Use the Comparison Test to determine whether the integral $\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is convergent or divergent.

The first step is to decide what to do, that is should we try to prove that the integral is convergent, or should we try to prove that it is divergent?

We make this decision by counting the powers of x. We consider the power of x in $\sqrt{x^5-32}$ to be $\sqrt{x^5} = x^{5/2}$, thus the entire fraction has power $\frac{x}{x^{5/2}} = \frac{1}{x^{3/2}}$

We know that $\int_3^\infty \frac{1}{x^{3/2}}\,dx$ is convergent (i.e. $p>1$). Thus we conclude that $\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ should be convergent.

In the next step we have to make everything fit the pattern of the comparison test, i.e. we have to find a suitable function $g(x)$ such that

$$\frac{2x+1}{\sqrt{x^5-32}} \le g(x) \text{ for all } x \text{ such that } x \geq 3,$$

and $\int_3^\infty g(x)\,dx$ is convergent. Then we will be able to conclude by the Comparison Test that

$$\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx \text{ is also convergent.}$$

What should the $g(x)$ be? Again we take a clue from the Step 1: $g(x)$ should be related to $\frac{1}{x^{3/2}}$. To figure out a more exact relationship, we consider

$$\lim_{x\to\infty} \frac{\frac{2x+1}{\sqrt{x^5-32}}}{\frac{1}{x^{3/2}}} = \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^5-32}} \cdot x^{3/2}$$

$$= \lim_{x\to\infty} \sqrt{\frac{(2x+1)^2 \cdot x^3}{x^5-32}} =$$

$$= \lim_{x\to\infty} \sqrt{\frac{\frac{(2x+1)^2 \cdot x^3}{x^5}}{\frac{x^5-32}{x^5}}} =$$

$$= \lim_{x\to\infty} \sqrt{\frac{\left(\frac{2x+1}{x}\right)^2 \cdot \frac{x^3}{x^3}}{\frac{x^5}{x^5} - \frac{32}{x^5}}} =$$

$$= \lim_{x\to\infty} \sqrt{\frac{\left(2+\frac{1}{x}\right)^2}{1-\frac{32}{x^5}}} = 2$$

But if the quantity $\dfrac{\frac{2x+1}{\sqrt{x^5-32}}}{\frac{1}{x^{3/2}}}$

becomes arbitrarily close to 2,

as x becomes arbitrarily large, it follows that this quantity is < 10 when x becomes sufficiently large. Thus for some $a \geq 3$, it is true

$$\frac{\dfrac{2x+1}{\sqrt{x^5-32}}}{\dfrac{1}{x^{3/2}}} < 10 \quad \text{for all } x \geq a$$

Hence

$$\frac{2x+1}{\sqrt{x^5-32}} < 10 \cdot \frac{1}{x^{3/2}} \text{ for all } x \geq a.$$

We now state some basic facts about convergent and divergent integrals which allow us to amplify the basic comparison test.

(a) If $\int_a^\infty f(x)\,dx$ is convergent and c is any constant, then $\int_a^\infty c f(x)\,dx$ is convergent.

(b) If $\int_a^\infty f(x)\,dx$ is divergent and c is any constant $\neq 0$, then $\int_a^\infty c f(x)\,dx$ is divergent.

(c) If $f(x)$ is cont. on $[a,\infty)$ and $b>a$, then $\int_a^\infty f(x)\,dx$ is convergent if and only if $\int_b^\infty f(x)\,dx$ is convergent.

(d) same as (c), except "convergent" replaced by "divergent".

Using the facts (a) – (d) above we can amplify the Comparison Test in the Book, on pages 191, 192 in these Notes:

Comparison Test, amplified.

Let $f(x)$, $g(x)$ be cont. on $[a, \infty)$,

$b \geq a$, and $c > 0$.

Let $f(x)$, $g(x)$ both ≥ 0 on $[b, \infty)$

(a) If $\int_b^\infty g(x)\,dx$ is convergent

and $f(x) \leq c\,g(x)$ for all $x \geq b$,

then $\int_a^\infty f(x)\,dx$ is convergent

(b) If $\int_b^\infty g(x)\,dx$ is divergent,

and $f(x) \geq c\,g(x)$ for all $x \geq b$,

then $\int_a^\infty f(x)\,dx$ is divergent.

Thus returning to the question on p. 195, whether the integral $\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is convergent or divergent:

We have shown that for some $a \geq 3$,

$$\frac{2x+1}{\sqrt{x^5-32}} < 10 \cdot \frac{1}{x^{3/2}}, \text{ for all } x \geq a.$$

(The number 10 is just any number > 2; we could choose 2.1 instead of 10, or we could choose 100 instead of 10)

Thus by part (a) of the amplified Comparison Test on p. 200,

$\int_3^\infty \frac{2x+1}{\sqrt{x^5-32}}\,dx$ is convergent:

We set $f(x) = \frac{2x+1}{\sqrt{x^5-32}}$, $g(x) = \frac{1}{x^{3/2}}$

We know that $\int_a^\infty \frac{1}{x^{3/2}}\,dx$ is convergent, and we can choose $c = 10$, obtaining $f(x) \leq c \cdot \frac{1}{x^{3/2}}$ for $x \geq a$.

