that this claim implies the desired inclusion $\tau(\Omega(U)) \subset \Lambda_{out}$. Indeed, since $\tau(\Omega(U))$ is a compact subset of $\mathbb{R}^2$, our claim implies that there is a neighborhood of $(0,0)$ disjoint from $\tau(\Omega(U))$. Consequently, since the equilibrium $(0,0)$ is a center for (2.2), there is a nonconstant periodic orbit $\mathcal{O}$ of (2.2) in this neighborhood (with $(0,0) \in \mathcal{I}(\mathcal{O})$). We then have $\tau(\Omega(U)) \subset \mathbb{R}^2 \setminus \overline{\mathcal{I}}(\mathcal{O})$, and a direct application of Lemma 3.8 (taking (5.1) into account) gives $\tau(\Omega(U)) \subset \Lambda_{out}$.

We prove our claim by contradiction. Assume $(0,0) \in \tau(\Omega(U))$. This means that $0 \in \Omega(U)$, and hence there is a sequence (x_n, t_n) with $t_n \to \infty$ such that $U(\cdot + x_n, t_n) \to 0$ in $L^\infty_{loc}(\mathbb{R})$. Pick a nonconstant periodic solution ψ of (2.1) with $\tau(\psi) \subset \Pi_0$, so that $(0,0) \in \mathcal{I}(\tau(\psi))$ and $\hat{p} < \min \psi < \max \psi < \hat{q}$. Then ψ has infinitely many zeros and therefore $\psi - U(\cdot, t_n)$ has infinitely many zeros for all large enough n. On the other hand, each of the values $\Theta_\pm(t) = U(\pm\infty, t)$ is either equal to γ or, if nonconstant, approaches γ as $t \to \infty$. So both $\Theta_\pm(t)$ are outside the interval $[\min \psi, \max \psi]$ for large t. Therefore, by Lemma 3.1, $z(U(\cdot, t) - \psi)$ is finite and bounded for large t. We have obtained a contradiction, which proves our claim. □

The following lemma completes the proof of Proposition 5.1 in the case (T1).

Lemma 5.7. $\alpha(U) = \{0\}$.

For the proof of this result, we need the following lemma.

Lemma 5.8. *The following statements are valid:*

(j) The zero number $z(U(\cdot, t))$ is independent of $t \in \mathbb{R}$.

(jj) If, for some $t \in \mathbb{R}$, J is a nodal interval of $U(\cdot, t)$ (that is, $U(\cdot, t)$ is nonzero everywhere in J and vanishes on ∂J), then $U(\cdot, t)$ has at most one critical point in J (the critical point is nondegenerate by (ci)).

Proof. For (j), it is sufficient to prove that $t \mapsto z(U(\cdot, t))$ is locally constant. Since the limits $\Theta_\pm(t) = U(\pm\infty, t)$ are nonzero, given any $T > 0$ we can find $R > 0$, $\epsilon > 0$ such that $U(x, t) \neq 0$ for any $(x, t) \in \mathbb{R}^2 \setminus [-R, R] \times [T - \epsilon, T + \epsilon]$. Since, by (ci), the zeros of $U(\cdot, t)$ are all simple, we obtain that $z(U(\cdot, t))$ is independent of t, for $t \in [T - \epsilon, T + \epsilon]$.

Statement (jj) follows from statement (j) and conditions (ci), (civ). □

Proof of Lemma 5.7. If the functions $\Theta_\pm$ are both constant (so that statements (i),(ii), and (iii)(a) of Lemma 5.5 apply), the conclusion can be proved by nearly the same arguments as those given in the proof of [18, Lemma 4.10]. We omit the details in this case, just mention the following simple changes that need to made in the proof: take $\beta_- = \beta_+ = 0$, and replace all references to Lemmas 4.4 and 4.8 of [18] by references to Lemmas 5.8 and 5.3 of the present paper, respectively.

We now consider the case when at least one of the functions $\Theta_\pm$ is nonconstant. For definiteness, we assume that Θ_+ is nonconstant. Here, too, there are two analogous possibilities: $\Theta_+(t) \in (\hat{p}, 0)$ for all t and $\Theta_+(t) \in (0, \hat{q})$ for all t. We just consider the former. Note that, since $\Theta_+(t)$ converges to a stable equilibrium of $\dot{\xi} = f(\xi)$ as $t \to \infty$, we necessarily have $f(\hat{p}) = 0$.

