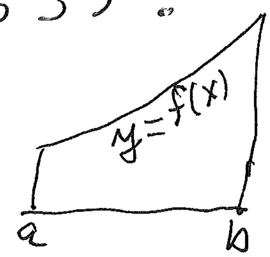
Sect. 8.1, 8.2.

The key formula for the length of the curve $y = f(x)$, $a \le x \le b$, i.e. the length of the graph of $f(x)$, $a \le x \le b$, is the formula in the large box on p. 539:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx.$$



Examples (Exercises, Exam Questions) involve calculating the length of the curve $y = f(x)$, $a \le x \le b$, for various functions f. This requires strong integration skills as one should have learned from Ch. 7, as well as strong algebra skills to be able to simplify integrals of the form near the top of this page.

HW due 2/18: None

HW due 2/20: 7.8: 41, 49, 51, 63
8.1: 1, 3, 7, 9, 17, 23; 8.2: 3, 5, 7.

<u>For Example</u> : Find the length of the curve $y = \frac{1}{2}x^2$, $0 \le x \le 2$.

We have $\frac{dy}{dx} = x$, hence

$$L = \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \sqrt{1+x^2}\,dx$$

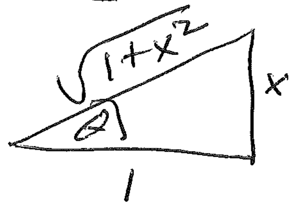
So we first calculate the antiderivative

(See Also Top of p. 550 in the Book)

$$\boxed{\int \sqrt{1+x^2}\,dx}$$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$



$$= \int \sqrt{1+(\tan\theta)^2}\,\sec^2\theta\,d\theta$$

$$= \int \sec\theta \sec^2\theta\,d\theta = \int \sec^3\theta\,d\theta$$

This is a somewhat complicated integral which we learned to do in the section 7.2 (Trigon. Integrals), and

$$= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C$$

$$= \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|\sqrt{1+x^2} + x\right|\right] + C$$

The integral $\int \sec^3\theta\,d\theta$ is calculated on p. 51 of the Notes.

Hence $\int_0^2 \sqrt{1+x^2}\,dx$

$$= \frac{1}{2}\Big[x\sqrt{1+x^2} + \ln\left|\sqrt{1+x^2}+x\right|\Big]_0^2$$

$$= \frac{1}{2}\big[2\sqrt{5} + \ln\left|\sqrt{5}+2\right|\big]$$

$$- \frac{1}{2}\big[0 + \underbrace{\ln 1}_{=0}\big]$$

$$= \underline{\underline{\frac{1}{2}\big[2\sqrt{5} + \ln(\sqrt{5}+2)\big]}} \quad *$$

An Easy Example, but one you should learn (as all Examples in the Book) is Example 1, p. 539 in the Book. Example 2, p. 540, is similar to the Example started at the top of page 230 of the Notes, but requires substitution $y = \frac{1}{2}\tan\theta$ (the variable of integration is y), not just $y = \tan\theta$.

Example 4, p. 542, in the Book does not require any special Integration skills, but requires good algebra skills.

Let's work out a similar Example, namely Exercise #10, p. 543:

Find the exact length of the curve

$$x = \frac{y^4}{8} + \frac{1}{4y^2}, \quad 1 \le y \le 2 :$$

$$\frac{dx}{dy} = 4 \cdot \frac{y^3}{8} + \frac{1}{4}(-2)\frac{1}{y^3} =$$

$$= \frac{1}{2}y^3 - \frac{1}{2y^3} = \frac{1}{2}\left(y^3 - \frac{1}{y^3}\right)$$

We now calculate the quantity

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}\left(y^3 - \frac{1}{y^3}\right)^2 =$$

$$= 1 + \frac{1}{4}\left(y^6 - 2y^3 \cdot \frac{1}{y^3} + \frac{1}{y^6}\right) =$$

$$= 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{y^6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{y^6}$$

$$= \frac{1}{4}\left(y^6 + 2 + \frac{1}{y^6}\right) = \frac{1}{4}\left(y^3 + \frac{1}{y^3}\right)^2$$

