Sect. 7.7, Approx. Integration.

For this section, you can use calculators for the Homework, but you don't have to.

On the Exams, of course, as stated on the syllabus, calculators are not allowed at all.

What Needs to Be Memorized in this section:

You Need to MEMORIZE the content of 5 (five) Boxes:

(1) The Formula for the Midpoint Rule; Box, bottom p. 507 in the Book

(2) The Formula for the Trapezoidal Rule; Box, top p. 508 in the Book

(3) The Formulas for the Error Bounds for the Midpoint Rule and the Trapezoidal Rule; Box near top of p. 510 in the Book

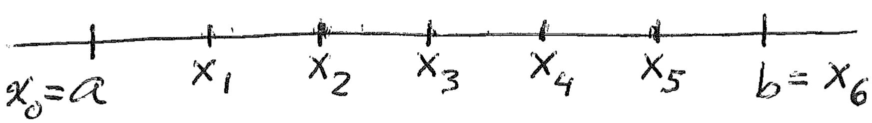
(4) The Formula for Simpson's Rule; Box, top p. 513 in the Book

(5) The Formula for the Error Bound for Simpson's Rule; Box, bottom p. 514.

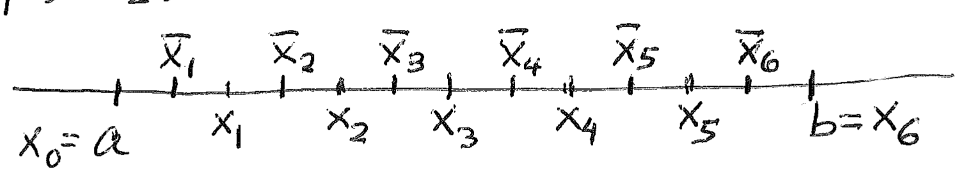
The three Rules are used to calculate an approximate value of $\int_a^b f(x)dx$.

To calculate such an approximate value we divide the interval $[a, b]$ into a specified number n of equal lengths intervals (of length $\Delta x = \frac{1}{n}(b-a)$). The endpoints of these intervals are denoted $x_0, x_1, x_2, \ldots, x_n$ where $x_0 = a$ and $x_n = b$. For Simpson's Rule, the n has to be even.

The following picture is for $n = 6$:



For the Midpoint Rule we need to use the midpoints of the intervals in the picture above, which are denoted by $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n$



The formula for x_i is

$$\boxed{x_i = a + \underbrace{i(\Delta x)}_{i \text{ times } \Delta x} = a + \frac{i(b-a)}{n}} \qquad 0 \le i \le n$$

This formula is in the Box at the top of p. 108.

The Formula that we use for $\bar{x}_i$ is

$$\boxed{\bar{x}_i = a + (2i-1)\frac{\Delta x}{2} = a + (2i-1)\frac{b-a}{2n} \qquad 1 \le i \le n}$$

$\bar{x}_i$ belongs to the interval $[x_{i-1}, x_i]$,
i.e., $\bar{x}_i$ is the midpoint of

Thus for the picture at the bottom of p. 151 we obtain

$$\bar{x}_1 = a + \frac{b-a}{12}, \quad \bar{x}_2 = a + 3 \cdot \frac{b-a}{12},$$

$$\bar{x}_3 = a + 5 \cdot \frac{b-a}{12}, \quad \bar{x}_4 = a + 7 \cdot \frac{b-a}{12},$$

$$\bar{x}_5 = a + 9 \cdot \frac{b-a}{12}, \quad \bar{x}_6 = a + 11 \cdot \frac{b-a}{12}$$

Thus if we partition the interval $[a, b]$ into n equal intervals using points $x_0, x_1, x_2, \dots x_n$ as explained above, and use the formula for one of the three Rules (in one of the Boxes as explained on p. 150 of these Notes), we obtain the corresponding approximate value of $\int_a^b f(x)dx$. This approximate value is denoted M_n if we use the Midpoint Rule, T_n for Trapez. Rule, and S_n for Simpson's Rule.

The Approximations become more accurate as n gets larger.

The difference between the exact value and an approximate value is called the error: Thus

$$E_M = \int_a^b f(x)dx - M_n,$$ which

is the Error if the Midpoint Rule is used (we don't put n on the E).

Similarly $E_T = \int_a^b f(x)dx - T_n$,

if we use the Trapez. Rule,

and $E_S = \int_a^b f(x)dx - S_n$

for Simpson's Rule.