If $U < 0$, we get the conclusion $\alpha(U) = \{0\}$ immediately from Lemma 5.3(i). Henceforth assume that $U(\cdot, t)$ has at least one zero for all t and denote by $\eta(t)$ the largest of these zeros. Since we are assuming $\Theta_+(t) \in (\hat{p}, 0)$, $U(\cdot, t) < 0$ in $(\eta(t), \infty)$. Lemma 5.8(j) and condition (ci) imply that $\eta(t)$ is a C^1 function of $t \in \mathbb{R}$. By Lemma 5.8(jj), $U(\cdot, t)$ has at most one critical point in $(\eta(t), \infty)$.

Due to the monotonicity of $t \mapsto z_{(\eta(t),\infty)}(U_x(\cdot, t))$—note that Lemma 3.2 is applicable, as (ci) gives $U_x(\eta(t), t)) \neq 0$—one of the following possibilities occurs:

(p1) There is $t_0 \in \mathbb{R}$ such $U(\cdot, t)$ has unique critical point in $(\eta(t), \infty)$ for all $t < t_0$; this critical point is the global minimizer of $U(\cdot, t)$ in $(\eta(t), \infty)$).

(p2) $U_x(x, t) < 0 \quad (x > \eta(t))$.

If (p1) holds, then, by Lemma 3.6, there is m such $\eta(t) < m$ for all $t \in (-\infty, t_0]$. Since $U(\cdot, t) < 0$ in $(\eta(t), \infty)$, condition (a1) of Lemma 5.4 applies and we obtain $\alpha(U) = \{0\}$.

Assume now that (p2) holds. If $U(\cdot, t) > \Theta_+(t)$ for all t, then, since $\Theta_+(t) \to 0$ as $t \to -\infty$, we obtain the desired conclusion $\alpha(U) = \{0\}$ immediately from Lemma 5.3(ii). Otherwise, $z(U(\cdot, t) - \Theta_+(t)) \geq 1$ for some t, and consequently for all large negative t (note that $U(\cdot, t) - \Theta_+(t)$ is a solution of a linear parabolic equation (3.1) on $\mathbb{R}^2$). The zero number is finite and bounded uniformly in t due the bound on the number of critical points of $U(\cdot, t)$, see (ci). Hence $z(U(\cdot, t) - \Theta_+(t))$ is independent of t for large negative t, say for $t < t_0$; and the zeros of $U(\cdot, t) - \Theta_+(t)$ are then all simple for $t < t_0$. We denote by $\xi(t)$ the largest of these zeros; $\xi(t)$ is a C^1 function of t. Clearly, for any $t < t_0$, we have $U(\cdot, t) - \Theta_+(t) > 0$ on $(\xi(t), \infty)$ and $U(\cdot, t)$ is not monotone in the interval $(\xi(t), \infty)$. Therefore, by Lemma 3.6, there is $m \in \mathbb{R}$ such that $\xi(t) \leq m$ for all $t < t_0$. Using this and the fact that $\Theta_+(t) \to 0$ as $t \to -\infty$, we obtain

$$\liminf_{x \in [m, \infty),\, t \to -\infty} U(x, t) \geq 0. \tag{5.15}$$

Applying Lemma 5.4 , we conclude that $\alpha(U) = \{0\}$ in this case as well. □

5.3 Case (T2): $\Theta_\pm \equiv 0$

Throughout this subsection, we assume that $\Theta_\pm \equiv 0$. Hence, $(U(x, t), U_x(x, t)) \to (0, 0)$ as $x \to \pm\infty$ for all t.

If $U > 0$ or $U < 0$, then the desired conclusion (5.4) follows from Lemma 5.3(i). Henceforth we therefore assume that there is t_0 such that

$$z(U(\cdot, t_0)) \geq 1. \tag{5.16}$$

The following then holds.

