**Table of Integration Formulas** Constants of integration have been omitted.

p. 495 in the book

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \dfrac{1}{x}\,dx = \ln|x|$

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \dfrac{a^x}{\ln a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

Correct in the box

$\boxed{\int \cos x\,dx = \sin x + C}$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \csc^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \csc x \cot x\,dx = -\csc x$

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\int \tan x\,dx = \ln|\sec x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sinh x\,dx = \cosh x$

16. $\int \cosh x\,dx = \sinh x$

17. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right), \quad a > 0$

*19. $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$

*20. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Optional (can be derived)

also $= \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

also $= \arcsin\left(\frac{x}{a}\right)$

The Book uses both notations $\sin^{-1}x$, $\arcsin x$ (which both mean the same thing, i.e. inverse sin)

and Likewise $\tan^{-1}x$, $\arctan x$ which both mean inverse tan

As the comment at the top of the page states, the constant of integration is omitted to make the list look more streamlined. But Correct Answer on an Exam/Homework etc. MUST include C: e.g. $\boxed{\int \sec^2 x\,dx = \tan x + C}$ is the correct Answer.

Reference Page 2 (Front of the Book).

## Addition and Subtraction Formulas

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

## Double-Angle Formulas

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

## Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

## Fundamental Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

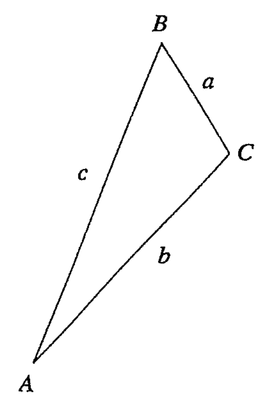
$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

## The Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$




## The Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Wed. 1/22

Integration by Parts.

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$ formula 1, p. 464

$$\boxed{\int u\,dv = uv - \int v\,du}$$ formula 2, p. 464

The connection between the two formulas:

$u = f(x)$, $v = g(x)$

$du = f'(x)dx$, $dv = g'(x)dx$

So

$$\int \overbrace{f(x)}^{u}\,\overbrace{g'(x)\,dx}^{dv} = \overbrace{f(x)}^{u}\,\overbrace{g(x)}^{v} - \int \overbrace{g(x)}^{v}\,\overbrace{f'(x)dx}^{du}$$

hence $\int u\,dv = uv - \int v\,du$

Examples done in the Book usually use the formula 2, so I'll try to follow that practice.

## Examples with ln.

These are examples of the form

$\int f(x)\ln x\,dx$, or even $\int f(x)\ln(x+1)\,dx$,

$\int f(x)\ln(2x+5)\,dx$, $\int f(x)\ln(3x^2+5)\,dx$,

$\int f(x)\ln(\tan x)\,dx$, etc.

Furthermore some examples of the form

$\int f(x)(\ln x)^2\,dx$, $\int f(x)(\ln x)^3\,dx$, etc.

These Examples we try to handle by Integration by Parts where we always set $\boxed{dv = f(x)dx}$, and $\boxed{u = \text{the ln-expression}}$.

There is <u>one Exception</u>: When $f(x)$ is the derivative of the expression inside $\ln(\ )$, i.e. when the integral has the form $\int g'(x)\ln(g(x))\,dx$

$\int g'(x) \ln(g(x))dx$

In this case (of course!) we make the substitution $u = g(x)$, $du = g'(x)dx$, hence we obtain

$$= \boxed{\int \ln u \, du}$$

one of FIRST new things (i.e. integrals) This Semester

This is a VERY BASIC (!) Integral worked in Example 2, p. 465 in the Book (with $x$ in place of $u$). To avoid confusion with $u, v$ in formula 2 (for Integration by Parts), I'll use the formula 1: STEP I ~~INTEGR.~~ BY PARTS

$$\int \ln u \, du := \int \underset{f(u)}{(\ln u)} \cdot \underset{g'(u)}{1} \, du =$$

$$= \underset{f(u)}{(\ln u)} \underset{g(u)}{u} - \int \underset{g(u)}{u} \cdot \underset{f'(u)}{\frac{1}{u}} \, du =$$

