Some Discussion of Integrals of the form $\int \cot^m x \csc^n x \, dx$.

Let's first discuss the analogy with the integrals of the type $\int \tan^m x \sec^n x \, dx$ which are discussed on pages 44 – 52 of these Notes.

We have used the identity

$$\sec^2 x = 1 + \tan^2 x \quad (\text{or } \tan^2 x = \sec^2 x - 1)$$

and the formulas

$$\int \tan x \, dx = \ln|\sec x| + C,$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C,$$

$$\int \sec^2 x \, dx = \tan x + C \quad (\text{i.e. } (\tan x)' = \sec^2 x),$$

$$\int \sec x \tan x \, dx = \sec x + C \quad (\text{i.e. } (\sec x)' = \sec x \tan x)$$

To work integrals of the type at the heading of this page we need

$$\csc^2 x = 1 + \cot^2 x \quad (\text{or } \cot^2 x = \csc^2 x - 1)$$

which is listed on Reference Page 2 at the Front of the Book, and also

$\int \cot x\,dx = \ln|\sin x| + C,$

$\int \csc x\,dx = \ln|\csc x - \cot x| + C,$

$\int \csc^2 x\,dx = -\cot x,$ (i.e. $(\cot x)' = -\csc^2 x$),

$\int \csc x \cot x\,dx = -\csc x + C,$

(i.e. $(\csc x)' = -\csc x \cot x$)

For Example,

Calculate $\int \csc^3 x \cot^3 x\,dx$:

$= \int (\csc x \cot x) \csc^2 x \cot^2 x\,dx =$

$= \int (\csc x \cot x) \csc^2 x (\csc^2 x - 1)\,dx$

Since $(\csc x)' = -\csc x \cot x$,

we can use the subst. $u = \csc x$,

$du = -\csc x \cot x\,dx$,

so $\csc x \cot x\,dx = -du$

$= \int u^2(u^2 - 1)(-du) = -\int (u^4 - u^2)\,du$

$= -\frac{u^5}{5} + \frac{u^3}{3} + C = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

*

Yet another type of trig. Integral is done in Example 9, p. 476, namely integrals of the form

$\int \sin ax \cos bx\, dx$, $\int \sin ax \sin bx\, dx$,

and $\int \cos ax \cos bx\, dx$

For Example,

Calculate $\int \sin 5x \sin 2x\, dx$

Then the relevant formula to be used is the formula (b) in the box above the Example 9, p. 476:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

Using this formula, we obtain

$$\sin 5x \sin 2x = \frac{1}{2}\left[\cos(5x-2x) - \cos(5x+2x)\right]$$

$$= \frac{1}{2}\left[\cos 3x - \cos 7x\right]$$

Hence $\int \sin 5x \sin 2x\, dx =$

$$\frac{1}{2}\int(\cos 3x - \cos 7x)\, dx =$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x\right) + C$$

*

Integration by parts could also be used but is more work.

What you need to learn about this type of problem is how to do it if you are given the formulas in the box. (Of course the formulas would not be provided directly with the question itself, but rather on a general list of formulas that you would need to examine and find what's relevant for any given question.)

More Ideas.

We will not study a general method for calculating integrals of rational functions of $\sin x, \cos x$. However, there is an easy useful idea for fractions whose denominator contains $1 \pm \sin x$ or $1 \pm \cos x$.

For Example, Calculate $\int \frac{\cos x}{(1+\cos x)^2}dx$

We multiply both the numerator and denominator by $(1-\cos x)^2$:

$$\frac{\cos x}{(1+\cos x)^2} = \frac{\cos x\,(1-\cos x)^2}{(1+\cos x)^2(1-\cos x)^2} =$$

$$= \frac{\cos x\,(1-\cos x)^2}{((1+\cos x)(1-\cos x))^2} =$$

$$= \frac{\cos x\,(1-2\cos x+\cos^2 x)}{(1-\cos^2 x)^2} =$$

$$= \frac{\cos x - 2\cos^2 x + \cos^3 x}{(\sin^2 x)^2} =$$

$$= \boxed{\frac{\cos x}{\sin^4 x} - \frac{2\cos^2 x}{\sin^4 x} + \frac{\cos^3 x}{\sin^4 x}}$$

We obtain $\int \frac{\cos x}{\sin^4 x}dx = -\frac{1}{3\sin^3 x} + C$

(which also $= -\frac{1}{3}\csc^3 x + C$)

which is done by substituting $u = \sin x$, since $\cos x = (\sin x)'$.