Lemma 5.9. *Relation* (5.16) *implies that for all* $t < t_0$ *one has*

$$z(U(\cdot, t)) = 1. \tag{5.17}$$

Proof. By monotonicity, the zero number in (5.17) is at least 1 for all $t < t_0$. We prove that it is exactly 1, or, equivalently, that for any $t < t_0$ the spatial trajectory $\tau(U(\cdot, t))$ intersects the v-axis at exactly one point. We go by contradiction, assume that for some $t < t_0$ the function $U(\cdot, t)$ has at least two zeros and denote by $x_1 < x_2$ the two smallest ones. We also assume that $U(x, t) > 0$ when $x < x_1$; the case $U(x, t) < 0$ when $x < x_1$ being analogous. Clearly, by property (civ) and the fact that $U(x, t) \to 0$ as $x \to -\infty$, $U(\cdot, t)$ has a unique critical point in each of the intervals $(-\infty, x_1)$, (x_1, x_2); we denote them by y_1, y_2, respectively. We have $U_x(\cdot, t) > 0$ on $(-\infty, y_1) \cup (y_2, x_2)$ and $U_x(\cdot, t) < 0$ on (y_1, y_2).

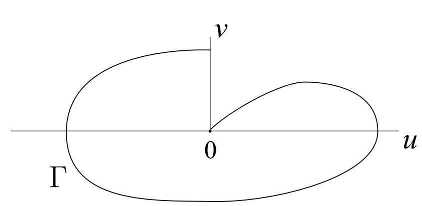

Figure 3: The curve $\Gamma \subset \tau(U(\cdot, t))$ whose existence is ruled out in the proof of Lemma 5.9

Consider now the following curve, a part of the spatial trajectory $\tau(U(\cdot,t))$:

$$\Gamma := \{(U(x,t), U_x(x,t)) : x < x_2\}.$$

By (ciii), Γ is a simple curve. It is the union of the points $(U(y_1,t),0)$, $(U(y_2,t),0)$ on the u-axis, with $U(y_1,t) > 0 > U(y_2,t)$, and the sets $\{(U(x,t),U_x(x,t)) : x \leq y_1\}$, $\{(U(x,t),U_x(x,t)) : y_1 < x < y_2\}$, $\{(U(x,t),U_x(x,t)) : y_2 < x < x_2\}$, which are contained, respectively, in the quadrant $\{(u,v) : u,v > 0\}$, the half-plane $\{(u,v) : v < 0\}$, and the quadrant $\{(u,v) : u < 0 < v\}$ (cp. Figure 3). Since $(U(x,t),U_x(x,t)) \to (0,0)$ as $x \to \pm\infty$, the union of Γ and the closed segment of the v-axis between the points $(0,U_x(x_2,t)$ and $(0,0)$, is a Jordan curve. Using now the facts that $\tau(U(\cdot,t)$ has no self-intersections (cp. (ciii)) and that $U(x,t)$ increases with x when $(U(x,t),U_x(x,t))$ is in the quadrant $\{(u,v) : u,v > 0\}$, we obtain that $(U(x,t),U_x(x,t))$ cannot converge to $(0,0)$ as $x \to \infty$. We have thus found a contradiction, completing the proof. $\square$

Whenever (5.17) holds, conditions (civ), (ci), and (T2) imply that the function $U(\cdot,t)$ has exactly two critical points, both nondegenerate; one of them is the global maximum point of $U(\cdot,t)$, further denoted by $\bar{\xi}(t)$, the other one is the global minimum point of $U(\cdot,t)$, denoted by $\underline{\xi}(t)$ (cp. Figure 4). The unique zero of $U(\cdot,t)$ is denoted by $\xi(t)$; it is between the two critical points.

By the monotonicity of the zero number,

$$z(U(\cdot,t)) \leq 1 \quad (t \in \mathbb{R}). \tag{5.18}$$

If $U(\cdot,t) > 0$ for some t, then $U(\cdot,t)$ has a unique critical point, the point of global maximum; we still denote it by $\bar{\xi}(t)$. If $U(\cdot,t) < 0$ for some t, then the unique critical point of $U(\cdot,t)$ is the point of global minimum, still denoted by $\underline{\xi}(t)$.

