Sect. 7.8, Improper Integrals.

An ordinary type of definite integral has the form $\int_a^b f(x)dx$ where f is continuous on $[a,b]$, i.e. "proper", and a, b are finite numbers.

Improper integrals are of the type

Type 1

where one of a, b is an infinity (i.e. a finite, $b=+\infty$) or $a=-\infty$, b finite, or both are infinity, i.e. $a=-\infty$, $b=+\infty$); or a, b both finite and f is discontinuous, where the discontinuity can be at a, or at b, or at some point c such that $a<c<b$.

Type 2

Proper Integrals ("Ordinary" Integrals) always have a value (i.e. are equal to some number).

Improper Integrals may, or may not have a value. If an improper integral has a value (can be assigned a value), it is called convergent. If it cannot be assigned a value, it is called divergent. By a value we mean here a finite number.

Example. Determine whether $\int_2^\infty \frac{1}{x^2}dx$ is convergent or divergent, and if convergent, evaluate it.

We start by evaluating the ordinary integral $\int_2^t \frac{1}{x^2}dx$ where t is any number ≥ 2.

Firstly, the antiderivative,

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + C, \quad \underbrace{-\frac{1}{t} - \left(-\frac{1}{2}\right)}$$

hence $\int_2^t \frac{1}{x^2}dt = \left(-\frac{1}{x}\right)\Big]_2^t = -\frac{1}{t} + \frac{1}{2}$

We now consider the limit

$$\lim_{t\to\infty}\int_2^t \frac{1}{x^2}\,dt = \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{2}\right)=\frac{1}{2}$$

Since this limit exists and is finite,

we say that $\int_2^\infty \frac{1}{x^2}\,dt$ is convergent,

and its value $= \frac{1}{2}$

Let's consider now the improper integral $\int_3^\infty \frac{1}{x}\,dx$. Again

$$\int \frac{1}{x}\,dx = \ln|x| + C,$$

hence $\int_3^t \frac{1}{x}\,dx = \ln|t| - \ln 3 =$

$\ln t - \ln 3$ since we want to assume that $t \geq 3$, hence $|t| = t$.

Hence $\lim_{t\to\infty}\int_3^t \frac{1}{x}\,dx = \lim_{t\to\infty}(\ln t - \ln 3) = \infty$

Hence the limit is not finite, and thus the integral $\int_3^\infty \frac{1}{x}\,dx$ is <u>divergent</u>.

Thus no value is assigned to this integral.

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The Example 4, p. 522 in the Book shows that for <u>any</u> $a > 0$,

$$\int_a^\infty \frac{1}{x^p}\,dx$$

is convergent if $p > 1$,

and divergent if $p \leq 1$.

This fact is stated in the Box near the top of p. 523. It is stated there for $a = 1$, however it remains true for every $a > 0$.

It is very important to <u>Memorize</u> this Fact.

Determine if the improper integral

$$\int_{\pi^2}^{\infty} \sqrt{x}\cos\sqrt{x}\,dx$$ is convergent or divergent; if convergent, evaluate it.

$\int \sqrt{x}\cos\sqrt{x}\,dx$: As you should know by now, we make the substitution

$$u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,dx,$$

hence $dx = 2u\,du$.

Thus

$$= \int (u\cos u)\,2u\,du = 2\int u^2\cos u\,du$$

Integrate by Parts

$$= 2u^2\sin u - 2\int 2u\sin u\,du$$

$$= 2u^2\sin u - 4\left[u(-\cos u) - \int(-\cos u)\,du\right]$$

$$= 2u^2\sin u + 4u\cos u - 4\sin u + C$$

$$= 2x\sin\sqrt{x} + 4\sqrt{x}\cos\sqrt{x} - 4\sin\sqrt{x} + C$$

Hence

$$\int_{\pi^2}^{t} \sqrt{x}\cos\sqrt{x}\,dx =$$

$$= 2t\sin\sqrt{t} + 4\sqrt{t}\cos\sqrt{t} - 4\sin\sqrt{t}$$

$$-\left(2\pi^2\sin\pi + 4\pi\cos\pi - 4\sin\pi\right) =$$

$$= 2t\sin\sqrt{t} + 4\sqrt{t}\cos\sqrt{t} - 4\sin\sqrt{t} - 4\pi$$

Hence we can investigate

$$\lim_{t\to\infty} \int_{\pi^2}^{t} \sqrt{x}\cos\sqrt{x}\,dx =$$

$$= \lim_{t\to\infty} \overbrace{\left(2t\sin\sqrt{t} + 4\sqrt{t}\cos\sqrt{t} - 4\sin\sqrt{t} - 4\pi\right)}$$