Hence $\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{\frac{1}{4}\left(y^3+\frac{1}{y^3}\right)^2}$

$= \frac{1}{2}\left(y^3+\frac{1}{y^3}\right)$

Hence the length of the curve in Exercise #10 is

$$\int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^2 \frac{1}{2}\left(y^3+\frac{1}{y^3}\right)dy$$

$$= \left[\frac{1}{8}y^4 + \frac{1}{2}\left(-\frac{1}{2y^2}\right)\right]\Bigg|_1^2$$

$$= \left(\frac{1}{8}y^4 - \frac{1}{4y^2}\right)\Bigg|_1^2$$

$$= \left(\frac{1}{8}\cdot 16 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) =$$

$$= 2 - \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \underline{\underline{\frac{33}{16}}}$$

The idea of simplifying the square root of $\sqrt{1+(f'(x))^2}$ in the manner shown in the preceding Example occurs in many problems on calculating the length of a curve. As mentioned already, Example 4 on p. 542 is like that.

Another frequent example is:

$$y = \frac{1}{2}(e^x + e^{-x}),$$

hence $\dfrac{dy}{dx} = \dfrac{1}{2}(e^x - e^{-x})$,

and $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1}{4}(e^{2x} - 2e^x \cdot e^{-x} + e^{-2x})$

$= 1 + \dfrac{1}{4}e^{2x} - \dfrac{1}{2} + \dfrac{1}{4}e^{-2x}$

$= \dfrac{1}{4}e^{2x} + \dfrac{1}{2} + \dfrac{1}{4}e^{-2x} = \dfrac{1}{4}(e^{2x} + 2 + e^{-2x})$

$= \dfrac{1}{4}(e^x + e^{-x})^2$, Hence $\sqrt{\dfrac{1}{4}(e^x + e^{-x})^2}$

$= \dfrac{1}{2}(e^x + e^{-x})$.

Section 8.2 deals with the Area of the surface obtained by rotating a graph / curve about the x- or y-axis. Thus if the curve

$$y = f(x), \quad a \le x \le b$$

is rotated about the x-axis, the Area of the surface obtained in this manner is

$$\underbrace{\int_a^b 2\pi \underbrace{f(x)}_{=y} \sqrt{1+[f'(x)]^2}\,dx}_{\text{Box } \boxed{4},\ \text{p. } 547.}$$

If the curve is given by $x = g(y)$, $c \le y \le d$, the Area is given by

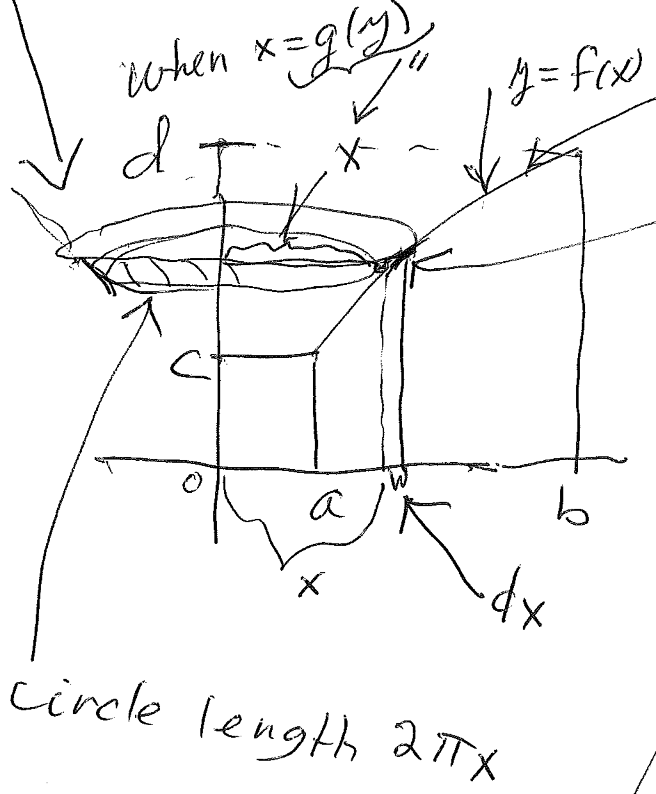
$$\int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy, \quad \text{Box } \boxed{6},$$

p. 547.

