Let's now evaluate $\int \frac{59x-48}{x^2+2x-3}\,dx$

(bottom p. 96). As stated on p. 97, we begin with factoring the Denominator,

$x^2+2x-3 := (x-1)(x+3)$

We now find a Partial Fraction Decomposition of $\frac{59x-48}{x^2+2x-3} = \frac{59x-48}{(x-1)(x+3)}$

This Partial Fraction Decomposition has the form $\frac{A}{x-1} + \frac{B}{x+3}$, where we need to find the values of A and B. To do this, we set

$$\frac{59x-48}{(x-1)(x+3)} = \underbrace{\frac{A}{x-1} + \frac{B}{x+3}}_{\text{Put on Common Denominator:}}$$

$$= \frac{Ax+3A+Bx-B}{(x-1)(x+3)} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

Since the circled expressions are equal, we obtain

$$A+B=59$$
$$3A-B=-48$$

We need to solve the system (bottom p.98):

Adding the Equations, we obtain $4A = 11$, hence $A = \frac{11}{4}$. Thus $B = 59 - A = 59 - \frac{11}{4}$, i.e. $B = \frac{225}{4}$. Hence

$$\int \frac{59x-48}{(x-1)(x+3)}\,dx = \int \left(\frac{11}{4}\cdot\frac{1}{x-1} + \frac{225}{4}\cdot\frac{1}{x+3} \right) dx =$$

$$= \frac{11}{4}\ln|x-1| + \frac{225}{4}\ln|x+3| + C \quad *$$

Another Example: Evaluate $\int \frac{3}{x^3-2x}\,dx$

We start by factoring x^3-2x:

$$= x(x^2-2) = x(x+\sqrt{2})(x-\sqrt{2})$$

thus we need to find a Partial Fraction Decomposition:

$$\frac{3}{x^3-2x} = \frac{3}{x(x+\sqrt{2})(x-\sqrt{2})} = \frac{A}{x} + \frac{B}{x+\sqrt{2}} + \frac{C}{x-\sqrt{2}}$$

Putting a Common Denominator:

$$= \frac{A(x^2-2) + Bx(x-\sqrt{2}) + Cx(x+\sqrt{2})}{x(x+\sqrt{2})(x-\sqrt{2})} =$$

$$= \frac{(A+B+C)x^2 + (C\sqrt{2} - B\sqrt{2})x - 2A}{x(x^2-2)}$$

Since the circled expressions are equal, we must have

$$A+B+C=0$$
$$C\sqrt{2}-B\sqrt{2}=0$$
$$-2A=3$$

Hence $A=-\frac{3}{2}$. Thus $-\frac{3}{2}+B+C=0$, i.e.

$$B+C=\frac{3}{2}$$

and $C\sqrt{2}-B\sqrt{2}=0$ (keeping the 2nd eq.)

Hence $C\sqrt{2}=B\sqrt{2}$, i.e. $B=C$

Thus $B+C=\frac{3}{2}$, $B=C$ yields

$2B=\frac{3}{2}$, $B=\frac{3}{4}$, hence also $C=\frac{3}{4}$.

Thus $\displaystyle\int\frac{3}{x^2-2x}dx=\int\left(-\frac{3}{2x}+\frac{3}{4}\cdot\frac{1}{x+\sqrt{2}}+\frac{3}{4}\cdot\frac{1}{x-\sqrt{2}}\right)dx$

$$=-\frac{3}{2}\ln|x|+\frac{3}{4}\ln|x+\sqrt{2}|+\frac{3}{4}\ln|x-\sqrt{2}|+C$$

$$=-\frac{3}{2}\ln|x|+\frac{3}{4}\ln|x^2-2|+C$$

The last two Examples show how to proceed when the denominator is a product of distinct linear factors (i.e. no repeated roots).

Let's do an Example with a quadratic Polynomial with complex Roots being a factor of the Denominator:

Evaluate $\displaystyle\int \frac{x}{(2x-1)(x^2+2x+2)}\,dx$

Since x^2+2x+2 has no real roots (only complex roots), we do not factor it any further. Instead we look for a Partial Fraction Decomposition in the form

$$\frac{x}{(2x-1)(x^2+2x+2)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+2x+2}$$