$= (\ln u)u - \int 1\,du =$

$= \boxed{u \ln u - u + C} =$ Answer for $\int \ln u\,du$

Thus $\int g'(x) \ln(g(x))\,dx =$

$= \int \ln(g(x))\, g'(x)\,dx$

$= \int (\ln u)\,du$ $\Big\}$ $u = g(x)$, $du = g'(x)\,dx$

$= u \ln u - u + C$

So combination of old (substitution) and new (Integr. by Parts)

$= g(x) \ln g(x) - g(x) + C$ $\quad *$

As we just explained, Example 2, p. 465, which is $\int \ln x\,dx$, is the case $\int f(x) \ln x\,dx$ with $f(x) = 1$.

The following Examples are handled similarly:

$\int (\ln x)^2 dx, \quad \int (\ln x)^3 dx, \quad \int (\ln x)^4 dx$, etc.

$\int \ln(2x+5)\,dx, \quad \int \ln(3x^2+5)\,dx,$

For Example:
Calculate
$\int (\ln x)^3 dx$ :

STEP I (INTEGR. BY PARTS)

$$= \underbrace{\int \underbrace{(\ln x)^3}_{u} \underbrace{1\,dx}_{dv}}$$

$u = (\ln x)^3,$

$du = 3(\ln x)^2 \cdot \frac{1}{x}\,dx$

$dv = 1\,dx$

$v = x$

$$= \underbrace{x}_{v}\underbrace{(\ln x)^3}_{u} - \int \underbrace{x}_{v} \underbrace{\left(3(\ln x)^2 \cdot \frac{1}{x}\right) dx}_{du}$$

$$= x(\ln x)^3 - 3\int (\ln x)^2 dx$$

So we need to calculate

$\int (\ln x)^2 dx$ : $\quad u = (\ln x)^2,$

$du = 2(\ln x)\frac{1}{x}\,dx$

$dv = 1\,dx, \quad v = x$

$$\int \underbrace{(\ln x)^2}_{u} \underbrace{dx}_{dv} = \underbrace{x}_{v} \underbrace{(\ln x)^2}_{u} - \int \underbrace{x}_{v} \underbrace{(2\ln x)\frac{1}{x}dx}_{du}$$

$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$$= x(\ln x)^2 - 2(x\ln x - x) + C$$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

Thus $\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx =$

$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2x\ln x + 2x\right] + C$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C \quad *$$

---

Find

$\int \ln(2x+5)dx$ : STEP I ← INTEGR, BY PARTS:

$1\,dx = dv$,

$v = x$,

$u = \ln(2x+5)$, $du = \dfrac{2}{2x+5}dx$

$$= \underbrace{x}_{v} \underbrace{\ln(2x+5)}_{u} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{2}{2x+5}dx}_{du}$$

Thus

$$\int \ln(2x+5)\,dx =$$

STEP I

$$= \boxed{x\ln(2x+5) - \int \frac{2x}{2x+5}\,dx}$$

Thus we need to evaluate

$$\int \frac{2x}{2x+5}\,dx:$$

STEP II

Use <u>Substitution</u>

$$2x+5 = u$$

$$du = 2\,dx, \quad dx = \tfrac{1}{2}\,du$$

$$2x = u-5$$

$$= \int \frac{u-5}{u}\cdot\frac{1}{2}\,du =$$

$$= \frac{1}{2}\int\left(\frac{u}{u} - \frac{5}{u}\right)du = \frac{1}{2}\int\left(1-\frac{5}{u}\right)du =$$

$$= \frac{1}{2}\left(u - 5\ln|u| + C\right) =$$

$$= \frac{1}{2}\left(2x+5 - 5\ln|2x+5|\right) + C =$$

$$= x + \frac{5}{2} - \frac{5}{2}\ln|2x+5| + C$$

This substitution is familiar from Ch. 5: E.g. Exercises #46, 68, p. 414; #46: $u = 2+x$, #68: $u = 1+2x$

Hence

$$\int \ln(2x+5)\,dx =$$

$$= x\ln(2x+5) - x - \frac{5}{2} + \frac{5}{2}\ln|2x+5| + C$$

Since $\ln(2x+5)$ is defined only when $2x+5>0$, we must have $2x+5>0$, hence we can omit the absolute value sign $|\ |$ in $\ln|2x+5|$.