We need to decide whether this limit exists and is finite. For this limit to exist and be finite means that as t becomes very large, the quantity in the parentheses should be getting closer and closer to some finite number. That this is not the case can be seen by setting $t = 4n^2\pi^2$, where n is any positive integer. For such values of t we see that the quantity in the parentheses

$$= 8n^2\pi^2 \underbrace{\sin 2n\pi}_{=0} + 4(2n\pi)\underbrace{\cos 2n\pi}_{1} \underbrace{- 4\overbrace{\sin 2n\pi}^{=0}}_{-4\pi}$$

$$= 8n\pi - 4\pi = (8n-4)\pi$$

But as $n \to \infty$, the quantities $(8n-4)\pi$ approach infinity. Thus there are arbitrarily large values of t such that

$$\int_{\pi^2}^{t} \sqrt{x}\cos\sqrt{x}\, dx$$

likewise become arbitrarily large. Hence $\lim\limits_{t\to\infty} \int_{\pi^2}^{t} \sqrt{x}\cos\sqrt{x}\, dx$

could at best be $= \infty$ (if it exists as an infinite limit).

It thus follows that $\int_{\pi^2}^{t} \sqrt{x}\cos\sqrt{x}\,dx$ is divergent.

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Determine whether the improper integral $\int_0^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx$ is convergent or divergent, and if convergent, evaluate it.

This is an improper integral of type 2, since the interval $[0,1]$ is finite, but the function $\frac{e^{-\frac{1}{x}}}{x^3}$ is not defined at 0, hence not continuous at 0. (Actually, there is an extra subtlety here, which we will comment on later, but shall initially treat the integral as improper.)

We need to consider the integrals

$$\int_t^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx \quad , \quad \text{where} \quad 0 < t < 1 ,$$

which are ordinary integrals (of a continuous function), then we consider $\lim\limits_{t \to 0+} \int_t^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx$

Thus we evaluate $\int \frac{e^{-\frac{1}{x}}}{x^3}\,dx$

We make the substitution $u = \frac{1}{x}$, hence $du = -\frac{1}{x^2}dx = -\left(\frac{1}{x}\right)^2 dx$

$= -u^2 dx$, hence $dx = -\frac{1}{u^2}du$.

Thus $\int \frac{e^{-\frac{1}{x}}}{x^3}\,dx = \int e^{-\frac{1}{x}} \cdot \left(\frac{1}{x}\right)^3 dx$

$= \int e^{-u} u^3 \left(-\frac{1}{u^2}\right) du = -\int u e^{-u} du$

$= -\left(u(-e^{-u}) - \int (-e^{-u}) du \right) =$

$$= ue^{-u} - \int e^{-u}\,du = ue^{-u} + e^{-u} + C$$

$$= \frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}} + C$$

Hence $\displaystyle\int_t^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx$

$$= \frac{e^{-1}}{1^3} + e^{-1} - \left(\frac{1}{t}e^{-\frac{1}{t}} + e^{-\frac{1}{t}}\right)$$

$$= 2e^{-1} - e^{-\frac{1}{t}} - \frac{1}{t}e^{-\frac{1}{t}}$$

We need to evaluate the limit of this expression as $t \to 0+$

Firstly, as $t \to 0+$, $\frac{1}{t} \to \infty$, and thus $-\frac{1}{t} \to -\infty$, hence $e^{-\frac{1}{t}} \to 0$.

Thus we need to evaluate the limit of third term $\frac{1}{t}e^{-\frac{1}{t}}$ (as $t \to 0+$).

$$= \frac{e^{-\frac{1}{t}}}{t}$$

Both numerator and denominator $\to 0$,

hence we can use L'Hospital's Rule.
However, this is not immediately productive since differentiating $e^{-\frac{1}{t}}$ yields the factor $\frac{1}{t^2}$. Instead we make the same kind of substitution $\frac{1}{t} = u$, as we did on p. 181 to evaluate $\int \frac{e^{-\frac{1}{x}}}{x^3}\,dx$.

Thus $u \to \infty$ as $t \to 0+$.