The integral $\int \frac{\cos^3 x}{\sin^4 x}\,dx$ is handled similarly — i.e. these integrals are handled as the case $\int \sin^m x \cos^n x\,dx$, when one of m, n is odd (p. 42 in these Notes).

More explicitly,

$$\frac{\cos^3 x}{\sin^4 x} = \frac{\cos x\,(\cos^2 x)}{\sin^4 x} =$$

$$= \frac{\cos x\,(1-\sin^2 x)}{\sin^4 x} =$$

$$= \cos x\left(\frac{1}{\sin^4 x} - \frac{1}{\sin^2 x}\right) =$$

$$= \frac{\cos x}{\sin^4 x} - \frac{\cos x}{\sin^2 x},$$

hence $\int \frac{\cos^3 x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C$

$\left(= -\frac{1}{3}\csc^3 x + \csc x + C \right)$

Finally $\int \frac{\cos^2 x}{\sin^4 x}\,dx =$

$$= \int \frac{1-\sin^2 x}{\sin^4 x}\,dx = \int\left(\frac{1}{\sin^4 x} - \frac{1}{\sin^2 x}\right)dx$$

$$= \int (\csc^4 x - \csc^2 x)\,dx$$

We know $\int (-\csc^2 x)\,dx = \cot x + C$

(since $(\cot x)' = -\csc^2 x$).

An even power of $\csc x$ is handled as on p. 44 of these Notes (i.e. similarly):

$$\int \csc^4 x\,dx = \int \csc^2 x \csc^2 x\,dx =$$

$$= \int \csc^2 x\,(1+\cot^2 x)\,dx = \int \csc^2 x\,dx +$$

$$+ \underbrace{\int \csc^2 x \cot^2 x\,dx}_{\text{i.e. use subst. } u=\cot x,} = -\cot x \underbrace{- \tfrac{1}{3}\cot^3 x + C}_{\text{to obtain}}$$

Thus the final answer to the Problem at the top of p. 60 is obtain by putting together the integrals of the three expressions in the box on p. 60 *

Another technique we should have in our repertoire: If the product of trigonometric functions does not fit any of the patterns we discussed, we express all the functions involved using the basic functions $\sin x$, $\cos x$ and simplify: For Example,

Calculate $\int \tan^3 x \csc^3 x \, dx$.

This is not one of the basic types, so,

$$\tan^3 x \csc^3 x = \left(\frac{\sin x}{\cos x}\right)^3 \cdot \left(\frac{1}{\sin x}\right)^3 =$$

$$= \frac{1}{\cos^3 x} = \sec^3 x$$

But we know how to calculate $\int \sec^3 x \, dx$ (bottom p. 51 of these Notes, or Example 8, p. 475 in the Book).

*

Additional Recommended Exercises from sect. 7.2:

p. 476: #6, 13, 14, 16, p. 477: 17, 18, 19, 20, 33, 34, 42, 43, 44, 45, 46, 47, 48, 49

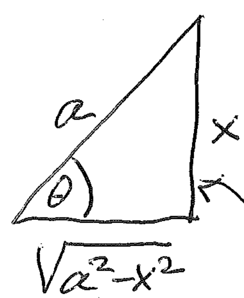
If you study the Notes, pages 42–63, and also have mastered integration by parts (and, of course, the substitution method), that should be helpful with the exercises listed above.

If you have difficulty come to the Office Hours.

Sect. 7.3, TRIG. SUBSTITUTIONS.

There are three trig. substitutions that we use to find integrals involving expressions $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$. These substitutions are listed in the box at the bottom of p. 478. Each time we use one of these substitutions, it is important to draw a right triangle, with an annotation, that goes with that substitution:

$\sqrt{a^2-x^2}$:

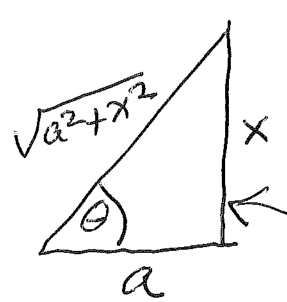


$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

Note that $\sin\theta = \dfrac{x}{a}$

$\sqrt{a^2+x^2}$:

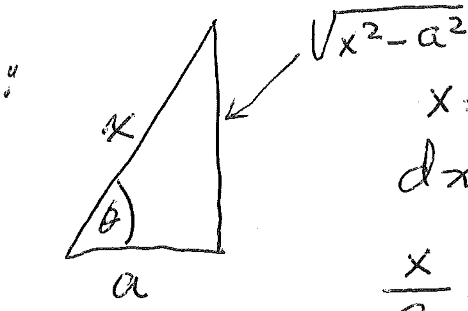


$x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$\dfrac{x}{a} = \tan\theta$

$\sqrt{x^2 - a^2}$:



$$x = a\sec\theta$$
$$dx = a\sec\theta\tan\theta\, d\theta$$
$$\frac{x}{a} = \sec\theta \left(= \frac{1}{\cos\theta}\right)$$

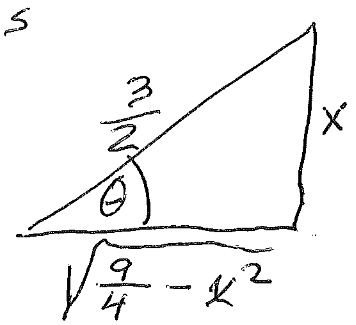
Calculate $\int x^2 (9 - 4x^2)^{3/2}\, dx$

First Note that the substitutions are formulated for the case when the coefficient of x^2 is ± 1. So we start with a preparatory step

$$= \int x^2 \left[4\left(\frac{9}{4} - x^2\right)\right]^{3/2} dx$$
$$= 4^{3/2} \int x^2 \left(\frac{9}{4} - x^2\right)^{3/2} dx$$
$$= 8 \int x^2 \left(\frac{9}{4} - x^2\right)^{3/2} dx$$

Since $\frac{9}{4} = a^2$, we obtain $a = \sqrt{\frac{9}{4}} = \frac{3}{2}$

Thus



$$x = \frac{3}{2}\sin\theta$$
$$dx = \frac{3}{2}\cos\theta\, d\theta$$

Thus $x^2\left(\frac{9}{4}-x^2\right)^{3/2} =$

$$= \left(\frac{3}{2}\sin\theta\right)^2\left(\frac{9}{4}-\left(\frac{3}{2}\sin\theta\right)^2\right)^{3/2} =$$

$$= \frac{9}{4}\sin^2\theta\left(\frac{9}{4}-\frac{9}{4}\sin^2\theta\right)^{3/2} =$$

$$= \frac{9}{4}(\sin^2\theta)\left(\frac{9}{4}\right)^{3/2}(1-\sin^2\theta)^{3/2} =$$

$$= \frac{9}{4}\cdot\left(\frac{3}{2}\right)^3\sin^2\theta\,(\cos^2\theta)^{3/2} =$$

$$= \left(\frac{3}{2}\right)^5\sin^2\theta\cos^3\theta$$

Also $dx = \frac{3}{2}\cos\theta\,d\theta$, hence

$$\underline{\underline{\int x^2(9-4x^2)^{3/2}dx}} = 8\int x^2\left(\frac{9}{4}-x^2\right)^{3/2}dx =$$

$$= 8\int\left(\frac{3}{2}\right)^5\sin^2\theta\cos^3\theta\cdot\frac{3}{2}\cos\theta\,d\theta =$$

$$= 8\cdot\left(\frac{3}{2}\right)^6\int\sin^2\theta\cos^4\theta\,d\theta =$$

$$= \underline{\underline{\frac{729}{8}\int\sin^2\theta\cos^4\theta\,d\theta}}$$

Now we need calculate

$$\int \sin^2\theta \cos^4\theta \, d\theta$$

So we proceed as in the Example $\int \sin^4 x \cos^6 x \, dx$ on p. 43, using the formulas $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ (These formulas are on the Reference Page 2 in front of the Book.

