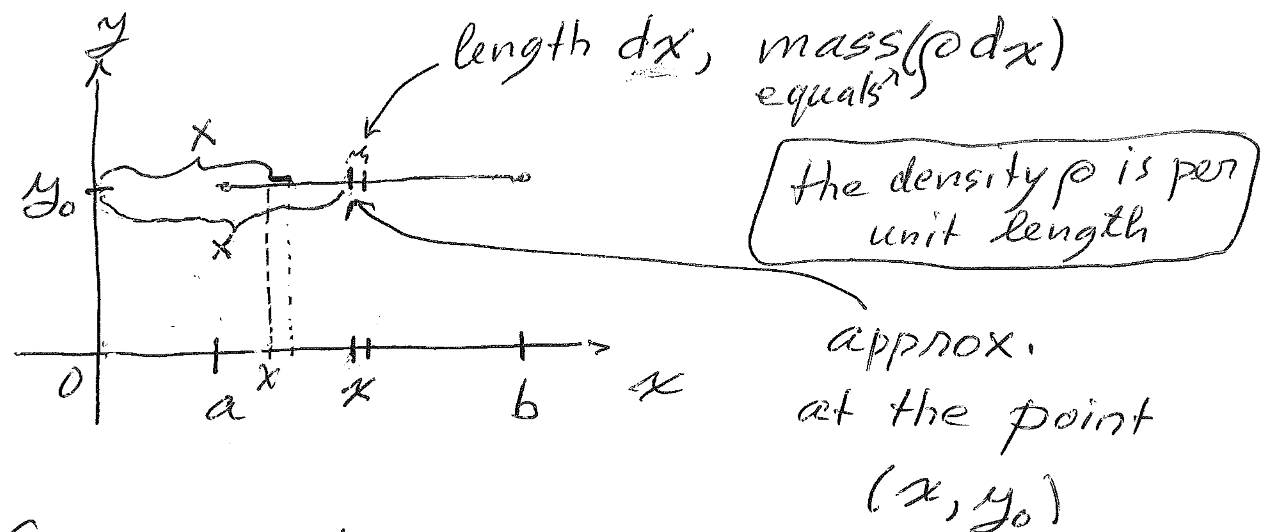
The Moment of a horizontal thin wire of uniform density ρ about the y-axis:



So similarly to the formula

$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n$$

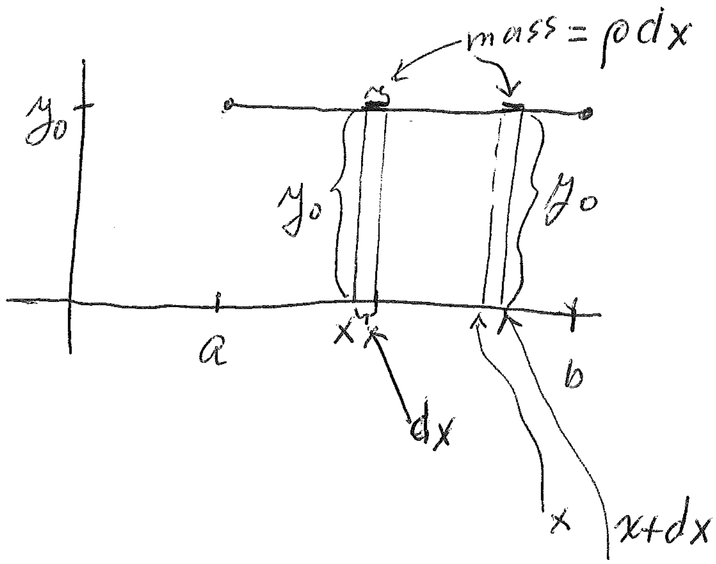
we have $\cdots\cdots\boxed{(\rho dx)x}\cdots\cdots$ which is a part of the "sum" which is the integral

$$M_y = \int_a^b (\rho dx)x = \int_a^b \rho x dx$$

$$= \rho \int_a^b x dx = \rho\left(\tfrac{1}{2}x^2\right)\Big|_a^b$$

$$= \tfrac{1}{2}\rho(b^2 - a^2) = \underbrace{\rho(b-a)}_{\text{mass}}\overbrace{\left[\tfrac{1}{2}(a+b)\right]}^{\text{x-coord. of center of mass}}$$