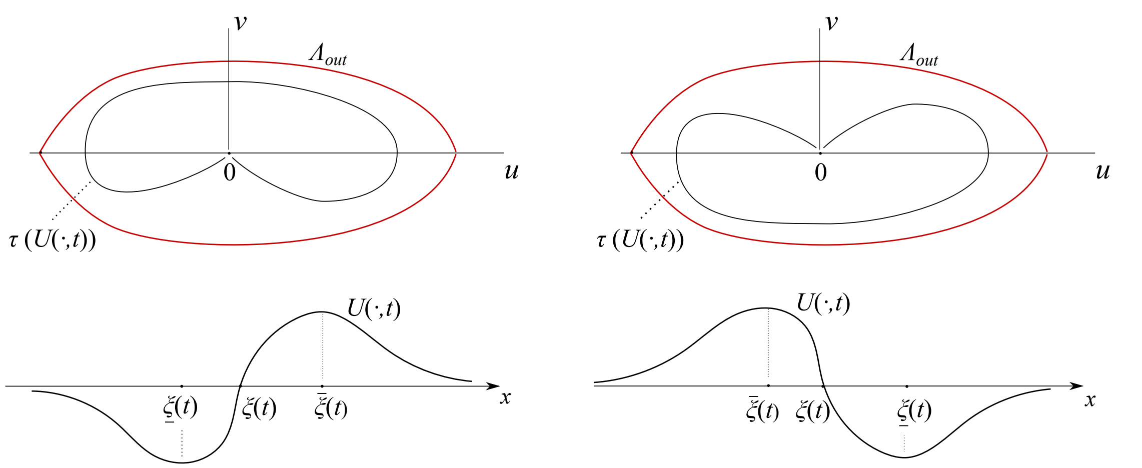

Figure 4: The spatial trajectories (top figures) and graphs of $U(\cdot,t)$ when $U(\cdot,t)$ has a zero $\xi(t)$ (necessarily unique). The figures depict both possibilities $U(\cdot,t) < 0$ in $(-\infty,\xi(t))$ (the figures on the left) and $U(\cdot,t) > 0$ in $(-\infty,\xi(t))$.

We now prove that the desired conclusion holds when Λ_{out} is a homoclinic loop:

Lemma 5.10.
If Λ_{out} is a homoclinic loop as in (A1)*, then* (5.4) *holds:* $\alpha(U)) = \{0\}$, $\tau(\omega(U)) \subset \Lambda_{out}$.

Proof. For definiteness, we assume that $\gamma = \hat{p}$, so Φ is a ground state at level $\hat{p}$ and $\hat{q} = \Phi(0)$ is its maximum; the case $\gamma = \hat{q}$ is similar. We prove that for some $\vartheta > 0$

$$\sup_{x\in\mathbb{R}} U(x,t) < \hat{q} - \vartheta \quad (t \in \mathbb{R}). \tag{5.19}$$

Once this is done, the desired conclusion follows immediately from Lemma 5.3(i).

Assume that (5.19) is not true for any $\vartheta > 0$. Then there is a sequence $t_n \in \mathbb{R}$ such that $\sup_{x\in\mathbb{R}} U(\cdot, t_n) \nearrow \hat{q}$. In particular, $U(\cdot, t_n)$ takes positive values (for large enough n), so its global maximum point $\bar{\xi}(t_n)$ exists. We can therefore take $x_n = \bar{\xi}(t_n)$ and thus have $U_x(x_n, t_n) = 0$. Using Lemma 5.2(i), we obtain that, passing to a subsequence if necessary, $U(\cdot + x_n, t_n) \to \Phi$ in $C^1_{loc}(\mathbb{R})$. But this implies that for large n the function $U(\cdot+, t_n)$ has at least two zeros, a contradiction to (5.18). Thus (5.19) holds and the proof is complete. $\square$

Next, we treat the case when Λ_{out} is a heteroclinic loop.

Lemma 5.11. *Assume that Λ_{out} is a heteroclinic loop as in* (A2)*. Then* (5.4) *holds.*

Proof. We first show that

$$A(U) = \{0\}. \tag{5.20}$$

Note that this conclusion—stronger than the needed $\alpha(U) = \{0\}$—is equivalent to the convergence $U(\cdot, t) \to 0$ in $L^\infty(\mathbb{R})$ (not just in $L^\infty_{loc}(\mathbb{R})$) as $t \to -\infty$.

With $\bar{\xi}(t)$, $\underline{\xi}(t)$, $\xi(t)$ as above (cp. Figure 4), and with both $\bar{\xi}(t)$ and $\underline{\xi}(t)$ defined for $t < t_0$, we assume that

$$U(x,t) > 0 \quad (x \in (-\infty, \xi(t)),\, t \in \mathbb{R}), \qquad U(x,t) < 0 \quad (x \in (\xi(t), \infty),\, t \in \mathbb{R}). \tag{5.21}$$

The case with the reversed inequalities is analogous.