Hence $\lim_{t \to 0+} \frac{e^{-\frac{1}{t}}}{t} = \lim_{u \to \infty} \frac{e^{-u}}{\frac{1}{u}}$

$= \lim_{u \to \infty} u e^{-u} = \lim_{u \to \infty} \frac{u}{e^u} \underset{\text{L'H. Rule}}{=}$

$= \lim_{u \to \infty} \frac{1}{e^u} = 0.$

Hence $\lim_{t \to 0+} \int_t^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx = \lim_{t \to 0+} (2e^{-1} - \underbrace{e^{-\frac{1}{t}}}_{\to 0} - \underbrace{\tfrac{1}{t}e^{-\frac{1}{t}}}_{\to 0})$

$= 2e^{-1}$, hence the improper integral $\int_0^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx$ is convergent, and $= 2e^{-1}$. ✗

Comment. The integrand $\frac{e^{-\frac{1}{x}}}{x^3}$ in the improper integral $\int_0^1 \frac{e^{-\frac{1}{x}}}{x^3}\,dx$ has the property that $\lim_{t\to 0+} \frac{e^{-\frac{1}{x}}}{x^3} = 0$ which can be shown similarly as we showed on p. 183 that $\lim_{t\to 0+} \frac{e^{-\frac{1}{t}}}{t} = 0$.

Thus if we define

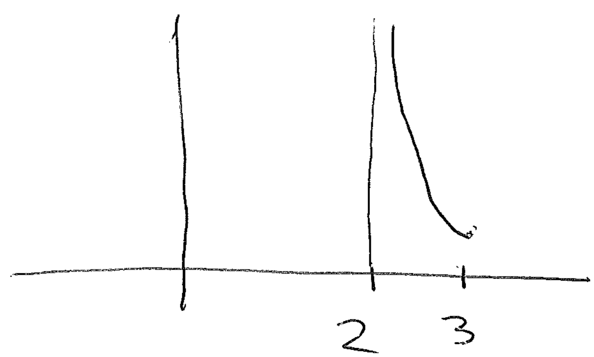
$$f(x) = \begin{cases} \frac{e^{-\frac{1}{x}}}{x^3}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

we obtain a continuous function, and $\int_0^1 f(x)\,dx$ is then an ordinary proper integral which has the same value as the original improper integral. However, in order to evaluate $\int_0^1 f(x)dx$ we need to proceed as we did on pages 181-183, evaluating the original improper integral. *

Evaluate the improper integral

$$\int_2^3 \frac{1}{\sqrt{x^2-4}}\,dx$$ (First determining whether it is convergent or divergent)

This integral is improper since the function $\frac{1}{\sqrt{x^2-4}}$ is not continuous at 2 — $\lim_{x\to 2+} \frac{1}{\sqrt{x^2-4}} = \infty$, and also the function is undefined when $x = 2$.



Thus we need to evaluate the proper integral $\int_t^3 \frac{1}{\sqrt{x^2-4}}\,dx$

for t such that $2 < t < 3$

and decide whether or not the

$\lim_{t\to 2+} \int_t^3 \frac{1}{\sqrt{x^2-4}}\,dx$ exists and is finite.

The antiderivative $\int \frac{1}{\sqrt{x^2-4}}\,dx$

$= \ln|x+\sqrt{x^2-4}| + C$

which is evaluated in Example 5, p. 481 in the Book (using the trig. substitution $x = 2\sec\theta$).

Hence $\int_t^3 \frac{1}{\sqrt{x^2-4}}\,dx =$

$= \ln|3+\sqrt{5}| - \ln|t+\sqrt{t^2-4}|$

$= \ln(3+\sqrt{5}) - \ln(t+\sqrt{t^2-4})$

since the quantities inside absolute value signs are > 0. Hence

$\lim_{t\to 2+} \int_t^3 \frac{1}{\sqrt{x^2-4}}\,dx =$

$\lim_{t\to 2+} \left[\ln(3+\sqrt{5}) - \ln(t+\sqrt{t^2-4})\right]$

$= \ln(3+\sqrt{5}) - \ln 2$

Hence $\lim_{t \to 2+} \int_t^3 \frac{1}{\sqrt{x^2-4}}\,dx$ exists

and is finite. Hence the improper

integral $\int_2^3 \frac{1}{\sqrt{x^2-4}}\,dx$ is convergent

and $= \ln(3+\sqrt{5}) - \ln 2$.

Determine whether $\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx$

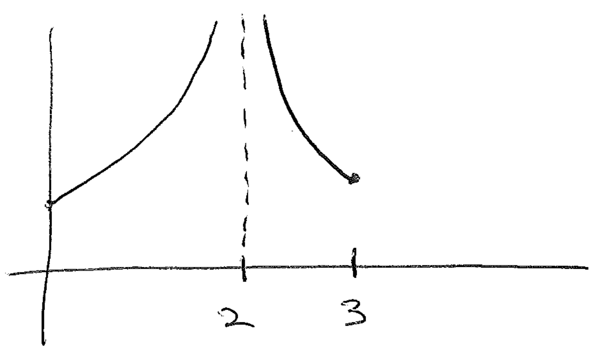
is convergent or divergent, and if

convergent, evaluate it.