The Moment of a horizontal thin wire of uniform density ρ about the x-axis:



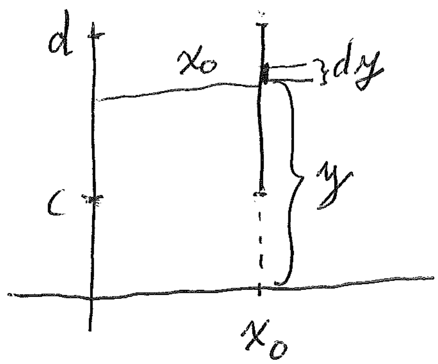
$\boxed{\text{the density } \rho \text{ is per unit length}}$

y_0 = the distance from x-axis

Thus the wire can be thought as being built from very small pieces of mass ρdx each of which has distance y_0 from the x-axis, hence it has moment $y_0(\rho dx)$ about the x-axis. The total moment of the entire wire about the x-axis is obtained by adding together all these moments $y_0(\rho dx)$, over all x between a and b, i.e. the integral

$$M_x = \int_a^b y_0 \rho \, dx = \rho y_0 (b-a) = \underbrace{y_0}_{\text{distance from x-axis}} \overbrace{\rho(b-a)}^{\text{total mass}}$$

Moments of a vertical thin wire of uniform density ρ (per unit length):



We proceed by analogy with the work for a horizontal wire:

$$M_x = \int_c^d \rho y \, dy = \frac{1}{2}\rho(d^2 - c^2) =$$

$$= \frac{1}{2}\rho(c+d)(d-c)$$

$$= \underbrace{[\rho(d-c)]}_{\text{mass}} \underbrace{\left[\frac{1}{2}(c+d)\right]}_{\text{y-coord. of the center}}$$

$$M_y = \int_c^d x_0 \rho \, dy = \rho x_0 (d-c)$$

$$= \underbrace{x_0}_{\text{distance from y-axis}} \underbrace{\rho(d-c)}_{\text{mass}}$$

In general the moment of an extended masslive) object about an axis (which can be any line (not just the x-axis or the y-axis) is calculated by dividing the object into very small pieces (i.e. of small size), multiplying the mass of each of them by its distance from the axis, (algebraic distance) which yields the moment of that small piece (about the given axis), then adding up the moments of those small pieces. Of course, the "adding up" usually means integrating.

A Principle for Calculating Moments.

The Moment of an object about an Axis = (The total Mass) times (the Algebraic Distance of Center of Mass from the Axis)

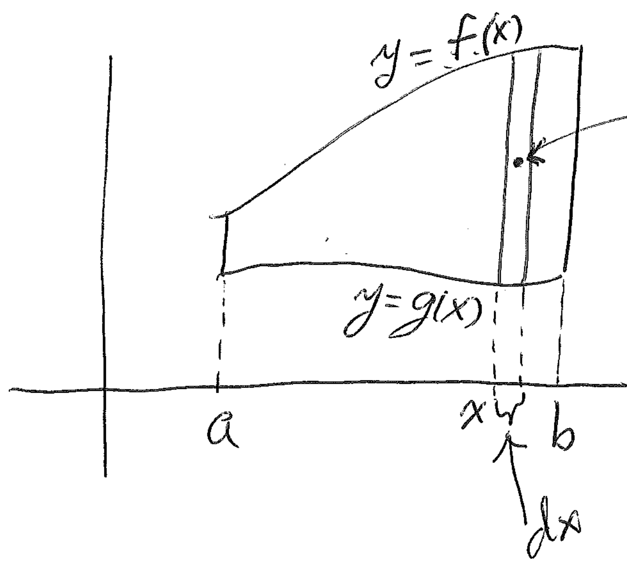
Another Principle for Calculating the Moment About an Axis:

If an object is the union of a collection of disjoint pieces, then the moment of the object equals the sum of the moments of those pieces.