It is sufficient to prove that the constants $\gamma_\pm$ are not contained in $A(U)$. Indeed, if this holds, then $A(U)$ does not contain any shifts of the standing waves $\Phi_\pm$ either (by compactness and translation invariance of $A(U)$). Consequently, by Lemma 5.2, $\operatorname{dist}(\tau(A(U)), \Lambda_{out}) > 0$, and (5.20) follows upon an application of Lemma 3.8.

Assume, for a contradiction that $\gamma_+ \in A(U)$ (arguments to rule out the possibility $\gamma_- \in A(U)$ are similar and are omitted). Since the function $U(\cdot, t)$ is monotone neither on $(-\infty, \xi(t))$ nor on $(\xi(t), \infty)$, Lemma 3.6 tells us that for some $K > 0$ one has

$$|\xi(t)| < K, \quad (t < t_0). \tag{5.22}$$

From the assumption that $\gamma^+ \in A(U)$ we obtain that there are sequences x_n, t_n, with $t_n \to -\infty$, such that $U_n := U(\cdot + x_n, \cdot + t_n) \to \gamma^+$. This, (5.22), and (5.21) in particular imply that $x_n \to -\infty$.

Let ψ be any periodic solution of (2.1) with $\tau(\psi) \subset \Pi_0$ and $\psi(0) > 0$, $\psi'(0) = 0$. Let $2\rho > 0$ be the minimal period of ψ, so $\psi(0)$ is the maximum of ψ, and $\psi(-\rho) = \psi(\rho) < 0$ is the minimum of ψ. Obviously, for all large enough n, say for all $n > n_0$, we have

$$U(\cdot + x_n, t_n) > \psi \text{ on } [-\rho, \rho].$$

Also, due to (5.22) and the convergence $x_n \to -\infty$, we have, making n_0 larger if necessary,

$$U(\pm\rho + x_n, t) > 0 > \psi(-\rho) = \psi(\rho) \quad (n > n_0,\ t \in (t_n, t_0]).$$

Therefore, by the comparison principle, for $n > n_0$,

$$U(x + x_n, t) > \psi(x) \quad (x \in [-\rho, \rho], \quad t \in (t_n, t_0]).$$

In particular, at $t = t_0$, we obtain

$$\max_{x\in[-\rho,\rho]} U(x + x_n, t_0) \geq \max \psi > 0 \quad (n > n_0).$$

Since $x_n \to -\infty$, we have a contradiction to the relation $U(-\infty, t_0) = \Theta_- = 0$. This contradiction completes the proof of (5.20).

We now prove the second needed conclusion:

$$\tau(\omega(U)) \subset \Lambda_{out}. \tag{5.23}$$

If $U(\cdot, t) > 0$ for some t, then $U(\cdot, t) \to \gamma^+$ as $t \to \infty$ (with the convergence in $L^\infty_{loc}(\mathbb{R})$) by the well known property of solutions with range in the monostable interval $(0, \gamma^+)$. Similarly, if $U(\cdot, t) < 0$ for some t, then $U(\cdot, t) \to \gamma^-$ as $t \to \infty$. In these cases we are done. We proceed assuming that $U(\cdot, t)$ changes sign for all t, that is, the equality holds in (5.18) and $\bar{\xi}(t)$, $\underline{\xi}(t)$, $\xi(t)$ are defined for all $t \in \mathbb{R}$. For definiteness, we again assume that (5.21) holds; the case with the reversed inequalities is analogous. For this part of the proof, we adapt some arguments from [18, Proof of Lemma 4.13].

We claim that the following alternative holds:

$$\tau(\omega(U)) = \{(0, 0)\} \text{ or } \tau(\omega(U)) \subset \Lambda_{out}. \tag{5.24}$$

Indeed, relations (5.21) imply that hypothesis (ii) of Theorem 3.7 holds (namely, by (5.21), $V_\lambda U(x, t) > 0$ for $x \approx \pm\infty$ and the finiteness of the zero number then follows from Lemma 3.1(i)). From Theorem 3.7 we obtain that $\omega(U)$ consists of steady states. It does not contain nonconstant periodic steady states (due to (ci)), so $\tau(\omega(U)) \subset \bar{\Pi}_0 \setminus \mathcal{P}_0$. Since $\tau(\omega(U))$ is connected, (5.24) must hold.