$*$

From 1st Semester Calculus:

$\int \frac{\ln(\ln x)}{x\ln x}\,dx$ : Since $(\ln(\ln x)) = \frac{1}{x\ln x}$, we make the substitution $u = \ln(\ln x))$, $du = \frac{1}{x\ln x}\,dx$, hence

$$\int \frac{\ln(\ln x)}{x\ln x}\,dx = \int (\ln(\ln x))\frac{1}{x\ln x}\,dx =$$

$$= \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\ln(\ln x))^2 + C$$

$*$

Also from 1st Semester Calculus:

$$\int \frac{\ln(\ln(\ln x))}{x(\ln x)(\ln(\ln x))}\,dx \ :$$

Similar to the preceding Example (bottom p. 8). Answer: $\frac{1}{2}(\ln(\ln(\ln x)))^2 + C$

Combination of Substitution and Integr. by Parts

Find

$$\int \frac{\ln(\ln(\ln x))}{x\ln x}\,dx \ :$$

Now $\dfrac{1}{x\ln x} = (\ln(\ln x))'$

So we make the substitution

$$u = \ln(\ln x), \quad du = \frac{1}{x\ln x}\,dx$$

Hence

$$= \int (\underbrace{\ln(\ln(\ln x))}_{u}) \underbrace{\frac{1}{x\ln x}dx}_{du} =$$

$$= \int \ln u\,du = u\ln u - u + C$$

Example 2, p. 465, or bottom p. 3, top p. 4 these Notes

$$= \overbrace{(\ln(\ln x))}^{u}\overbrace{(\ln(\ln(\ln x)))}^{\ln u} - \ln(\ln x) + C$$

*

Calculate $\int \frac{(\ln x)\ln(\ln x)}{x}\,dx$ (ON YOUR OWN)

Calculate $\int \ln(3x^2+5)\,dx$ :

Again, $\int \ln(3x^2+5)dx = \int \underbrace{(\ln(3x^2+5))}_{u} \underbrace{1dx}_{dv}$

$dv = dx$
$v = x$

$u = \ln(3x^2+5)$

$du = \frac{1}{3x^2+5} \cdot 6x\,dx$

So Integration by Parts gives

$$= \underset{u}{(\ln(3x^2+5))}\,\underset{v}{x} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{3x^2+5} \cdot 6x\,dx}_{du}$$

$$= x\ln(3x^2+5) - \int \frac{6x^2}{3x^2+5}\,dx$$

To find

we use a simple trick (Need to Learn it)

$$\frac{6x^2}{3x^2+5} = \frac{6x^2+10-10}{3x^2+5} = \frac{6x^2+10}{3x^2+5} - \frac{10}{3x^2+5}$$

$= 2 - \frac{10}{3x^2+5}$ ; so $\int \frac{6x^2}{3x^2+5}\,dx =$

$$= \int \left(2 - \frac{10}{3x^2+5}\right)dx = 2x - 10 \cdot \int \frac{1}{3x^2+5}\,dx$$

Next $\int \frac{1}{3x^2+5}\,dx$ : This was learned in 1-st Semester Calculus, as it says at the beginning of Ch. 7 on p. 463:

We use the Formula $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

$$\int \frac{1}{3x^2+5}\,dx = \int \frac{1}{3\left(x^2+\frac{5}{3}\right)}\,dx =$$

$$= \frac{1}{3}\int \frac{1}{x^2+\frac{5}{3}}\,dx = \frac{1}{3}\cdot\frac{1}{\sqrt{\frac{5}{3}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{5}{3}}}\right)+C$$

$$= \frac{1}{\sqrt{15}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{5}}\right)+C$$