The Exercise #57, p.528 in the Book:

Find the values of p for which the integral $\int_0^1 \frac{1}{x^p}\,dx$ is convergent:

Everything works exactly the same way if we replace 1 by any number $a>0$, so we consider instead $\int_0^a \frac{1}{x^p}\,dx$:

We find the antiderivative $\int \frac{1}{x^p}\,dx$:

$$\int \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{-p+1}\, x^{-p+1} + C & \text{if } p \neq 1 \\ \ln|x| \underset{x>0}{=} \ln x + C, & p=1. \end{cases}$$

First suppose $p \neq 1$.

Then for $t>0$, $t<a$.

$$\int_t^a \frac{1}{x^p}\,dx = \frac{1}{-p+1}\, a^{-p+1} - \frac{1}{-p+1}\, t^{-p+1}$$

We have to consider the limit as $t \to 0+$.

Thus focus on $\lim_{t\to 0+} t^{-p+1}$:

Since we are assuming $p \neq 1$,
we obtain $-p+1 \neq 0$, hence
$-p+1 > 0$ or $-p+1 < 0$

When $-p+1>0$, i.e. $p<1$, we obtain

$$\lim_{t \to 0+} t^{-p+1} = 0$$

(E.g. if $p = \frac{1}{2}$, then $\lim_{t \to 0+} t^{-\frac{1}{2}+1} =$
$\lim_{t \to 0+} t^{1/2} = 0$;
if $p = \frac{1}{3}$, then $\lim_{t \to 0+} t^{-\frac{1}{3}+1} =$
$= \lim_{t \to 0+} t^{2/3} = 0$ etc.)

Thus if $p<1$, then

$$\lim_{t \to 0+} \int_t^a \frac{1}{x^p} dx = \lim_{t \to 0+} \left(\frac{1}{-p+1} a^{-p+1} - \frac{1}{-p+1} t^{-p+1} \right)$$

$$= \frac{1}{-p+1} a^{-p+1} - \frac{1}{-p+1} \underbrace{\lim_{t \to 0+} t^{-p+1}}_{=0}$$

$$= \frac{1}{-p+1} a^{-p+1}$$

$$\boxed{\text{Thus if } p<1, \text{ then the improper integral } \int_0^a \frac{1}{x^p}\,dx \text{ converges.}}$$

Now consider $-p+1<0$ i.e. $p>1$:

Then $\lim\limits_{t\to 0+} t^{-p+1} = \lim\limits_{t\to 0+} \frac{1}{t^{p-1}}$

Since $-p+1<0$, we obtain $p-1>0$, hence $\lim\limits_{t\to 0+} t^{p-1} = 0$, while t^{p-1} is >0. Hence

$$\lim_{t\to 0+} \frac{1}{t^{p-1}} = +\infty.$$

Hence $\lim\limits_{t\to 0+} \int_t^a \frac{1}{x^p}\,dx =$

$$= \lim_{t\to 0+} \left(\frac{1}{-p+1} a^{-p+1} - \frac{1}{-p+1} t^{-p+1} \right)$$

$$= \underbrace{\underbrace{\frac{1}{-p+1}}_{<0} a^{-p+1} - \frac{1}{-p+1}}_{>0} \underbrace{\lim_{t \to 0+} t^{-p+1}}_{=+\infty}$$

$$= +\infty$$

$\boxed{\text{Hence } \int_0^a \frac{1}{x^p}\,dx \text{ diverges when } p>1.}$

It remains to consider the case

$\boxed{p=1, \text{ i.e. } \boxed{\int_0^a \frac{1}{x}\,dx}}$ DIVERGENT

In this case, for $t>0$, we obtain $\int_t^a \frac{1}{x}\,dx = \ln a - \ln t$,

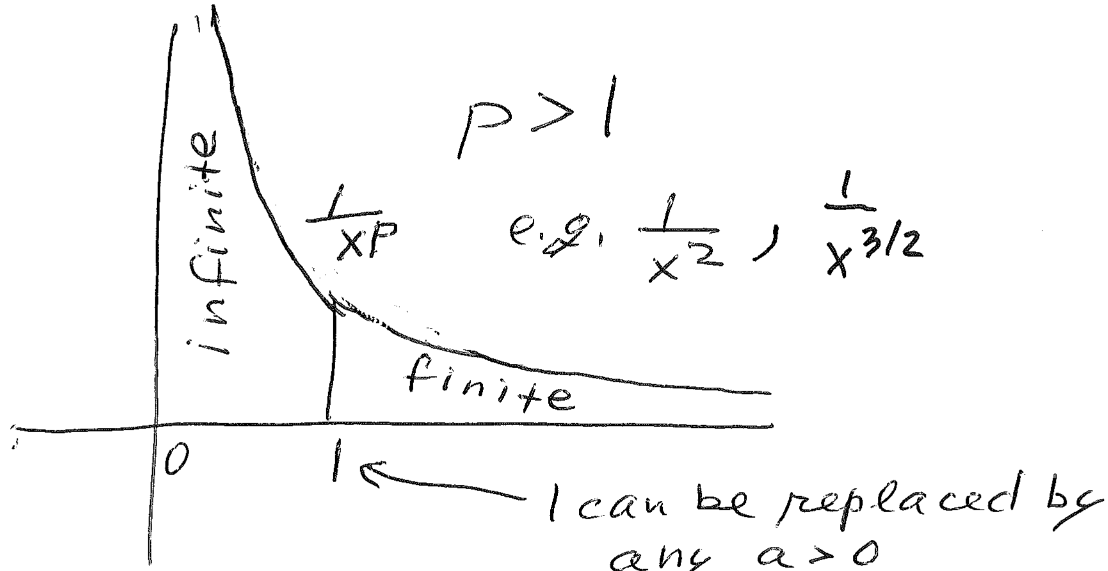
hence $\lim_{t \to 0+} \int_t^a \frac{1}{x}\,dx =$