Thus, to complete the proof of (5.23), we just need to rule out the possibility

$$\omega(U) = \{0\}. \tag{5.25}$$

Assume it holds. We derive a contradiction. Pick any $\varepsilon > 0$ with $\epsilon < \min\{-\gamma^-, \gamma^+\}$. Relation (5.25) in particular implies that

$$\text{for any } M > 0 \text{ there is } T = T(M) \text{ such that } -\varepsilon < U(x, t) < \varepsilon \quad (x \in (-M, M),\ t > T(M)). \tag{5.26}$$

By Lemma 5.2(ii), $\Omega(U) \cup A(U)$ contains one of the constants $\gamma_\pm$ (or a shift of one of the standing waves $\Phi_\pm$, and, consequently, also both constants $\gamma_\pm$). Since we have proved that $A(U) = \{0\}$, $\Omega(U)$ must contain one of these constants. We only consider the case $\gamma_+ \in \Omega(U)$, the case $\gamma_- \in \Omega(U)$ being similar. Hence, there is a sequence (x_n, t_n) with $t_n \to \infty$ such that

$$U(\cdot + x_n, t_n) \underset{n\to\infty}{\longrightarrow} \gamma_+, \tag{5.27}$$

with the convergence in $L^\infty_{loc}(\mathbb{R})$. In view of (5.26), we have $|x_n| \to \infty$. We claim that necessarily $x_n \to -\infty$. To show this, first observe that, by (5.21) and the simplicity of the zero $\xi(t)$, we have $U_x(\xi(t),t) < 0$. Using this and Lemma 3.6, we obtain the following monotonicity relations for each $t > 0$:

$$\text{if } \xi(t) > \xi(0), \text{ then } U_x(\cdot,t) < 0 \text{ on } (\xi(0),\xi(t)]\,, \tag{5.28}$$

$$\text{if } \xi(t) < \xi(0), \text{ then } U_x(\cdot,t) < 0 \text{ on } [\xi(t),\xi(0))\,. \tag{5.29}$$

From (5.21) and (5.27), it follows that there is n_1 such that $\xi(t_n) > x_n$ for all $n > n_1$. If for some $n > n_1$ it is also true that $x_n > \xi(0)$, then the relations $\xi(t_n) > x_n > \xi(0)$ and (5.28) give $U(\xi(0),t_n) > U(x_n,t_n)$. This inequality can hold only for finitely many n, due to (5.26), (5.27). Thus for all large enough n we have $x_n \le \xi(0)$. Since $|x_n| \to \infty$, it must be true that $x_n \to -\infty$, as claimed.

Pick now a periodic solution ψ of (2.1) with $\tau(\psi) \subset \Pi_0$ such that $\min\psi < -\varepsilon$ and $\max\psi > \varepsilon$. We shift ψ such that $\max\psi = \psi(0)$. Let $2\rho > 0$ be the minimal period of ψ; so

$$\min\psi = \psi(\pm\rho) < -\varepsilon, \qquad \max\psi = \psi(0) > \varepsilon.$$

By (5.27), for n large enough,

$$U(x_n + x, t_n) > \psi(0) > \varepsilon \quad (x \in (-\rho,\rho)). \tag{5.30}$$

This and (5.21) in particular imply that $\xi(t_n) > x_n + \rho$. We now show that for some large enough n_0, the following must hold in addition to (5.30):

$$U(x_{n_0} \pm \rho, t) > \psi(\pm\rho) \quad (t > t_{n_0}). \tag{5.31}$$

Indeed, if not, then there exists arbitrarily large n such that for some $\tilde{t}_n > t_n$ one has

$$U\left(x_n + \bar{\rho}, \tilde{t}_n\right) = \psi(\bar{\rho}) < -\varepsilon,$$

where $\bar{\rho}$ is either $-\rho$ or ρ. Since $U(\cdot,t) > 0$ on $(-\infty,\xi(t))$, it follows that $\xi\left(\tilde{t}_n\right) < x_n + \bar{\rho}$. But, due to $x_n \to -\infty$, we also have $x_n + \bar{\rho} < \xi(0)$ if n is large enough; so, by (5.29), $U\left(\xi(0),\tilde{t}_n\right) < U\left(x_n + \bar{\rho},\tilde{t}_n\right) < -\varepsilon$. This cannot be true for arbitrarily large n, due to (5.26), so (5.30),(5.31) both hold for some n_0.