Thus continuing from the bottom of p. 10, $\int \frac{6x^2}{3x^2+5}\,dx = 2x - \frac{10}{\sqrt{15}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{5}}\right)+C$

and finally $\int \ln(3x^2+5)\,dx =$

$$x\ln(3x^2+5) - 2x + \frac{10}{\sqrt{15}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{5}}\right)+C$$

<u>Comment.</u> We could also write $\arctan\left(\frac{x\sqrt{3}}{\sqrt{5}}\right)$ instead of $\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{5}}\right)$ *

We now come to Examples of the type at the top of p. 2 — so far we did mostly ones with $f(x) = 1$:

Calculate $\int x^5 (\ln x)^3 dx$

As stated on p. 2, we set $dv = x^5 dx$, $u = (\ln x)^3$, hence $v = \frac{1}{6}x^6$, $du = 3(\ln x)^2 \cdot \frac{1}{x} dx$, and thus

$$\int x^5 (\ln x)^3 dx = \int \underbrace{(\ln x)^3}_{u} \underbrace{x^5 dx}_{dv}$$

$$= \underbrace{\frac{1}{6}x^6}_{v} \underbrace{(\ln x)^3}_{u} - \int \underbrace{\frac{1}{6}x^6}_{v} \cdot \underbrace{3(\ln x)^2 \cdot \frac{1}{x} dx}_{du}$$

$$= \frac{1}{6}x^6 (\ln x)^3 - \frac{1}{2}\int x^5 (\ln x)^2 dx$$

Thus we need to calculate $\int x^5 (\ln x)^2 dx$, which is simpler than the original integral, but essentially of the same kind.

13

Thus
$$\int x^5 (\ln x)^2 dx = \int \underbrace{(\ln x)^2}_{u} \underbrace{x^5 dx}_{dv}$$

$$= \overbrace{\frac{1}{6}x^6}^{v} \overbrace{(\ln x)^2}^{u} - \int \overbrace{\frac{1}{6}x^6}^{v} \underbrace{2(\ln x)\cdot \frac{1}{x} dx}_{du}$$

$$= \frac{1}{6}x^6 (\ln x)^2 - \frac{1}{3}\int x^5 \ln x \, dx$$

Thus we need to calculate $\int x^5 \ln x dx$:

$$\int x^5 \ln x \, dx = \int \underbrace{(\ln x)}_{u} \underbrace{x^5 dx}_{dv}$$

$$= \underbrace{\frac{1}{6}x^6}_{v} \underbrace{\ln x}_{u} - \int \underbrace{\frac{1}{6}x^6}_{v} \underbrace{\frac{1}{x}dx}_{du}$$

$$= \frac{1}{6}x^6 \ln x - \int \frac{1}{6}x^5 dx =$$

$$= \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C$$

Thus we can now evaluate the integral $\int x^5 (\ln x)^2 dx$ at the top of this page:

$$\int x^5(\ln x)^2 dx =$$

$$\frac{1}{6}x^6(\ln x)^2 - \frac{1}{3}\left(\frac{1}{6}x^6\ln x - \frac{1}{36}x^6\right) + C$$

$$= \frac{1}{6}x^6(\ln x)^2 - \frac{1}{18}x^6\ln x + \frac{1}{108}x^6 + C$$

Thus we can evaluate the integral

$\int x^5(\ln x)^3 dx$ on p. 12

$$= \frac{1}{6}x^6(\ln x)^3 - \frac{1}{2}\int x^5(\ln x)^2 dx =$$

$$= \frac{1}{6}x^6(\ln x)^3 - \frac{1}{2}\left(\frac{1}{6}x^6(\ln x)^2 - \frac{1}{18}x^6\ln x + \frac{1}{108}x^6\right) + C$$

$$= \frac{1}{6}x^6(\ln x)^3 - \frac{1}{12}x^6(\ln x)^2 + \frac{1}{36}x^6\ln x - \frac{1}{216}x^6 + C$$