The Moments of a Thin Plate laid flat on the x,y-plane:

We also assume uniform density ρ per unit area.

Shape of the Plate:



center of mass of a rectangle is the center of the rectangle (unif. density assumed) thus the center of mass of the rectangle in the picture on preceding page has y-coord. $= \frac{1}{2}(f(x)+g(x))$, and the x-coord. approx. equals x, since dx is very small.

Moreover the area of the rectangle is $(f(x)-g(x))dx$, hence its mass $\rho(f(x)-g(x))dx$

Thus the moment of the rectangle about x-axis equals

$$\underbrace{\underbrace{\frac{1}{2}(f(x)+g(x))}_{\substack{\text{distance of its}\\ \text{center of mass.}\\ \text{from x-axis}}}\underbrace{[\rho(f(x)-g(x))\,dx]}_{\substack{\text{mass of}\\ \text{the rectangle}}}}_{\text{moment about x-axis.}}$$

Moment of the Plate $a \le x \le b$, $g(x) \le y \le f(x)$ about x-axis

equals $M_x = \int_a^b \frac{1}{2}(f(x)+g(x))\rho(f(x)-g(x))\,dx$

$$= \rho \int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$

Thus the moment M_x of the plate of density 1 equals

$$\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))\right] dx$$

which is the called the moment of the region $\{(x,y): a \le x \le b,\ g(x) \le y \le f(x)\}$ about the x-axis.

<u>The Moment about the y-axis:</u>

For the rectangle in the picture on p. 249,

$$\underbrace{x}_{\text{distance of center of mass from } y\text{-axis}} \ \underbrace{\rho(f(x)-g(x))\,dx}_{\text{mass}}$$

The Moment about the y-axis
For the plate of density ρ,
and $a \le x \le b$, $g(x) \le y \le f(x)$:

$$M_y = \int_a^b x\rho(f(x) - g(x))\,dx$$

$$= \rho \int_a^b x(f(x) - g(x))\,dx$$

If density $\rho = 1$, we obtain the moment M_y of the region

$$\{(x,y) : a \le x \le b,\ g(x) \le y \le f(x)\}$$

$$M_y = \int_a^b x(f(x) - g(x))\,dx .$$

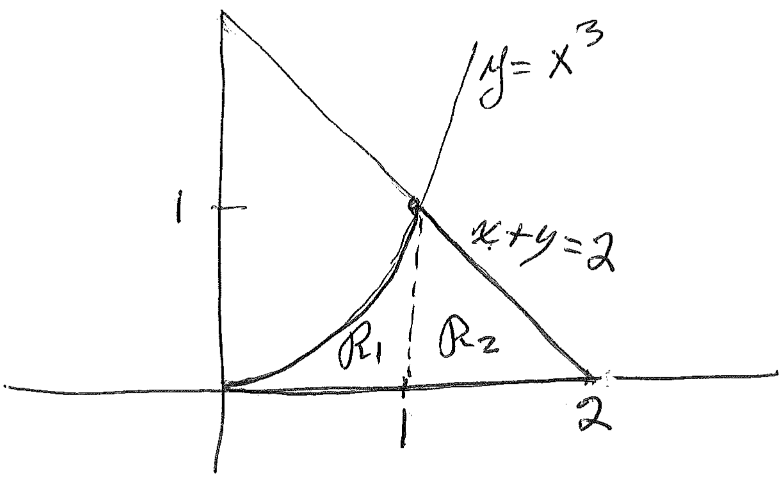
The x and y coordinates of the <u>centroid</u> of the region are obtained by dividing by the area:

$$\bar{x} = \frac{M_y}{A}, \quad \bar{y} = \frac{M_x}{A}$$

Compare this with the formulas near the top of p. 243. ∴

We will do Exercise #32, p. 561 in the Book:

Find the centroid of the region bounded by the curves $y = x^3$, $x + y = 2$, $y = 0$



We have to solve
$$y = x^3$$
$$x + y = 2$$

Hence $x + x^3 = 2$, thus $x = 1, y = 1$ is a solution. Hence the drawing above shows the region. We divide it into the two regions R_1, R_2. Thus the moments M_x, M_y are obtained by adding the corresp. moments of R_1 and R_2.

First we calculate $\underline{M_x \text{ and } M_y}$ $\underline{\text{for } R_1:}$ We have $0 \le x \le 1$, $f(x) = x^3$, $g(x) = 0$. Hence

$$M_x = \int_0^1 \frac{1}{2}[f(x)]^2 dx = \int_0^1 \frac{1}{2} x^6 dx$$

$$= \left(\frac{1}{2} \cdot \frac{1}{7} x^7\right)\Big|_0^1 = \frac{1}{14}$$

$$M_y = \int_0^1 x f(x) dx = \int_0^1 x \cdot x^3 dx$$

$$= \int_0^1 x^4 dx = \frac{1}{5} x^5 \Big|_0^1 = \frac{1}{5}$$

M_x and M_y for R_2:

$f(x) = 2 - x$ (From $x + y = 2 \Rightarrow$
$y = 2 - x = f(x)$)

$g(x) = 0$

$1 \le x \le 2$

$$M_x = \int_1^2 \frac{1}{2}[f(x)]^2 dx = \int_1^2 \frac{1}{2} \overbrace{(2-x)^2}^{(x-2)^2} dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} (x-2)^3 \Big|_1^2 = \frac{1}{6}$$

$$M_y = \int_1^2 x f(x)\,dx = \int_1^2 x(2-x)\,dx$$

$$= \int_1^2 (2x - x^2)\,dx = \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_1^2 =$$

$$= \left(4 - \tfrac{8}{3}\right) - \left(1 - \tfrac{1}{3}\right) =$$

$$= 3 - \tfrac{8}{3} + \tfrac{1}{3} = 3 - \tfrac{7}{3} = \tfrac{2}{3}$$

Hence $(M_x$ for $R)$ equals

$(M_x$ for $R_1) + (M_x$ for $R_2)$

$$= \frac{1}{14} + \frac{1}{6} = \frac{3+7}{42} = \frac{10}{42} = \underline{\frac{5}{21}}$$

$(M_y$ for $R)$ equals

$(M_y$ for $R_1) + (M_y$ for $R_2)$

$$= \frac{1}{5} + \frac{2}{3} = \frac{3+10}{15} = \underline{\frac{13}{15}}$$

Now we need to calculate the area of $\mathcal{R}$: equals area of $\mathcal{R}_1$ plus Area of $\mathcal{R}_2$.

$$(\text{Area of } \mathcal{R}_1) = \int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$$

$$(\text{Area of } \mathcal{R}_2) \text{ obviously } = \frac{1}{2}$$

Hence $(\text{The Area of } \mathcal{R}) = \underline{\frac{3}{4}}$

Finally $$\bar{x} = \frac{M_y}{A} = \frac{\frac{13}{15}}{\frac{3}{4}} = \underline{\underline{\frac{52}{45}}}$$

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{5}{21}}{\frac{13}{15}} = \frac{5}{21}\cdot\frac{15}{13} =$$

$$= \frac{5}{7}\cdot\frac{5}{13} = \underline{\underline{\frac{25}{91}}}$$

Hence the centroid of $\mathcal{R}$ is

$$(\bar{x}, \bar{y}) = \left(\frac{52}{45}, \frac{25}{91}\right)$$