Hence

$$= \int \frac{1}{2}(1-\cos 2\theta) \cdot \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \frac{1}{8}\int (1+\cos 2\theta)^2 d\theta - \frac{1}{8}\int (\cos 2\theta)(1+\cos 2\theta)^2 d\theta$$

$$= \frac{1}{8}\int (1+2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$- \frac{1}{8}\int \cos 2\theta \,(1+2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \frac{1}{8}\int \left(1+\cos 2\theta - \cos^2 2\theta - \cos^3 2\theta\right) d\theta =$$

$$= \frac{1}{8}\int (1-\cos^2 2\theta)\,d\theta$$

$$+ \frac{1}{8}\int (\cos 2\theta - \cos^3 2\theta)\,d\theta =$$

$$= \frac{1}{8}\int \sin^2 2\theta\, d\theta + \frac{1}{8}\int \cos 2\theta\,(1-\cos^2 2\theta)\,d\theta$$

$$= \frac{1}{8}\int \sin^2 2\theta\, d\theta + \frac{1}{8}\int \cos 2\theta \sin^2 2\theta\, d\theta$$

We calculate each of the two integrals above separately:

$$\int \sin^2 2\theta\, d\theta = \int \frac{1}{2}(1-\cos 4\theta)\,d\theta =$$

$$= \frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{4}\sin 4\theta + C$$

$$= \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta + C\,;$$

To calculate $\int \cos 2\theta \sin^2 2\theta\, d\theta$

we make the substitution

$u = \sin 2\theta$, $du = 2\cos 2\theta\, d\theta$,

hence $\cos 2\theta\, d\theta = \frac{1}{2}du$

$$\int \cos 2\theta \sin^2 2\theta \, d\theta = \int u^2 \cdot \frac{1}{2} \, du$$

$$= \frac{1}{6} u^3 + C = \frac{1}{6} \sin^3 2\theta + C$$

Hence $\int \sin^2\theta \cos^4\theta \, d\theta =$

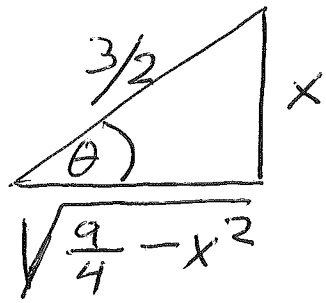
$$= \frac{1}{8} \int \sin^2 2\theta \, d\theta + \frac{1}{8} \int \cos 2\theta \sin^2 2\theta \, d\theta$$

$$= \frac{1}{8} \left(\frac{1}{2}\theta - \frac{1}{8} \sin 4\theta \right) + \frac{1}{8} \cdot \frac{1}{6} \sin^3 2\theta + C$$

$$= \boxed{\frac{1}{16}\theta - \frac{1}{64} \sin 4\theta + \frac{1}{48} \sin^3 2\theta + C}$$

We can now see why the picture at the bottom of p. 66 is important. Namely, we will use it to express the result in the box above in terms of x (using x).

Picture from bottom p. 66:



$x = \frac{3}{2}\sin\theta$,

hence $\sin\theta = \frac{2}{3}x$,

$$\boxed{\theta = \sin^{-1}\tfrac{2}{3}x = \arcsin\tfrac{2}{3}x}$$

The formula in the box on p. 70 also has $\sin^3 2\theta$, so we need to express $\sin 2\theta$ using x.

We use $\sin 2\theta = 2\sin\theta\cos\theta$:

We have known all along that $\sin\theta = \frac{2}{3}x$. But we also conclude from the drawing above that $\cos\theta = \dfrac{\sqrt{\frac{9}{4}-x^2}}{\frac{3}{2}} =$

$= \frac{2}{3}\sqrt{\frac{9}{4}-x^2}$

Hence we obtain

$$\underline{\underline{\sin 2\theta}} = 2\sin\theta\cos\theta =$$

$$= 2 \cdot \frac{2}{3}x \cdot \frac{2}{3}\sqrt{\frac{9}{4} - x^2}$$

$$= \underline{\underline{\frac{8}{9}x\sqrt{\frac{9}{4} - x^2}}}$$

Thus (for use in the box on p. 70)

we obtain

$$\sin^3 2\theta = \left(\frac{8}{9}x\sqrt{\frac{9}{4} - x^2}\right)^3 =$$

$$= \frac{8^3}{9^3}x^3\left(\frac{9}{4} - x^2\right)^{3/2}$$

We also need to express $\sin 4\theta$ using x. Similarly as before,

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta$$

We have already expressed $\sin 2\theta$, so we need to express $\cos 2\theta$.