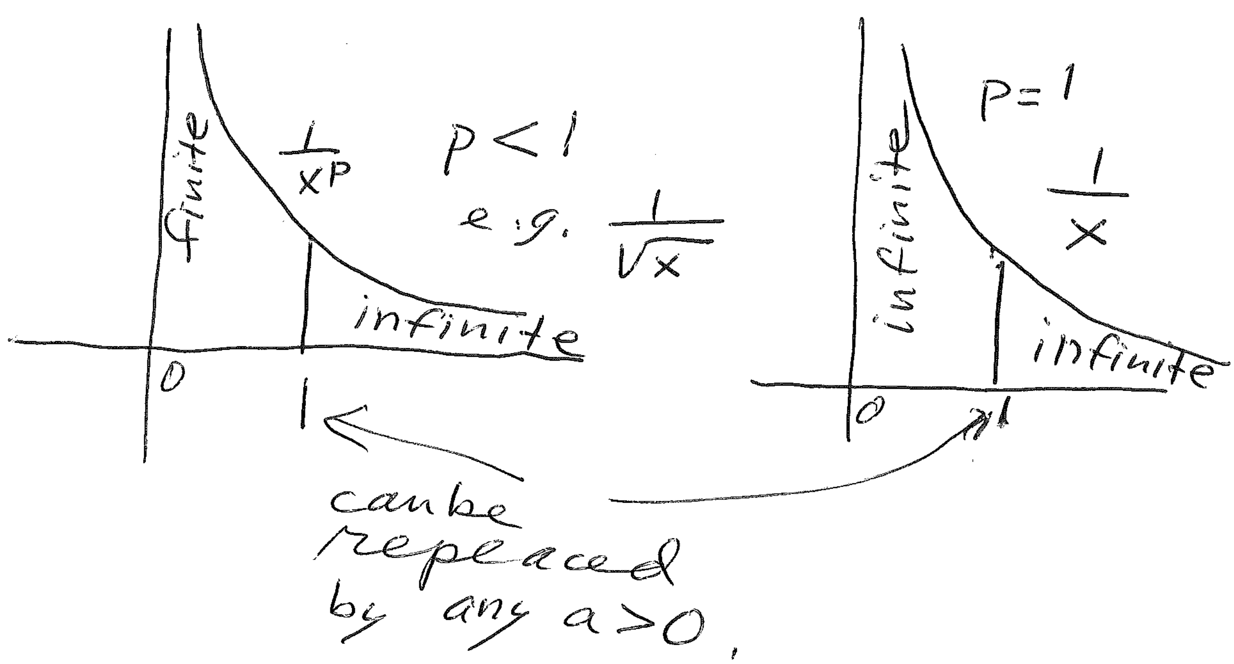
$$= \lim_{t \to 0+} (\ln a - \underbrace{\ln t}_{\to -\infty}) = +\infty$$

$\times$

It is best to see the Facts in Boxes on p. 204, 205 in combination with Facts on p. 176:

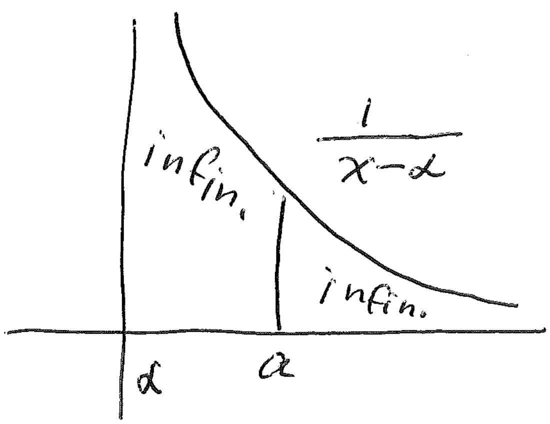
<u>All These Facts are called p-Tests.</u> Pictures.





FURTHERMORE:

All p-Tests Facts apply to the case when the "reference point" 0 is replaced by any α, i.e. we replace x by $x-\alpha$:



(i.e. $p=1$)