Put on a Common Denom.:

$$= \frac{Ax^2+2Ax+2A + 2Bx^2+2Cx-Bx-C}{(2x-1)(x^2+2x+2)}$$

$$= \frac{(A+2B)x^2+(2A+2C-B)x+2A-C}{(2x-1)(x^2+2x+2)}$$

Thus setting equal the coefficient of x^2, x, and the const. terms in the original expression $\frac{x}{(2x-1)(x^2+2x+2)}$ and in the one at the bottom of p. 101, we obtain

$$A+2B=0$$
$$2A+2C-B=1$$
$$2A-C=0$$

Thus $C=2A$, and substituting in the first two eqs., we obtain

$A+2B=0$, $2A+4A-B=1$,

i.e. $A+2B=0$, $6A-B=1$

Thus $A=-2B$, $-12B-B=1$,

$B=-\frac{1}{13}$, $A=\frac{2}{13}$, $C=\frac{4}{13}$

Hence

$$\frac{x}{(2x-1)(x^2+2x+2)} =$$

$$\frac{2}{13}\cdot\frac{1}{2x-1} + \frac{-\frac{1}{13}x+\frac{4}{13}}{x^2+2x+2} =$$

$$= \frac{2}{13(2x-1)} + \frac{4-x}{13(x^2+2x+2)}$$

Thus we need to evaluate

$$\int \frac{4-x}{x^2+2x+2}\,dx$$

We simplify x^2+2x+2 by completing to a square and making a substitution:

$$x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1,$$

$x+1=u$, $du=dx$ $\quad = u^2+1$

$x = u-1$

$$= \int \frac{4-(u-1)}{u^2+1}\,du =$$

$$= \int \left(\frac{5}{u^2+1} - \underbrace{\frac{u}{u^2+1}}\right) du$$

$$= 5\arctan u - \frac{1}{2}\ln|u^2+1| + C$$

$$= 5\arctan(x+1) - \frac{1}{2}\ln(x^2+2x+2) + C$$

substitute $t=u^2+1$, $u\,du = \frac{1}{2}dt$

Hence, from p. 102,

$$\int \frac{x}{(2x-1)(x^2+2x+2)}\,dx = \int\left(\frac{2}{13(2x-1)} + \frac{4-x}{13(x^2+2x+2)}\right)dx$$

$$= \frac{1}{13}\ln|2x-1| + \frac{5}{13}\arctan(x+1) - \frac{1}{26}\ln(x^2+2x+2) + C$$

Since finding the values of all the coefficients in a partial fraction decomposition can be quite time consuming, it is useful to practice writing out the form of the partial fraction decomposition (without finding the values of the coefficients). Example 7, p. 491 in the Book, is of this type. Exercises 1–6, p. 492, also are of this type. Also problems of this type will be on the Exams.

Example. Write out the form of the partial fraction decomposition for

$$\frac{x^3+5x+3}{\underbrace{(2x^2+x)}_{=x(2+x)}\underbrace{(x^2+2)}\underbrace{(x^2-2x+2)}}$$

only complex roots,
no real roots.
So we do not factor any further

Thus there will be a partial fraction with each of the denominators x, $2+x$, x^2+2, x^2-2x+2

Hence a partial fraction Decomposition has to have the form

$$\frac{x^3+5x+3}{x(2+x)(x^2+2)(x^2-2x+2)}$$

$$= \frac{A}{x} + \frac{B}{2+x} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{x^2-2x+2}$$

We will now consider cases when the * denominator has repeated roots.

Evaluate $\int \frac{x^2+5x-2}{x^3-2x^2}\,dx$

A complete factorization of the denominator is $x^3-2x^2 = x^2(x-2) (= x \cdot x \cdot (x-2))$

The partial fraction decomposition now has to be looked for in the form

$$\frac{x^2+5x-2}{x^2(x-2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-2}$$

Note that the factor x, which occurs in degree 2 in the denominator, now has two terms corresponding to it in the Decomposition, namely

$\frac{A}{x^2}$, $\frac{B}{x}$. (If x^3 occurred in the factorization, there would be $\frac{A}{x^3}$, $\frac{B}{x^2}$ and $\frac{C}{x}$). To find the numerical values of A, B, C we proceed as before: Putting on the common denom., we obtain

$$\frac{x^2+5x-2}{x^2(x-2)} = \frac{A(x-2)+Bx(x-2)+Cx^2}{x^2(x-2)}$$

$$= \frac{(B+C)x^2+(A-2B)x-2A}{x^2(x-2)}$$

Hence

$$B+C=1$$
$$A-2B=5$$
$$-2A=-2$$

Hence $A=1$, $\Rightarrow 1-2B=5 \Rightarrow B=-2$

$\Rightarrow C=1-B=3$

Hence $\int \frac{x^2+5x-2}{x^3-2x^2}dx = \int\left(\frac{1}{x^2}-\frac{2}{x}+\frac{3}{x-2}\right)dx$