We know $\cos 2\theta = \cos^2\theta - \sin^2\theta$, but also, more conveniently,

$$\cos 2\theta = 1 - 2\sin^2\theta, \text{ hence}$$

$$\underline{\cos 2\theta} = 1 - 2\left(\tfrac{2}{3}x\right)^2 = \underline{1 - \tfrac{8}{9}x^2}$$

Thus $\underline{\sin 4\theta} = 2\sin 2\theta\cos 2\theta =$

$$= 2\underbrace{\left(\frac{8}{9}x\sqrt{\frac{9}{4}-x^2}\right)}_{\text{top p. 72}}\underbrace{\left(1-\frac{8}{9}x^2\right)}$$

Thus we can now put together the final answer, using the formula at the bottom of p. 67:

$$\int x^2(9-4x^2)^{3/2}dx = \frac{729}{8}\int \sin^2\theta\cos^4\theta\, d\theta$$

and the formula near the top of p. 70 (answer in the box).

$$\int x^2(9-4x^2)^{3/2}dx =$$

$$\frac{729}{8}\left[\frac{1}{16}\arcsin\frac{2}{3}x - \frac{1}{64}\cdot\frac{16}{9}x\left(1-\frac{8}{9}x^2\right)\sqrt{\frac{9}{4}-x^2} + \frac{1}{48}\cdot\frac{8^3}{9^3}x^3\left(\frac{9}{4}-x^2\right)^{3/2}\right]$$

We would like to use the expression $(9-4x^2)$ appearing in the given integral, hence we substitute

$$\frac{9}{4}-x^2 = \frac{1}{4}(9-4x^2).$$

Hence $\sqrt{\frac{9}{4}-x^2} = \sqrt{\frac{1}{4}(9-4x^2)} = \frac{1}{2}\sqrt{9-4x^2}$

and $\left(\frac{9}{4}-x^2\right)^{3/2} = \left[\frac{1}{4}(9-4x^2)\right]^{3/2} = \frac{1}{8}(9-4x^2)^{3/2}$

We thus substitute into the formula at the top, ignoring for the moment the arcsin term and the factor $\frac{729}{8}$.

Hence

$$-\frac{1}{64}\cdot\frac{16}{9}x\left(1-\frac{8}{9}x^2\right)\sqrt{\frac{9}{4}-x^2}+\frac{1}{48}\cdot\frac{8^3}{9^3}x^3\left(\frac{9}{4}-x^2\right)^{3/2}=$$

$$=-\frac{1}{36}x\left(1-\frac{8}{9}x^2\right)\cdot\frac{1}{2}\sqrt{9-4x^2}+\frac{1}{3}\cdot\frac{32}{9^3}x^3\cdot\frac{1}{8}\left(9-4x^2\right)^{3/2}=$$

$$=\sqrt{9-4x^2}\left[-\frac{1}{72}x\left(1-\frac{8}{9}x^2\right)+\frac{4}{3}\cdot\frac{1}{9^3}x^3\left(9-4x^2\right)\right]$$

$$=\sqrt{9-4x^2}\left[-\frac{1}{72}x+\frac{1}{81}x^3+\frac{4}{3}\cdot\frac{1}{81}x^3-\frac{16}{3}\cdot\frac{1}{9^3}x^5\right]$$

$$=\sqrt{9-4x^2}\left[-\frac{1}{72}x+\frac{7}{3}\cdot\frac{1}{81}x^3-\frac{16}{3}\cdot\frac{1}{9^3}x^5\right]$$

This needs to be multiplied by $\frac{729}{8}$ (see bottom p. 74). Thus

$$\frac{1}{72}\cdot\frac{729}{8}=\frac{81}{64},\quad \frac{7}{3}\cdot\frac{1}{81}\cdot\frac{729}{8}=\frac{7\cdot 3}{8}=\frac{21}{8}$$

and $\frac{16}{3}\cdot\frac{1}{9^3}\cdot\frac{729}{8}=\frac{2}{3}$

We thus obtain the final answer for the integral at the top of p. 74:

$$=\frac{729}{128}\arcsin\left(\frac{2}{3}x\right)+\sqrt{9-4x^2}\left(-\frac{81}{64}x+\frac{21}{8}x^3-\frac{2}{3}x^5\right)$$

END OF PROBLEM BEGUN on p. 66

*

SUMMARY OF THE STEPS IN USING A TRIG. SUBSTIT.

