Calculate $\int x \arcsin x \, dx$:

We use Integration by Parts :

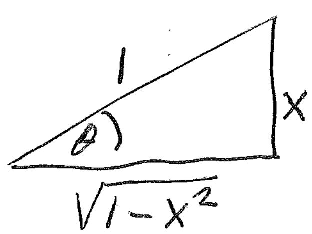
$$\int x \arcsin x \, dx = \int \underbrace{(\arcsin x)}_{u} \underbrace{x\,dx}_{dv} = \left(\begin{array}{l} u = \arcsin x \\ du = \dfrac{1}{\sqrt{1-x^2}}\,dx \end{array} \right)$$

$$= \underbrace{\tfrac{1}{2}x^2}_{v} \underbrace{\arcsin x}_{u} - \int \tfrac{1}{2}x^2 \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

So we now need to calculate $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$.

We use the trig. substitution $x = \sin\theta$

$dx = \cos\theta \, d\theta$



$$= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta = \int \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d\theta$$

$$= \int \underbrace{\tfrac{1}{2}(1-\cos 2\theta)}_{=\sin^2\theta}\, d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + C$$

Now we need to use the triangle to "undo" the substitution. Firstly $\theta = \sin^{-1}x = \arcsin x$;

Then we use the identity $\sin 2\theta = 2\sin\theta\cos\theta$,

and from the triangle we obtain

$\sin\theta = x$, $\cos\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

Hence $\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$

Hence $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin x - \frac{1}{4}\cdot 2x\sqrt{1-x^2} + C$

$= \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C$

and $\underline{\int x \arcsin x\,dx} = \frac{1}{2}x^2\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \frac{1}{2}x^2\arcsin x - \frac{1}{2}\left(\frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2}\right) + C$

$= \frac{1}{2}x^2\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$

You should Study on your own *
The substitution $x = a\sec\theta$ for dealing with integrals involving $\sqrt{x^2-a^2}$.
The ideas needed are sufficiently illustrated by the other two substitutions. This third substitution ($x = a\sec\theta$) is shown in Example 5, p. 481 in the Book, and some of the Homework Exercises on p. 483 also use this substitution. You are also welcome to ask Questions about it in Office Hours.

Calculate $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x+13}}\,dx$:

We start ~~with~~ completing to a square:

$$x^2+4x+13 = x^2+4x+4+9 = (x+2)^2+9$$

$$= \int \frac{1}{(x+2)\sqrt{(x+2)^2+9}}\,dx$$

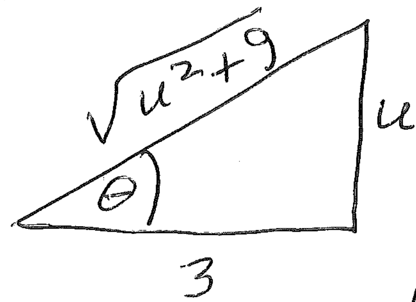
To simplify the expression we make the subst. $u = x+2$, $du = dx$

$$= \int \frac{1}{u\sqrt{u^2+9}}\,du$$

$9 = 3^2$, hence we substitute

$u = 3\tan\theta$,

$du = 3\sec^2\theta\,d\theta$



$$\sqrt{u^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)}$$

$$= 3\sec\theta$$

$$= \int \frac{1}{3\tan\theta \cdot 3\sec\theta}\,3\sec^2\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

The integral $\int \frac{\sec\theta}{\tan\theta}\, d\theta$ is not of a standard type, so we convert everything to $\sin\theta$, $\cos\theta$:

$$\frac{\sec\theta}{\tan\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = \frac{1}{\sin\theta} = \csc\theta$$

Hence $\int \frac{\sec\theta}{\tan\theta}\, d\theta = \int \csc\theta\, d\theta =$

$$= \ln|\csc\theta - \cot\theta| + C$$

We use the triangle to "undo" the substitution: $\csc\theta = \frac{\sqrt{u^2+9}}{u}$, $\cot\theta = \frac{3}{u}$

$$= \ln\left|\frac{\sqrt{u^2+9} - 3}{u}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4x+13} - 3}{x+2}\right| + C$$

Hence $\int \frac{1}{(x+2)\sqrt{x^2+4x+13}}\, dx =$

$$= \frac{1}{3} \ln\left|\frac{\sqrt{x^2+4x+13} - 3}{x+2}\right| + C$$

$*$

Also, concerning the Homework Question #33, p. 483, Note that the average value of f(x) on an interval [a, b] is defined by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

which is the formula in a box on p. 452.

Homework Due Tues. 2/4 is all the Exercises listed on the syllabus For sections 7.3, 7.4. So please start on it as soon as possible since some of the problems on partial fraction decompositions may be more time consuming.

Section 7.4

Integration of Rational Functions, Partial Fractions Decompositions

We want to learn how to evaluate integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$, $Q(x)$ are polynomials.

The Main Step, called Partial Fractions Decomposition, is applied when the Degree of $P(x)$ is less than the Degree of $Q(x)$, i.e. the Degree of the Numerator is less than the Degree of the Denominator. A preliminary step of dividing $Q(x)$ into $P(x)$ is required in the opposite case when Deg. of $P(x) \geq$ Deg. of $Q(x)$.

We first do some examples which require the Preliminary Step of Division of the Denominator into the Numerator. These examples may also be helpful in clarifying the reason for such a Preliminary Step.