Using (5.30), (5.31), and the comparison principle, we obtain $U(x_{n_0},t) > \psi(0) > \varepsilon$ for all $t > t_{n_0}$. This is a contradiction to (5.26).

We have shown that the assumption (5.25) leads to a contradiction, which concludes the proof of Lemma 5.11. □

5.4 Case (T3): $\Theta_+ \equiv 0$ and $\Theta_-(t) \neq 0$ for all t.

Dealing with case (T3), our assumption in this subsection is that $\Theta_+ \equiv 0$ and $\Theta_-(t) \neq 0$ for all t. The other possibility, $\Theta_- \equiv 0$ and $\Theta_+(t) \neq 0$ for all t is analogous. For definiteness, we also assume that $\Theta_- < 0$, hence $\Theta_-(t) \to \hat{p}$ as $t \to \infty$ (which includes the possibility that $\Theta_- \equiv \hat{p}$) and $f(\hat{p}) = 0$; the other possibility, $\Theta_- > 0$ can be treated similarly.

First, we prove the desired conclusion regarding the α-limit set of U.

Lemma 5.12. $\alpha(U) = \{0\}$.

Proof. If $U > 0$ or $U < 0$, then the conclusion follows from Lemma 5.3(i). Henceforth we therefore assume that $z(U(\cdot,t)) \geq 1$ holds for some t and, consequently, for any sufficiently large negative t. By (ci) and the monotonicity of the zero number, $z(U(\cdot,t))$ is finite and independent of t for large negative t and all zeros of $U(\cdot,t)$ are simple. We denote by $\xi(t)$ the largest of these zeros. Since $U(\infty,t) = \Theta_+(t) = 0$, the function $U(\cdot,t)$ is not monotone in $(\xi(t),\infty)$. Therefore, by Lemma 3.6, $\xi(t)$ is bounded from above as $t \to -\infty$. Thus, there are $t_0, m \in \mathbb{R}$ such that either $U(\cdot,t) < 0$ on (m,∞) for all $t < t_0$ or $U(\cdot,t) > 0$ on (m,∞) for all $t < t_0$. In either case, an application of Lemma 5.4 gives $\alpha(U) = \{0\}$. $\square$

We now consider the ω-limit set of U. The following lemma, in conjunction with Lemma 5.12, shows that (5.4) holds in the case (T3).

Lemma 5.13. $\tau(\omega(U)) \subset \Lambda_{out}$.

Proof. By Theorem 3.7 (which applies due to (T3)), U is quasiconvergent, so $\omega(U)$ contains only nonperiodic steady states or constant steady states (nonconstant periodic steady states are excluded by (ci)). By (c0), all $\varphi \in \omega(U)$ satisfy $\tau(\varphi) \in \bar{\Pi}_0$, thus Lemma 5.13 will be proved if we show that $0 \notin \omega(U)$. To this end, we use a similar comparison arguments as in the proof of [18, Lemma 4.14].

If Λ_{out} is a heteroclinic loop, as in (A2), choose an increasing continuous function $\tilde{u}_0$ such that $0 > \tilde{u}_0(-\infty) > \Theta_-(0)$, $\tilde{u}_0(\infty) = \gamma_+$, and $\tilde{u}_0 \geq U(\cdot,0)$. By the comparison principle, the corresponding solution $\tilde{u} = u(\cdot,\cdot,\tilde{u}_0)$ of (1.1) satisfies $\tilde{u}(\cdot,t) > U(\cdot,t)$ for all $t > 0$. By [9, Theorem 3.1], the (front-like) solution $\tilde{u}(\cdot,t)$ converges in $L^\infty(\mathbb{R})$ to a shift of the increasing standing wave Φ^+, say $\Phi^+(\cdot - \eta)$, as $t \to \infty$. This implies that $\varphi \leq \Phi^+(\cdot - \eta)$ for all $\varphi \in \omega(U)$; in particular $0 \notin \omega(U)$.