Thus let's take another look at the integral $\int x^2(9-4x^2)^{3/2}dx$.

Of course, to begin with we need to recognize that the integral includes an expression of one of the types

$$\sqrt{a^2-x^2},\ \sqrt{a^2+x^2},\ \sqrt{x^2-a^2}.$$

This may require a bit of algebra:

E.g., $(9-4x^2)^{3/2} = (9-4x^2)\sqrt{9-4x^2}$

(I did not need to write it in this form to do the actual calculations.)

Furthermore, the coefficient of x^2 has to be ± 1. Again this can be achieved with a little bit of algebra, as shown on p. 66.

Then we need to decide on the form of substitution (3 possibilities) as explained on pages 65, 66.

Then one needs to draw the corresponding triangle. This one has to learn — note that the quotient $\frac{x}{a}$ always equals the corresponding trig. function, the horizontal and vertical sides make the 90° angle, and θ is the angle between the hypothenuse and the horiz. side: (p. 66, p. 80)

Next one makes the chosen substitution; in our example

$$x = a\sin\theta = \frac{3}{2}\sin\theta, \quad dx = a\cos\theta\, d\theta$$
$$= \frac{3}{2}\cos\theta\, d\theta.$$

The benefit of the substitution is that the square root expression "disappears" — this is a consequence of the basic trig. identities:

For the substitution $x = a\sin\theta$,

we obtain $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta}$

$= \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{1-\sin^2\theta} = a\cos\theta$

(for θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$).

For the subst. $x = a\tan\theta$:

$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} =$

$\sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$

$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

For the subst. $x = a\sec\theta$

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} =$

$= \sqrt{a^2(\sec^2\theta - 1)} = a\tan\theta$

$(0 \le \theta < \pi/2)$

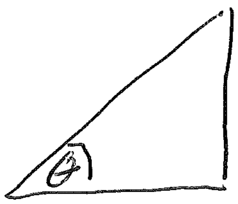
So again, after performing the substitution and simplifying <u>you should not have any square root signs</u> (if you do, you are doing something incorrect).

The Integral obtained from the substitution is a trigonom. integral — not necessarily of one of the most basic types discussed in section 7.2, but of type done in the exercises, perhaps. For our Example (started on p. 66), we obtained the trig. integral $\int \sin^2\theta \cos^4\theta \, d\theta$, which IS of the "basic type"! We evaluate this trig. integral. However at this stage we still have the variable θ

in the answer, <u>not x</u>.

So we need to, so to speak, <u>"undo" the substitution.</u>

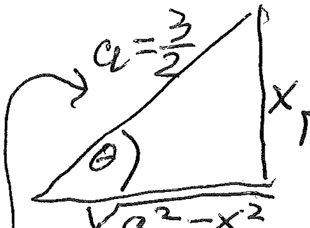
For this we use the triangle.



starting the triangle (see p. 77)

Substitution for the Problem on p. 66

$$\left.\begin{aligned} x &= a\sin\theta \\ &= \tfrac{3}{2}\sin\theta \end{aligned}\right\} \Leftrightarrow$$



$$\sin\theta = \frac{x}{a} = \frac{x}{\frac{3}{2}}$$

The answer to the trig. integral (box p. 70) may contain trig. functions of θ, which we can express in terms of x using the triangle.

Thus we $\sin\theta = \dfrac{x}{a} = \dfrac{x}{\frac{3}{2}} = \dfrac{2}{3}x$

(hence $\theta = \sin^{-1}\left(\frac{2}{3}x\right)$),

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a} = \frac{2}{3}\sqrt{\frac{9}{4}-x^2},$$

$$\tan\theta = \frac{x}{\sqrt{a^2-x^2}} = \frac{\frac{3}{2}}{\sqrt{\frac{9}{4}-x^2}}$$

$$\sec\theta = \frac{a}{\sqrt{a^2 - x^2}}, \text{ etc.}$$

Now the answer (in the box on p. 70) may contain trig. functions of integer multiples of θ – as on p. 70. So we now use trig. identities, as we did on pages 72, 73.