The integral is improper

because the function $\frac{1}{(x-2)^{2/3}}$

is discontinuous at $x=2$, and $0<2<3$.



We need to break it up into two improper integrals,

$$\boxed{\int_0^2 \frac{1}{(x-2)^{2/3}}\,dx \,, \quad \int_2^3 \frac{1}{(x-2)^{2/3}}\,dx}$$

The original improper integral

$$\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx \quad \text{is convergent}$$

if <u>both</u> integrals in the box are convergent,

and it is divergent if one or both of the improper integrals in the box is/are divergent.

Thus we find the antiderivative

$$\int \frac{1}{(x-2)^{2/3}}\,dx = \int (x-2)^{-2/3}\,dx$$

$$= \frac{(x-2)^{1/3}}{1/3} + C = 3(x-2)^{1/3} + C$$

We now deal separately with each of the integrals in the BOX. First $\int_0^2 \frac{1}{(x-2)^{2/3}}\,dx$. Thus, let $0<t<2$:

$$\int_0^t \frac{1}{(x-2)^{2/3}}\,dx = 3(x-2)^{1/3}\Big]_0^t$$

$$= 3(t-2)^{1/3} - 3(-2)^{1/3}$$

$$= 3(t-2)^{1/3} + 3(2)^{1/3}$$

Hence $\lim\limits_{t\to 2-} \int_0^t \frac{1}{(x-2)^{2/3}}\,dx =$

$$= \lim_{t\to 2-}\left(3(t-2)^{1/3} + 3(2)^{1/3}\right) = \underline{\underline{3(2)^{1/3}}}$$

Similarly $\lim\limits_{t\to 2+} \int_t^3 \frac{1}{(x-2)}\,dx$

$$= \lim_{t\to 2+} 3(x-2)^{1/3}\Big]_t^3 =$$

$$\lim_{t\to 2+}\left(3 - 3(t-2)^{1/3}\right) = \underline{\underline{3}}$$

Thus both improper integrals in the box on p. 188 are convergent.

Hence the original integral

$$\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx \text{ is convergent.}$$

The values of this improper integral is then defined to be the sum of the values of the two improper integrals: $\int_0^3 \frac{1}{(x-2)^{2/3}}\,dx =$

$$= \int_0^2 \frac{1}{(x-2)^{2/3}}\,dx + \int_2^3 \frac{1}{(x-2)^{2/3}}\,dx$$

$$= \underline{\underline{3(2)^{1/3} + 3}}$$

$*$

A Comparison Test.

We will refer to it as the Comparison Test, box p. 525 in the Book. There are 2 parts, part (a) for convergence, part (b) for divergence:

Let f, g be continuous functions and $f(x) \geq g(x)$ for all $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent, i.e. if the integral of the larger function is convergent, then so is the integral of the smaller function.

(b) If $\int_a^\infty g(x)dx$ is divergent,
then $\int_a^\infty f(x)dx$ is divergent,
i.e. if the integral of the smaller function is divergent, then the integral of the larger function is divergent.

<u>Example.</u> Determine whether the improper integral $\int_5^\infty \frac{1}{x^2(3+\cos x)}dx$ is convergent or divergent.

We have to find an integral which <u>we know</u> to be convergent or divergent. We consider the integral $\int_5^\infty \frac{1}{x^2}dx$ — see p. 176 of these Notes, where we stated that this integral is convergent.

Thus we would like to show that, for $x \geq 5$,

$$\frac{1}{x^2(3+\cos x)} \leq \frac{1}{x^2}$$

equivalently $x^2(3+\cos x) \geq x^2$

since $x^2 > 0$, we can cancel,

i.e. it suffices to show that

$$3 + \cos x \geq 1$$

i.e. $\cos x \geq -2$

which we know to be true since $\cos x \geq -1$ for all x.

Moreover $\frac{1}{x^2} \geq 0$ for all x,

and likewise $\frac{1}{x^2(3+\cos x)} \geq 0$,

since we just showed that $3+\cos x \geq 1$.

Thus we conclude that

$$\int_5^\infty \frac{1}{x^2(3+\cos x)}\,dx$$

is convergent. $*$

Comment. Using the Comparison test to prove that an Improper Integral is convergent does not, by itself, yield the value of the integral.