For Rotation about the y-axis the two corresponding formulas are :

$y = f(x)$, $a \le x \le b$: $\displaystyle\int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$x = g(y)$, $c \le y \le d$: $\displaystyle\int_c^d 2\pi \underbrace{g(y)}_{=x} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$



small portion of the graph :

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

OR : $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

which is width of a thin strip/band cut out from the surface

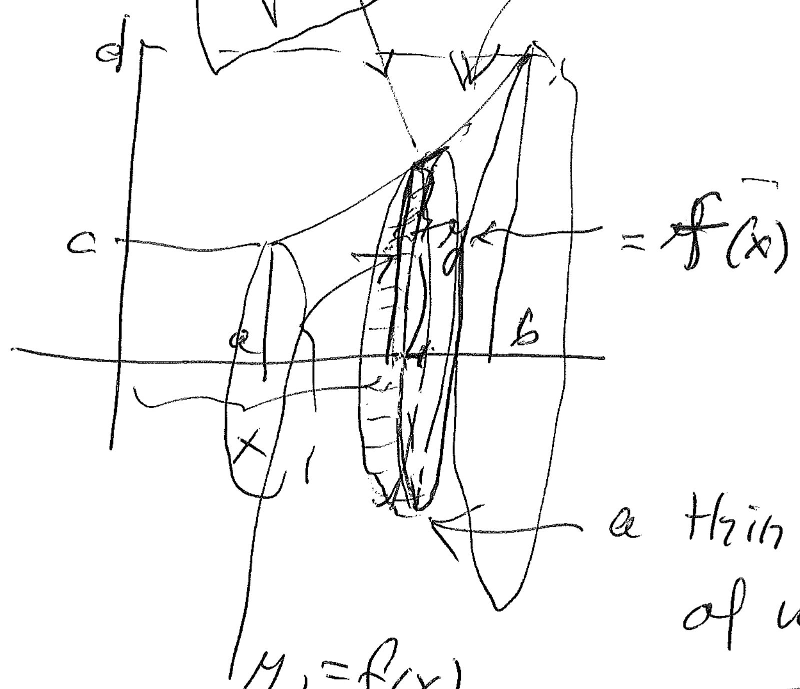
Picture for page 235:

$$y=f(x) \Rightarrow \int_a^b \underbrace{2\pi f(x)}_{=y} \sqrt{1+(\underbrace{f'(x)}_{\text{"}\frac{dy}{dx}\text{"}})^2}\,dx$$

$$\text{and } x=g(y) \Rightarrow \int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Two formulas for Area of surface obtained by Rotating a curve about the x-axis

$\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$ or $=$ $\sqrt{1+(f'(x))^2}dx$



$$= 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$\text{or } 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Example. Calculate the area of the surface obtained by rotating the curve $y = \frac{1}{2}x^2$, $0 \le x \le 2$, about the x-axis:

By the Formula

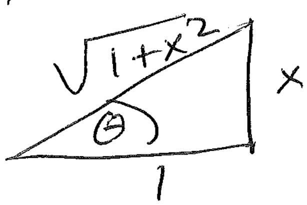
$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx :$$

$$S = \int_0^2 2\pi \cdot \frac{1}{2}x^2 \sqrt{1+x^2}\,dx$$

$$= \pi \int_0^2 x^2\sqrt{1+x^2}\,dx$$

We calculate $\int x^2\sqrt{1+x^2}\,dx$ first:

By subst. on p. 230, $x = \tan\theta$



we obtain $\int \tan^2\theta \sec^3\theta\, d\theta$

$$= \int (\sec^2\theta - 1)\sec^3\theta\, d\theta = \underbrace{\int \sec^5\theta\, d\theta}_{\text{on p. 52 in the Notes.}} - \underbrace{\int \sec^3\theta\, d\theta}_{\text{on p. 51 in the Notes}}$$

We will not continue this any further

$\times$

Example. Find the area of the surface obtained by rotating the curve $y = x^2 - \frac{1}{8}\ln x$, $1 \le x \le 2$ about the x-axis.