Thus if, for example, the answer contained $\tan 2\theta$, we would obtain, when $a = \frac{3}{2}$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{8}{9} x \sqrt{\frac{9}{4} - x^2}}{1 - \frac{8}{9} x^2} =$$

$$= \frac{8x\sqrt{\frac{9}{4} - x^2}}{9 - 8x^2} = \frac{8x\sqrt{\frac{1}{4}(9 - 4x^2)}}{9 - 8x^2} =$$

$$= \frac{4x\sqrt{9 - 4x^2}}{9 - 8x^2}, \text{ etc.}$$

*

Comments Concerning the Example on p. 43 of these Notes:

If I did not try to be clever I would obtain $\boxed{\text{as on p. 43}}$

$$\int \sin^4 x \cos^6 x\,dx =$$

$$= \boxed{\int \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 \left[\tfrac{1}{2}(1+\cos 2x)\right]^3 dx}$$

$$= \frac{1}{32}\int (1-\cos 2x)^2(1+\cos 2x)^3\,dx$$

continue in a simple-(minded(?)) way

$$= \frac{1}{32}\int (1-2\cos 2x+\cos^2 2x)(1+3\cos 2x+3\cos^2 2x+\cos^3 2x)\,dx$$

$$= \frac{1}{32}\int (1+3\cos 2x+3\cos^2 2x+\cos^3 2x-2\cos 2x-6\cos^2 2x-6\cos^3 2x$$
$$-2\cos^4 2x+\cos^2 2x+3\cos^3 2x+3\cos^4 2x+\cos^5 2x)\,dx$$

$$= \frac{1}{32}\int (-2\cos^2 2x+\cos^4 2x)\,dx \; +$$

$$+ \frac{1}{32}\underbrace{\int (1+\cos 2x)\,dx}_{\text{immediate}} + \frac{1}{32}\underbrace{\int (\cos^5 2x-2\cos^3 2x)\,dx}_{\text{odd powers, see p. 42}}$$

So it remains to deal with

$$\int (2\cos^2 2x + \cos^4 2x)\,dx$$

So we again use the half-angle formula $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$, this time with $\alpha = 2x$:

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x),$$

Hence

$$= \int \left[2\cos^2 2x + (\cos^2 2x)^2\right] dx$$

$$= \int \left[1+\cos 4x + \left(\tfrac{1}{2}(1+\cos 4x)\right)^2\right] dx$$

$$= \underbrace{\int (1+\cos 4x)\,dx}_{\text{immediate}} + \frac{1}{4}\underbrace{\int (1+\cos 4x)^2\,dx}_{\text{finish on your own}}$$

$\divideontimes$

Second Comment dealing the Example on p. 43: Note that by combining the factors $(1-\cos 2x)$, $(1+\cos 2x)$, i.e.

$$(1-\cos 2x)(1+\cos 2x) = 1-\cos^2 2x,$$

I am using the idea explained at the bottom of p. 59 and on p. 60, so on p. 43, I should have continued with $\boxed{1-\cos^2 2x = \sin^2 2x}$:

I.e., from page 43:

$$\int \sin^4 x \cos^6 x\,dx = \frac{1}{32}\int (1-\cos^2 2x)^2 (1+\cos 2x)\,dx$$

We now use the formula in the box above

$$= \frac{1}{32}\int (\sin^2 2x)^2 (1+\cos 2x)\,dx =$$

$$= \frac{1}{32}\int \sin^4 2x\,(1+\cos 2x)\,dx =$$

$$= \frac{1}{32}\int \sin^4 2x\,dx + \frac{1}{32}\underbrace{\int \sin^4 2x \cos 2x\,dx}$$

odd power of $\cos 2x$, so use p. 42, finish on your own

$\int \sin^4 2x\,dx$ is handled similarly as the $\cos^4 2x$ part of the integral at the top of p. 83, i.e. we use the identity

$$\sin^2 2x = \frac{1}{2}(1-\cos 4x).$$

So the conclusion is that if a product contains factors $(1-\cos mx)$, $(1+\cos mx)$, one should combine as many of them as possible and replace the resulting product by a power of $\sin mx$:

E.g.

$$(1-\cos 3x)^5(1+\cos 3x)^3 =$$

$$(1-\cos 3x)^2\left[(1-\cos 3x)(1+\cos 3x)\right]^3 =$$

$$= (1-\cos 3x)^2\left[1-\cos^2 3x\right]^3 =$$

$$= (1-\cos 3x)^2(\sin^2 3x)^3 =$$

$$= (1-\cos 3x)^2 \sin^6 3x$$

*