Examples. We will use all three Rules with $n=6$ to calculate an approximate value of $\int_0^3 e^{x^2}dx$.

Thus $\Delta x = \frac{3-0}{6} = 0.5 = \frac{1}{2}$

Furthermore, using the formula at the top of p.152, we obtain

$x_i = 0 + 0.5i$, $0 \le i \le 6$,

Hence $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$.

Furthermore $\bar{x}_i = 0 + (2i-1)\frac{3-0}{12} =$

$= (2i-1)(0.25)$, $1 \le i \le 6$,

Hence $\bar{x}_1 = 0.25$,

$\bar{x}_2 = (2*2-1)(0.25) = 0.75$,

$\bar{x}_3 = (2*3-1)(0.25) = 1.25$,

Then $\bar{x}_4 = 1.75$, $\bar{x}_5 = 2.25$, $\bar{x}_6 = 2.75$

We now calculate M_6, i.e. use the Midpoint Rule with $n=6$:

We use the formula in the Box on p. 507 with $n=6$:

$$M_6 = \Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+f(\bar{x}_4)+f(\bar{x}_5)+f(\bar{x}_6)\right)$$

$$= \frac{1}{2}\left(f(0.25)+f(0.75)+f(1.25)+f(1.75)+f(2.25)+f(2.75)\right)$$

$$= \frac{1}{2}\left(e^{(0.25)^2}+e^{(0.75)^2}+e^{(1.25)^2}+e^{(1.75)^2}+e^{(2.25)^2}+e^{(2.75)^2}\right)$$

If this is a Question on an Exam you can stop, since going any further requires the use of a calculator. You can also stop on the Homework unless you want to finish it using a calculator (e.g. if you want to check your answer using the answers in the back of the Book).

Next we calculate T_6

Using the formula in the Box on p. 508, we obtain

$$T_6 = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right]$$

$$= \frac{1}{4}\left[e^0 + 2e^{(0.5)^2} + 2e^{1^2} + 2e^{(1.5)^2} + 2e^{2^2} + 2e^{(2.5)^2} + e^{3^2}\right]$$

Again, if we can't use a calculator we can stop here.

Finally we use Simpson's Rule. Using the Formula in the Box on p. 513, we obtain

$$S_6 = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\right]$$

$$= \frac{1}{6}\left[e^0 + 4e^{(0.5)^2} + 2e^{1^2} + 4e^{(1.5)^2} + 2e^{2^2} + 4e^{(2.5)^2} + e^{3^2}\right]$$

and again stop, unless use calculator.

Let's change the Question very slightly, by changing the interval $[0,3]$ to $[0,4]$. I.e. let's find M_6, T_6, S_6 for $\int_0^4 e^{x^2} dx$. The only difference from what we did is that it is better if we don't use decimal notation since $\Delta x = \frac{4}{6} = 0.6666\ldots = 0.\overline{6}$, so it is easier to write things down if we use $\Delta x = \frac{4}{6} = \frac{2}{3}$.

Thus as before we need to work out x_i $(0 \le i \le 6)$ and $\bar{x}_i$ $(1 \le i \le 6)$. We find

$$x_i = 0 + i\Delta x = \frac{2}{3}i, \quad 0 \le i \le 6,$$

hence $x_0 = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = \frac{6}{3} = 2$, $x_4 = \frac{8}{3}$, $x_5 = \frac{10}{3}$, $x_6 = \frac{12}{3} = 4$

and $\bar{x}_i = 0 + (2i-1)\frac{\Delta x}{2} = \frac{2i-1}{3}$, $1 \le i \le 6$

hence $\bar{x}_1 = \frac{1}{3}$, $\bar{x}_2 = \frac{3}{3} = 1$, $\bar{x}_3 = \frac{5}{3}$, $\bar{x}_4 = \frac{7}{3}$, $\bar{x}_5 = \frac{9}{3} = 3$, $\bar{x}_6 = \frac{11}{3}$

