8571, Evolutionary equations

This course explores the semigroup approach to partial differential equations. The semigroup generated by an operator A (usually an elliptic operator or a matrix with such operators as entries) plays a similar role in the theory of evolution equations as the exponential matrix e^{At} does in the theory of ordinary differential equations: it gives solutions of the initial value problems for the linear homogeneous equations $x_t = Ax$ and is used in the variation of constants formula for the solutions of nonhomogeneous linear equations $x_t = Ax + f(t)$. This then provides a natural framework for an application of a fixed point theorem proving the existence of solutions of nonlinear equations.

The abstract semigroup theory allows one to treat, by a unified approach, standard parabolic and hyperbolic equations—such as those considered in [L.C. Evans, Partial differential equations], for example—along with equation that are not of the standard type. Take for instance a simple nonlocal perturbation of a parabolic equation:

$$u_t(x,t) = \Delta u(x,t) + a(x,t)u + b(x,t)\int_\Omega u(x,t)\,dx, \quad x \in \Omega,\ t > 0,$$

which is meaningful in applications.

Also, the semigroup approach to nonlinear equations is best suited for qualitative analysis of such equations, as in, for example, stability theory and invariant manifolds.

We will discuss basic elements of the theory, including strongly continuous semigroups and more specific analytic semigroups, interpolation spaces, and generation of semigroups by elliptic differential operators. More advanced topics will include optimal regularity results for solutions of linear equations and spectral mapping theorems (relations between the spectra of the semigroup and its generator). Building on the linear theory, we will establish basic existence-uniqueness-regularity properties of solutions of nonlinear equations.

Abstract results will be applied to specific evolution equations of parabolic and hyperbolic types.

The course will then continue with a selection of topics (depending on the available time) from the qualitative theory of evolution equations, such as stability by linear approximation, blowup of solutions, and large-time behavior of solutions.

Prerequisites: Knowledge of basic real analysis (Lebesgue spaces, Banach spaces, bounded linear operators) will be assumed. Some results from complex analysis (holomorphic functions, contour integrals, Cauchy theorem) will be recalled and used. An overview of the spectral theory for closed operators on Banach spaces will be given at the beginning of the course. Knowledge of some PDEs and Sobolev spaces would be beneficial, but is not required.

No textbook is required. Most of the material we will cover is contained in the monographs

A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer, New York 1983.

A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhauser, Basel 1995