This curve is considered in Example 4 on p. 542 in the Book and it is calculated that

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2x + \frac{1}{8x}$$

Hence the area of the surface of this Example (top of this page) is given by $\displaystyle\int_1^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= \int_1^2 2\pi\left(x^2 - \frac{1}{8}\ln x\right)\left(2x + \frac{1}{8x}\right)dx$$

$$= 2\pi\int_1^2\left(2x^3 - \frac{1}{4}x\ln x + \frac{1}{8}x - \frac{1}{64}\frac{\ln x}{x}\right)dx$$

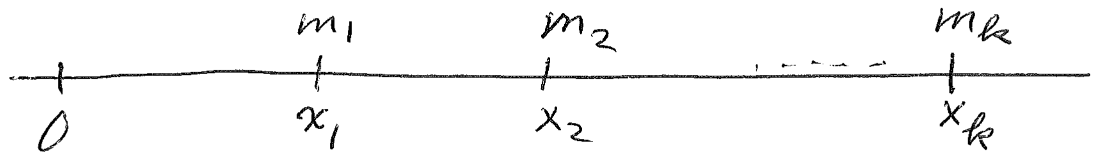
and we know how to find all antiderivatives occurring in this integral — Finish on your own.

Section 8.3

Moments and Centers of Mass.

(We will not cover the Problems on Hydrostatic Pressure and Force.)

Simplest Case: Point Masses.



Masses $m_1, m_2, \ldots m_k$ situated at points $x_1, x_2, \ldots x_k$ on the x-axis:
The center of mass $\bar{x}$:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_kx_k}{m_1 + m_2 + \cdots + m_k}$$

If the masses are equal, say all equal to m, we obtain

$$\bar{x} = \frac{mx_1 + \cdots + mx_k}{\underbrace{m + m + \cdots + m}_{k \text{ times}}} = \frac{m(x_1 + x_2 + \cdots + x_k)}{km},$$

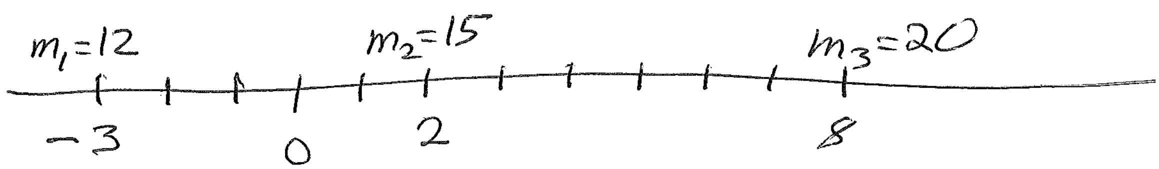
i.e. $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_k}{k}$,

i.e. the average of the coordinates.

$$M = m_1x_1 + m_2x_2 + \cdots + m_kx_k = \sum_{i=1}^{k} m_i x_i$$

which is the numerator of the fraction in the formula for $\bar{x}$, is called the moment of the given system of masses about the origin.

For Example, #22, bottom p. 561 in the Book: Find the moment M of the system about the origin and the center of mass $\bar{x}$:



$$M = m_1x_1 + m_2x_2 + m_3x_3 = 12(-3) + 15(2) + 20(8)$$

$$= -36 + 30 + 160 = 154$$

$$\bar{x} = \frac{M}{\underbrace{m_1+m_2+m_3}_{=m \text{ total mass}}} = \frac{154}{\underbrace{12+15+20}_{=\text{total mass}}} = \frac{154}{47}$$

Systems of masses in the plane.

$m_1, m_2, \cdots m_n$ located at $(x_1, y_1), (x_2, y_2), \cdots$
$\cdots (x\ , y\ )$

Then the moment of the system about the y-axis is

$$\boxed{M_y = \sum_{i=1}^{n} m_i x_i}$$

(page 555 in the Book

The moment of the system about the x-axis

$$\boxed{M_x = \sum_{i=1}^{n} m_i y_i}$$

(top p.556 in the Book)

The total mass $m = m_1 + m_2 + \cdots + m_n$

Note the switch of x, y, i.e.
M_y is calculated from the x_i,
i.e. the distances of the $m_1, \cdots m_n$ from the y-axis.
Similar switch occurs for M_x.