$$= -\frac{1}{x}-2\ln|x|+3\ln|x-2|+C$$

*

Evaluate $\int \frac{3x^2-2x+2}{x^4+2x^2}\,dx$

Factoring the denominator : $x^4+2x^2 = x^2(x^2+2)$

Hence the form of the P. F. D. is

$$\frac{3x^2-2x+2}{x^4+2x^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+2}$$

$$= \frac{A(x^2+2) + Bx(x^2+2) + x^2(Cx+D)}{x^2(x^2+2)}$$

$$= \frac{Ax^2+2A+Bx^3+2Bx+Cx^3+Dx^2}{x^2(x^2+2)}$$

$$= \frac{(B+C)x^3+(A+D)x^2+2Bx+2A}{x^2(x^2+2)}$$

Equating the corresponding Coefficients, we obtain

$$B+C=0$$
$$A+D=3$$
$$2B=-2$$
$$2A=2,$$

Hence $A=1$, $B=-1$, $C=-B=1$,

$D = 3-A = 2$. Hence

$$\frac{3x^2-2x+2}{x^4+2x^2} = \frac{1}{x^2} - \frac{1}{x} + \frac{x+2}{x^2+2}$$

Hence $\int \frac{3x^2-2x+2}{x^4+2x^2}\,dx =$

$$\int\left(\frac{1}{x^2}-\frac{1}{x}+\frac{x+2}{x^2+2}\right)dx$$

$$= -\frac{1}{x} - \ln|x| + \frac{1}{2}\ln(x^2+2) + \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

*

Evaluate $\int \frac{x^4-x^3+8x^2-3x+16}{x(x^2+4)^2}\,dx$

Since has only complex roots (no real roots), the denominator is already factored. Also, the degree of the denom. is $=5$, whereas the degree of the numerator is 4, hence the denom. has higher degree than numer., and no division needs to be performed. Thus the P.F.D. should be looked for in the form

$$\frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Putting again on the common denom.,

we obtain

$$\frac{A(x^2+4)^2+(Bx+C)x(x^2+4)+x(Dx+E)}{x(x^2+4)^2}$$

Working the numerator only, we obtain

$$A(x^4+8x^2+16)+Bx^4+4Bx^2+Cx^3+4Cx+Dx^2+Ex$$

$$=(A+B)x^4+Cx^3+(8A+4B+D)x^2+(4C+E)x+16A$$

Comparing with $x^4-x^3+8x^2-3x+16$,

we see that $C=-1$, $A=1$

Then $A+B=1$, hence $B=0$

Finally $8A+4B+D=8 \rightarrow D=0$,

and $4C+E=-3$

i.e. $4(-1)+E=-3 \Rightarrow E=1$.

Hence, from page 108,

$$\frac{x^4-x^3+8x^2-3x}{x(x^2+4)^2}$$

$$=\underbrace{\frac{1}{x}-\frac{1}{x^2+4}}+\frac{1}{(x^2+4)^2}$$

Integrating the first two terms is easy,

so we will focus on integrating $\dfrac{1}{(x^2+4)^2}$

There are two methods for evaluating

$$\int \frac{1}{(x^2+4)^2}\,dx$$

(1) We can use the trigonometric substitution $\underline{x = 2\tan\theta}$ we have learned in section 7.3 to deal with expressions containing $\sqrt{x^2+4}$: Thus $dx = 2\sec^2\theta\, d\theta$,

$$(x^2+4)^2 = (4\tan^2\theta + 4)^2 = (4\sec^2\theta)^2 = 16\sec^4\theta,$$

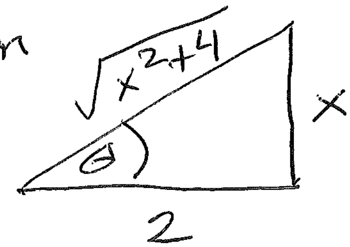
and $$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{16\sec^4\theta}\, 2\sec^2\theta\, d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\, d\theta =$$

$$= \frac{1}{8}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C$$

Now we use the triangle to "undo" the substitution



, $\theta = \tan^{-1}\left(\frac{x}{2}\right)$

Thus $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}$

$$= \frac{4x}{x^2+4}$$

Hence $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\,\frac{x}{x^2+4} + C$

Thus finally, from p. 109,

$$\int \frac{x^4 - x^3 + 8x^2 - 3x}{x(x^2+4)^2}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{1}{x^2+4} + \frac{1}{(x^2+4)^2}\right) dx$$

$$= \ln|x| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\,\frac{x}{x^2+4} + C$$

$$= \ln|x| - \frac{7}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{8}\,\frac{x}{x^2+4} + C \qquad *$$

Let us return to the integral $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$, showing a <u>Method (2)</u> for evaluating it. This second method is less direct, involving a bit of a trick like evaluating $\int e^{ax}\sin bx\,dx$, $\int e^{ax}\cos bx\,dx$, and $\int \sec^3 x\,dx$.