Evaluate $\displaystyle\int \frac{8x^3+3x}{2x+1}\,dx$:

We divide $2x+1$ into $8x^3+3x$:

$$\begin{array}{r l}
 & 4x^2-2x+\frac{5}{2} \quad \longleftarrow \text{ The Quotient} \\
2x+1 \,\big) & 8x^3+3x \\
 & \underline{8x^3+4x^2} \\
 & -4x^2 \\
 & \underline{-4x^2-2x} \\
 & 5x \\
 & \underline{5x+\frac{5}{2}} \\
 & -\frac{5}{2} \quad \longleftarrow \text{ The Remainder}
\end{array}$$

$$\boxed{\mathrm{Deg}\,(8x^3+3x) > \mathrm{Deg}\,(2x+1)}$$

Thus $4x^2-2x+\frac{5}{2}$ is the Quotient, $-\frac{5}{2}$ is the Remainder.

Thus we can write

$$\frac{8x^3+3x}{2x+1} = 4x^2-2x+\frac{5}{2}+\left(\frac{-\frac{5}{2}}{2x+1}\right)$$

$$= 4x^2-2x+\frac{5}{2}-\frac{5}{2}\cdot\frac{1}{2x+1}$$

and thus $\displaystyle\int \frac{8x^3+3x}{2x+1}\,dx =$

$$= \int\left(4x^2-2x+\frac{5}{2}-\frac{5}{2}\cdot\frac{1}{2x+1}\right)dx$$

$$= \frac{4}{3}x^3 - x^2 + \frac{5}{2}x - \frac{5}{2}\underbrace{\int\frac{1}{2x+1}\,dx}_{=\frac{1}{2}\ln|2x+1|+C}$$

$$= \frac{4}{3}x^3 - x^2 + \frac{5}{2}x - \frac{5}{4}\ln|2x+1| + C$$

$\ast$

Another Example:

Evaluate $\int \frac{8x^3+3x}{x^2+2}\,dx$

Again $\mathrm{Deg}(8x^3+3x) > \mathrm{Deg}(x^2+2)$, so we begin by doing the Division:

$$\begin{array}{r|l} & 8x \\ \hline x^2+2 & 8x^3+3x \\ & 8x^3+16x \\ \hline & -15x \end{array}$$

$8x \leftarrow$ The Quotient

$-15x \leftarrow$ the Remainder

Hence $\frac{8x^3+3x}{x^2+2} = 8x - \frac{15x}{x^2+2}$

and $\int \frac{8x^3+3x}{x^2+2}\,dx = \int\left(8x - \frac{15x}{x^2+2}\right)dx$

$= 4x^2 - 15\int \frac{x}{x^2+2}\,dx$

Substituting $u = x^2+2$, $du = 2x\,dx$,

$x\,dx = \frac{1}{2}du$

we obtain $\int \frac{x}{x^2+2}\,dx = \int \frac{1}{u}\cdot\frac{1}{2}du =$

$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+2) + C$

Hence $\int \frac{8x^3+3x}{x^2+2}\,dx = 4x^2 - \frac{15}{2}\ln(x^2+2) + C$

$\ast$

Evaluate $\int \frac{3x^2+2x+5}{x^2+2}\,dx$:

$$\begin{array}{r|l} & \;3 \\ x^2+2 & \overline{3x^2+2x+5} \\ & \underline{3x^2+6} \\ & 2x-1 \end{array}$$

Hence $\frac{3x^2+2x+5}{x^2+2} = 3 + \frac{2x-1}{x^2+2}$

and $\int \frac{3x^2+2x+5}{x^2+2}\,dx = \int\left(3+\frac{2x-1}{x^2+2}\right)dx$

$= 3x + \int \frac{2x}{x^2+2}\,dx - \int \frac{1}{x^2+2}\,dx$

$= 3x + \ln|x^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

Thus carrying out the Division is * helpful since in some cases we can use it to completely solve the problem. Sometimes the division can be carried out less formally: On p.7, $\frac{u-5}{u} = 1 - \frac{5}{u}$; On p.10, $\frac{6x^2}{3x^2+5} = 2 - \frac{10}{3x^2+5}$

Let's however consider a problem where the denominator is slightly more complicated:

Evaluate $\displaystyle\int \frac{8x^3+3x}{x^2+2x-3}\,dx$:

Carrying out the division:

$$
\begin{array}{r|l}
 & 8x - 16 \\
x^2+2x-3 & 8x^3 + 3x \\
 & \underline{8x^3+16x^2-24x} \\
 & -16x^2+27x \\
 & \underline{-16x^2-32x+48} \\
 & 59x-48
\end{array}
$$

Hence $$\frac{8x^3+3x}{x^2+2x-3} = 8x-16+\frac{59x-48}{x^2+2x-3}$$

Thus $$\int \frac{8x^3+3x}{x^2+2x-3}\,dx = 4x^2-16x+\int\frac{59x-48}{x^2+2x-3}\,dx$$

Now it is not so clear how to evaluate the remaining integral. Of course, the type of problem such as evaluating $\displaystyle\int\frac{59x-48}{x^2+2x-3}\,dx$ does not need to arise only as a result of division.

So we will now discuss how to evaluate integrals of the type $\int \frac{P(x)}{Q(x)} dx$ where Deg of $P(x)$ < Deg of $Q(x)$.

The <u>First Step</u> is to "completely" factor the Denominator $Q(x)$, i.e.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots$$
$$(c_1x^2+d_1x+e_1)(c_2x^2+d_2x+e_2)\cdots,$$

i.e. we write $Q(x)$ as a product of factors of degree 1 ($(a_1x+b_1)(a_2x+b_2)\cdots$), times a product of factors of degree 2, which have no real roots (have only imaginary roots), which is the second part, $(c_1x^2+d_1x+e_1)(c_2x^2+d_2x+e_2)\cdots$, of the factorization above. This is always possible.