The coordinates of the Center of Mass are

$$\bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m}$$

$$\bar{y} = \frac{M_x}{m} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m}$$

i.e. to calculate $\bar{x}$ we use $x_1, x_2, \cdots x_n$

" " $\bar{y}$ " " $y_1, y_2, \cdots y_n$

#24, p.561: Find the Moments M_x, M_y and the center of mass $(\bar{x}, \bar{y})$ if $m_1 = 5$, $m_2 = 4$, $m_3 = 3$, $m_4 = 6$, are situated at the points

$P_1(-4,2)$, $P_2(0,5)$, $P_3(3,2)$, $P_4(1,-2)$:

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4 =$$
$$= 5(2) + 4(5) + 3(2) + 6(-2) =$$
$$= 10 + 20 + 6 - 12 = \underline{\underline{24}}$$

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 =$$
$$= 5(-4) + 4(0) + 3(3) + 6(1) = \underline{\underline{-5}}$$

The total mass m:

$$m = 5+4+3+6 = 18$$

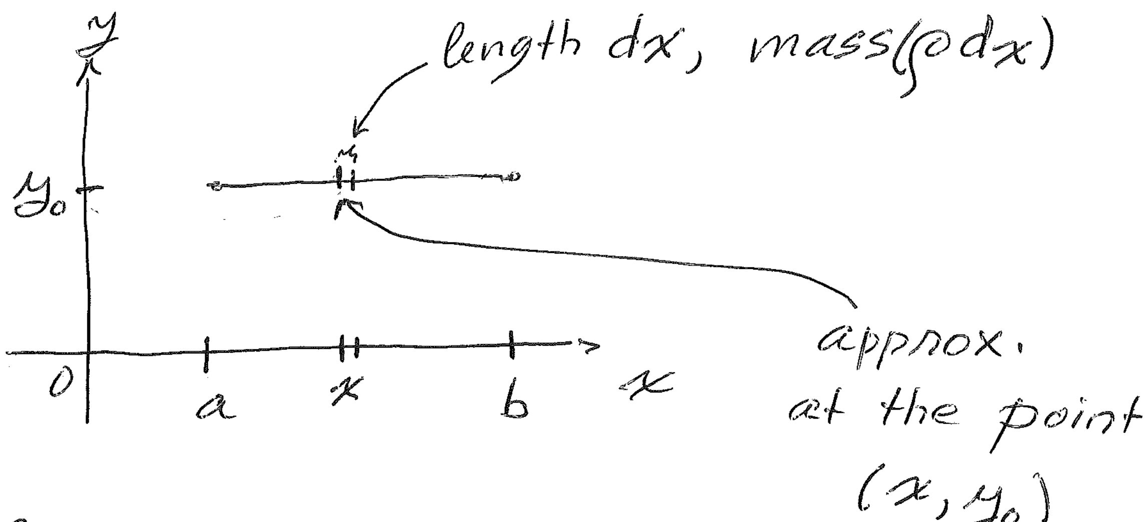
$$\bar{x} = \frac{M_y}{m} = \frac{-5}{18}$$

$$\bar{y} = \frac{M_x}{m} = \frac{24}{18} = \frac{4}{3}$$

Hence the center of mass is

$$(\bar{x}, \bar{y}) = \left(-\frac{5}{18}, \frac{4}{3}\right) \quad *$$

The Moment of a horizontal thin wire of uniform density ρ about the y-axis:



So similarly to the formula

$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$$

we have $\cdots\cdots (\rho dx)x \cdots\cdots$ which is a part of the "sum" which is the integral

$$M_y = \int_a^b (\rho dx)x = \int_a^b \rho x\,dx$$

$$= \rho\int_a^b x\,dx = \rho\left(\frac{1}{2}x^2\right)\Bigg|_a^b$$

$$= \frac{1}{2}\rho\,(b^2 - a^2) = \underbrace{\rho(b-a)}_{\text{mass}}\overbrace{\left[\frac{1}{2}(a+b)\right]}^{\boxed{\text{x-coord. of center of mass}}}$$