$\ast$

Calculate $\int \cos x \ln(\tan x)dx$ :

$$= \int \underbrace{\ln(\tan x)}_{u} \underbrace{\cos x dx}_{dv} \quad : \quad v = \sin x,$$

$$du = \frac{1}{\tan x} \cdot \sec^2 x \, dx$$

hence

$$\int \ln(\tan x)\cos x\,dx =$$

$$= \underbrace{\sin x}_{v}\;\underbrace{\ln(\tan x)}_{u} - \int \underbrace{\sin x}_{v}\cdot\underbrace{\frac{1}{\tan x}\cdot\sec^2 x\,dx}_{du}$$

Now $\sin x \cdot \dfrac{1}{\tan x}\cdot \sec^2 x =$

$$= \sin x \; \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x} = \frac{1}{\cos x} = \sec x$$

$$= \sin x \ln(\tan x) - \int \sec x\,dx$$

$$= \sin x \ln(\tan x) - \ln|\sec x + \tan x| + C$$

$*$

Other Examples of the type at the top of p. 2:

$$\int \frac{\ln x}{x^3}\,dx, \qquad \int \frac{(\ln x)^2}{x^5}\,dx$$

WORK OUT ON YOUR OWN!

Also, <u>Relevant Exercises in the Book that were not assigned to be turned in:</u>
(NOT Everything can be assigned to be turned in, but if you want to get a high grade you will need to practice more than the assigned Homeworks.)

p. 468: #1, 12, 27, 26,
p. 469: #32, 41, 42, 51, 55, 57, 64

Similar Approach to the one used for integrals involving ln (natural log) also works for integrals involving inverse trig functions

$\tan^{-1}x$ ($= \arctan x$),

$\sin^{-1}x$ ($= \arcsin x$),

$\cos^{-1}x$ ($= \arccos x$), etc.

Some Examples follow:

Calculate $\int x^2 \sin^{-1}x\,dx$:

$$\int x^2 \sin^{-1}x\,dx = \int \underbrace{(\sin^{-1}x)}_{u} \underbrace{x^2 dx}_{dv}$$

$$v = \frac{1}{3}x^3, \quad du = \frac{1}{\sqrt{1-x^2}}dx$$

$$= (\sin^{-1}x)\cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{\sqrt{1-x^2}}dx$$

Hence we need to evaluate

$$\int x^3 \cdot \frac{1}{\sqrt{1-x^2}}dx = \int \frac{x^2}{\sqrt{1-x^2}}\,x\,dx$$

We substitute $1-x^2 = u$, $du = -2x\,dx$, hence $x\,dx = -\frac{1}{2}du$, $x^2 = 1-u$

Thus

$$= \int \frac{1-u}{\sqrt{u}}\left(-\frac{1}{2}du\right) =$$

$$= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}du = \frac{1}{2}\int\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$$

$$= \frac{1}{2}\int\left(u^{1/2} - u^{-1/2}\right)du =$$

$$\frac{1}{2}\cdot\frac{2}{3}u^{3/2} - \frac{1}{2}\cdot 2u^{1/2} + C =$$

$$= \frac{1}{3} u^{3/2} - u^{1/2} + C$$

$$= \frac{1}{3}(1-x^2)^{3/2} - (1-x^2)^{1/2} + C$$

$$= (1-x^2)^{1/2}\left(\frac{1}{3}(1-x^2) - 1\right) + C$$

$$= (1-x^2)^{1/2}\left(-\frac{2}{3} - \frac{1}{3}x^2\right) + C$$

$$= \boxed{-\frac{1}{3}(1-x^2)^{1/2}(2+x^2) + C}$$

Thus $\int x^2 \sin^{-1} x \, dx =$ (from top p. 17)

$$= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3}\boxed{\int x^3 \cdot \frac{1}{\sqrt{1-x^2}}\, dx}$$

$$= \frac{1}{3}x^3 \sin^{-1} x - \frac{1}{3}\left(-\frac{1}{3}(1-x^2)^{1/2}(2+x^2)\right) + C$$

$$= \frac{1}{3}x^3 \sin^{-1} x + \frac{1}{9}(1-x^2)^{1/2}(2+x^2) + C$$

*

Calculate $\int(\arcsin x)^2 dx$:

This is, of course, the same as

$\int(\sin^{-1}x)^2 dx$ — just alternate notation.

$$\int \underbrace{(\arcsin x)^2}_{u} \underbrace{1\,dx}_{dv} \qquad v = x \qquad du = 2(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$= \underset{v}{x}\underset{u}{(\arcsin x)^2} - \int \underset{v}{x}\cdot\underbrace{2(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}dx}_{du}$$

$$= x(\arcsin x)^2 + \int \underbrace{(\arcsin x)}_{u}\cdot\underbrace{\frac{(-2x)}{\sqrt{1-x^2}}dx}_{dv}$$

So $v = \int \frac{(-2x)}{\sqrt{1-x^2}}dx$

To calculate this integral, we make the substitution $t = 1-x^2$

(we use $t$ instead of $u$ to avoid confusion with the $u$ in integration by parts above)

So $dt = (-2x)dx$

Thus $v = \int \frac{(-2x)}{\sqrt{1-x^2}}\,dx =$

$$= \int \frac{1}{\sqrt{t}}\,dt = \int t^{-1/2}\,dt = 2t^{1/2} + C$$

$$= 2\sqrt{1-x^2} + C$$

So we use $v = 2\sqrt{1-x^2}$

So returning to $\int \underbrace{(\arcsin x)}_{u} \cdot \underbrace{\frac{(-2x)}{\sqrt{1-x^2}}\,dx}_{dv}$

we have $v = 2\sqrt{1-x^2}$, $du = \frac{1}{\sqrt{1-x^2}}\,dx$,

hence

$$= \underbrace{2\sqrt{1-x^2}}_{v}\,\underbrace{\arcsin x}_{u} - \int \underbrace{2\sqrt{1-x^2}}_{v} \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}\,dx}_{du}$$

$$= 2\sqrt{1-x^2}\arcsin x - \int 2\,dx =$$

$$= 2\sqrt{1-x^2}\arcsin x - 2x + C$$

So now the integral at the top of p. 19:

$$\int (\arcsin x)^2 dx =$$

$$= x(\arcsin x)^2 + \int (\arcsin x) \cdot \frac{(-2x)}{\sqrt{1-x^2}} dx$$

$$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

*

Calculate $\int x \arctan x \, dx$:

$$= \int \underbrace{(\arctan x)}_{u} \underbrace{x\,dx}_{dv}$$

$$v = \frac{x^2}{2}, \quad du = \frac{1}{1+x^2}dx$$

$$= \underbrace{\frac{x^2}{2}}_{v} \underbrace{\arctan x}_{u} - \underbrace{\int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\frac{1}{1+x^2}dx}_{du}}$$

So we need to calculate

$$\frac{1}{2} \int \frac{x^2}{1+x^2}dx = \frac{1}{2} \int \frac{x^2+1-1}{1+x^2}dx$$

$$= \frac{1}{2} \int \left( \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx =$$

$$= \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx = \frac{1}{2}(x - \arctan x) + C$$

Hence

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C$$

$$= \frac{1}{2}(\arctan x)(x^2+1) - \frac{1}{2}x + C$$

$*$

The following Exercises in the Book are similar to those on pages 17 - 22 in these Notes:

on p. 468: # 10, 11, 22, 30, 31,

Also Example 5, p. 467

So we have learned that, typically, for integrals of the form

$$\int f(x)\ln x\,dx, \quad \int f(x)(\ln x)^2 dx, \ \ldots$$

$$\int f(x)(\arcsin x)\,dx, \quad \int f(x)(\arcsin x)^2 dx, \ \ldots$$

$$\int f(x)(\arctan x)\,dx, \quad \int f(x)(\arctan x)^2 dx, \ \ldots$$

(and so forth, look at the preceding pages for many examples ),

one should try Integration by parts $u = \ln x$, $u = (\ln x)^2$, ...

$u = \arcsin x$, $u = (\arcsin x)^2$, ...

$u = \arctan x$, $u = (\arctan x)^2$, ...

$dv = f(x)\,dx$.

Of course, this Rule is "off", if we can do just a basic