Assume now that Λ_{out} is a homoclinic loop, as in (A1). We have $\gamma = \hat{p}$ since, as noted above, $\Theta^- < 0$ implies that $f(\hat{p}) = 0$. The ground state Φ satisfies $\Phi > \gamma$, $\Phi(\pm\infty) = \gamma$. To prove that $0 \notin \omega(U)$, we again use a suitable comparison function $\tilde{u}_0$; specifically, a bounded C^1 function $\tilde{u}_0$ with the following properties:

(s1) $\tilde{u}_0$ has a unique critical point x_0 and $\tilde{u}_0(x_0)$ is the global maximum of $\tilde{u}_0$;

(s2) the limits $\tilde{u}_0(-\infty)$, $\tilde{u}_0(\infty)$, satisfy the following relations:

$$\Theta_-(0) < \tilde{u}_0(-\infty) < 0, \quad \tilde{u}_0(\infty) = \gamma;$$

(s3) as $t \to \infty$, the solution $\tilde{u}(\cdot,t) := u(\cdot,t,\tilde{u}_0)$ converges in $L^\infty(\mathbb{R})$ to $\Phi(\cdot - \mu)$, for some $\mu \in \mathbb{R}$.

The existence of such a function $\tilde{u}_0$ is a consequence of [21, Lemma 3.5]. Indeed, in [21, Lemma 3.5], the convergence to a shift $\Phi(\cdot - \mu)$ of the ground state Φ is proved for the solution $u(\cdot,t,g)$ whose initial datum g is given by the following relations (with $\gamma = \Phi(\pm\infty)$ as above)

$$g(x) = \begin{cases} \beta & (x \in (-\infty,-q), \\ \vartheta & (x \in [-q,0]), \\ \gamma & (x \in (0,\infty]), \end{cases} \tag{5.32}$$

where $\beta \in (\gamma,0)$, $\vartheta, q \in (0,\infty)$ are constants; $\beta \in (\gamma,0)$ can be chosen arbitrarily, while ϑ, q have to be selected suitably. We fix any β with $\Theta_-(0) < \beta < 0$ and take the corresponding constants $\vartheta, q > 0$. Then for any sufficiently small $t_0 > 0$, the function $\tilde{u}_0 := u(\cdot,t_0,g)$ is of class C^1 and satisfies (s1) (cp. [21, Remark 3.6]). For the limits at $x = \pm\infty$ we have the following relations: $u(\infty, t, g) = \gamma$ for each $t > 0$ and $u(-\infty, t, g) \to \beta$ as $t \searrow 0$ (the function $t \mapsto u(-\infty, t, g)$ is the solution of the equation $\dot{\xi} = f(\xi)$ with the initial condition $\xi(0) = \beta$, see [21, Lemma 3.3] or [27, Theorem 5.5.2]). Therefore, (s2) holds for any sufficiently small $t_0 > 0$ as well. Finally, since $u(\cdot, t, \tilde{u}_0) = u(\cdot, t_0 + t, g)$ for $t > 0$, condition (s3) is satisfied.

Now, by (T3) and the assumptions made at the beginning of this subsection, the function $U(x, 0)$ satisfies the following relations: $U(x, 0) \to \Theta_-(0)$, $U_x(x, 0) \to 0$ as $x \to -\infty$; $U(x, 0) \to 0$ as $x \to \infty$. If $\Theta_-(0) = \gamma$ (which holds if and only if $\Theta_-(t) = \gamma$ for all t), then also $U_x(x, 0) > 0$ for all sufficiently large negative x. These relations in conjunction with (s1) and (s2) imply that if $\eta > 0$ is sufficiently large, then the function $u_0(\cdot + \eta) - U(\cdot, 0)$ has only one zero and the zero is simple. It follows that $z(\tilde{u}(\cdot + \eta, t) - U(\cdot, t)) \leq 1$ for all $t > 0$. Since $\tilde{u}(\cdot + \eta, t) \to \Phi(\cdot - \mu + \eta)$ as $t \to \infty$, we obtain, taking into account that the difference of any two steady states (1.1) has only simple zeros, that $z(\Phi(\cdot - \mu + \eta) - \varphi) \leq 1$ for each $\varphi \in \omega(U)$. In particular, $0 \notin \omega(U)$. $\square$

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