Hence

$$M_6 = \Delta x \cdot \left(f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4) + f(\bar{x}_5) + f(\bar{x}_6)\right)$$

$$= \frac{2}{3}\left(f\left(\tfrac{1}{3}\right) + f(1) + f\left(\tfrac{5}{3}\right) + f\left(\tfrac{7}{3}\right) + f(3) + f\left(\tfrac{11}{3}\right)\right)$$

$$= \frac{2}{3}\left(e^{\left(\frac{1}{3}\right)^2} + e^{1^2} + e^{\left(\frac{5}{3}\right)^2} + e^{\left(\frac{7}{3}\right)^2} + e^{3^2} + e^{\left(\frac{11}{3}\right)^2}\right)$$

$$T_6 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6)\right]$$

$$= \frac{1}{3}\left(e^0 + 2e^{\left(\frac{2}{3}\right)^2} + 2e^{\left(\frac{4}{3}\right)^2} + 2e^{2^2} + 2e^{\left(\frac{8}{3}\right)^2} + 2e^{\left(\frac{10}{3}\right)^2} + e^{4^2}\right)$$

$$S_6 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$= \frac{2}{9}\left[e^0 + 4e^{\left(\frac{2}{3}\right)^2} + 2e^{\left(\frac{4}{3}\right)^2} + 4e^{2^2} + 2e^{\left(\frac{8}{3}\right)^2} + 4e^{\left(\frac{10}{3}\right)^2} + e^{4^2}\right]$$

*

The Error Bounds.

The Error Bounds for the Midpoint and Trapezoidal Rules are given in the Box on p. 510: If K is a bound on $|f''(x)|$ on $[a,b]$, then

$$|E_M| \le \frac{K(b-a)^3}{24n^2}, \quad |E_T| \le \frac{K(b-a)^3}{12n^2}$$

For Simpson's Rule: K a bound on $|f^{(4)}(x)|$ on $[a,b]$: $$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Thus let's find bounds on E_M, E_T for the first Question we did, i.e. $\int_0^3 e^{x^2}dx$, for arbitrary n (not just $n=6$)

We need to calculate $(e^{x^2})''$.

First $(e^{x^2})' = 2xe^{x^2}$

Hence $(e^{x^2})'' = (2xe^{x^2})' =$

$= 2e^{x^2} + 4x^2e^{x^2}$

This expression attains maximum value on $[0,3]$ at $x=3$, which is

$$2e^9 + 36e^9 = 38e^9 = K$$

Thus $\underline{\underline{|E_M| \le}} \frac{K(b-a)^3}{24n^2} = \frac{38e^9 \cdot 27}{24n^2} = \underline{\underline{\frac{171e^9}{4n^2}}}$

and $\underline{\underline{|E_T| \le}} \frac{K(b-a)^3}{12n^2} = \frac{38e^9 \cdot 27}{12n^2}$

$$= \frac{19e^9 \cdot 9}{n^2} = \underline{\underline{\frac{171e^9}{2n^2}}}$$

To calculate a bound on $|E_S|$ we need to calculate $(e^{x^2})''''$.

From the top of p. 160, we obtain

$$(e^{x^2})''' = (2e^{x^2} + 4x^2e^{x^2})' =$$

$$= 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} = 12xe^{x^2} + 8x^3e^{x^2}$$

Hence

$$(e^{x^2})'''' = (12xe^{x^2} + 8x^3e^{x^2})' =$$

$$= 12e^{x^2} + 24x^2e^{x^2} + 24x^2e^{x^2} + 16x^4e^{x^2}$$

$$= 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$$

and the max of $(e^{x^2})''''$ on $[0,3]$ is attained at $x = 3$, i.e.

$$K = 12e^9 + 48 * 9e^9 + 16 * 81e^9 = 1740e^9$$

Hence

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{1740 * 3^5 e^9}{180n^4} = \frac{174 * 3^5 e^9}{18n^4}$$

$$= \frac{87 * 3^3 e^9}{n^4} = \frac{2349\,e^9}{n^4}$$

<u>Question.</u> $\int_0^3 e^{x^2}dx$, use Trapez. Rule. How large should we choose n so that $|E_T| \le 0.1$?

Solution. From p. 160, we know

$$|E_T| \le \frac{171e^9}{2n^2}$$

Thus we can guarantee that $|E_T| \le 0.1$ if

$$\frac{171e^9}{2n^2} \le 0.1$$

i.e. $1710 \cdot e^9 \le 2n^2$,

i.e. $n^2 \ge \frac{1}{2} \cdot 1710 \cdot e^9$

i.e. $n \ge \sqrt{\frac{1}{2} \cdot 1710 \cdot e^9} \approx 2{,}632.14$

hence we should make $n = 2{,}633$

*