Method(2) for evaluating $\int \frac{1}{(x^2+4)^2}dx$

We begin with $\int \frac{1}{x^2+4}dx$. We of course know that $\int \frac{1}{x^2+4}dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

However, we try to evaluate this integral using Integration by Parts:

$\int \underbrace{\frac{1}{x^2+4}}_{u} \underbrace{dx}_{dv}$ $\qquad dv = dx \rightarrow v = x,$

$\qquad u = \frac{1}{x^2+4},\ du = -\frac{2x}{(x^2+4)^2}dx,$

$$= \underbrace{\frac{x}{x^2+4}}_{uv} - \int \underbrace{x}_{v} \cdot \underbrace{\left(-\frac{2x}{(x^2+4)^2}dx\right)}_{du}$$

$$= \frac{x}{x^2+4} + 2\int \frac{x^2}{(x^2+4)^2}dx$$

Next $\int \frac{x^2}{(x^2+4)^2}dx = \int \frac{x^2+4}{(x^2+4)^2}dx - \int \frac{4}{(x^2+4)^2}dx$

$$= \int \frac{1}{x^2+4}dx - 4\int \frac{1}{(x^2+4)^2}dx$$

Hence $\int \frac{1}{x^2+4}\,dx$

$$= \frac{x}{x^2+4} + 2\left(\int \frac{1}{x^2+4}\,dx - 4\int \frac{1}{(x^2+4)^2}\,dx\right)$$

$$= \frac{x}{x^2+4} + 2\int \frac{1}{x^2+4}\,dx - 8\int \frac{1}{(x^2+4)^2}\,dx$$

We now solve the above equation for $\int \frac{1}{(x^2+4)^2}\,dx$, obtaining

$$8\int \frac{1}{(x^2+4)^2}\,dx = \frac{x}{x^2+4} + \int \frac{1}{x^2+4}\,dx$$

$$= \frac{x}{x^2+4} + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C,$$

Hence $\int \frac{1}{(x^2+4)^2}\,dx$

$$= \frac{1}{8}\,\frac{x}{x^2+4} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C,$$

which agrees with the result on p. 111 obtained using the Method (1).

*

A Comment.

Let us consider the partial Fraction Decomposition of $\dfrac{1}{(4x-2)(1-4x^2)}$

Factoring the denominator, we obtain

$(4x-2)(1-4x^2) = (4x-2)(1+2x)(1-2x)$

So we have three distinct (?) $\underbrace{\text{factors}}_{\text{linear}}$

$4x-2,\ 1+2x,\ 1-2x$

So should we look for a Decomposition in the form

$$\frac{1}{(4x-2)(1-4x^2)} = \frac{A}{4x-2} + \frac{B}{1+2x} + \frac{C}{1-2x} \ ?$$

The Answer is NO!

We need to consider whether or not any of the linear factors have a common root, and the factors

$4x-2,\ 1-2x$ both have the common root $\frac{1}{2}$ ($4x-2=0 \Rightarrow x=\frac{1}{2},\ 1-2x=0 \Rightarrow x=\frac{1}{2}$)

Thus the denominator contains the factor $(2x-1)^2$; that is,

$$(4x-2)(1-4x^2) = (4x-2)(1+2x)(1-2x) =$$
$$= 2(2x-1)(1+2x)(-1)(2x-1) =$$
$$= -2(2x-1)^2(1+2x)$$

Thus the form of P.F.D. is

$$\frac{1}{(4x-2)(1-4x^2)} = -\frac{1}{2(2x-1)^2(1+2x)}$$

$$= \frac{A}{(2x-1)^2} + \frac{B}{2x-1} + \frac{C}{1+2x}$$

The fraction $\dfrac{1}{(x-\frac{1}{2})(1-4x^2)}$ would be treated similarly, and the form of the P.F.D. looks the same as above.

Similar comments apply to fractions whose denominator has quadratic factors with complex roots.