Section 9.1.

What can one say about a function $y = f(x)$ if one knows only the derivative $y' = f'(x)$? Of course, if one knows the derivative, one can integrate, i.e. if $f'(x) = g(x)$, then $y = f(x) = \int g(x)dx + C$, where C is an arbitrary constant, hence one can find all possible functions with a prescribed derivative, and any two such functions differ only by a constant.

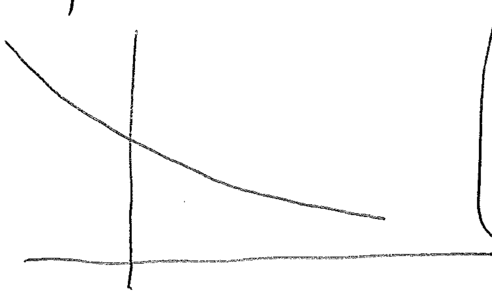
Nevertheless, let's consider some things we can do without integrating.

So suppose we are given

$$y' = e^x + 2$$

What can we say about functions $y = f(x)$ whose derivative $y' = e^x + 2$?

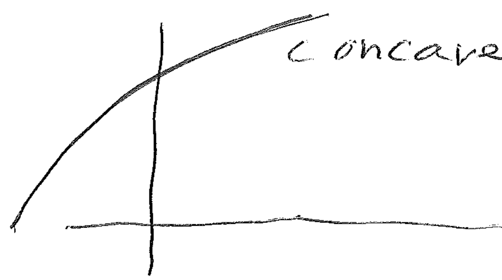
Question 1. Can the graph below be the graph of such a function?



HW for Tue. 2/25:
- 8.2: # 13, 19, 33
- 8.3: on syllabus
- 9.1: on syllabus

No, since the function with this graph is decreasing, whereas the derivative $y' = e^x + 2$ is > 0, hence $y = f(x)$ has to be increasing.

Question 2. What about the next graph?



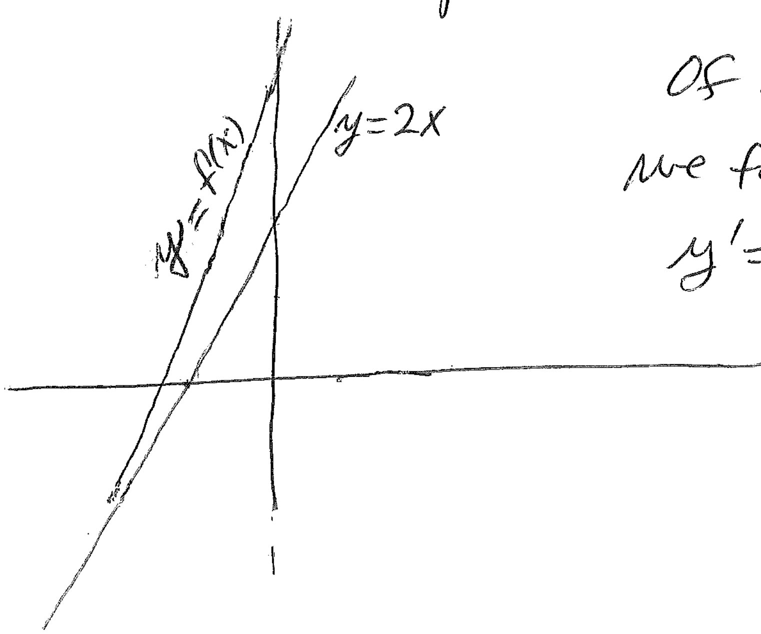
Again, the answer is no, since $y'' = e^x > 0$, hence any function with the property $y' = e^x + 2$ has to be concave up.

Hence any function $y = f(x)$ which has the property $y' = f'(x) = e^x + 2$ has to be increasing and concave up.

We can say even more:

Consider $\lim_{x\to-\infty}(e^x+2) = 2$, since $\lim_{x\to-\infty} e^x = 0$. Hence $\lim_{x\to-\infty} y' = 2$.

However, y' is the slope of the tangent to the graph (at any given point). Hence the slopes of tangents at $(x, f(x))$, as $x \to -\infty$, approach 2, while being slightly larger than 2. Hence the graph has a line of slope 2 as an asymptote as $x \to -\infty$. Also it is concave up, as already explained above:



Of course, by integrating we find that if $y' = e^x + 2$, then

$$y = e^x + 2x + C$$

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The equation $y' = g(x)$ is the simplest kind of differential equation. We can solve it by integrating: $y = \int g(x)\,dx + C$.

In general, a differential equation is any equation involving the independent variable, which is usually x or t, and the dependent variable which can be y, P (for population), or even x — e.g. x dependent, t independent.

Examples of differential equations:

(1) $y'' + 3y' + 2y = x$,

(2) $y'' + 3y' + 2y = t$

(3) $y'' + 3xy' + 2x^2y = 0$

(4) $y'' = 3y$ (the independent variable — x or t, does not have to be in the equation)

(5) $y''' - 3y = x$

The <u>order of a differential equation</u> is the order of <u>the highest derivative</u> appearing in the equation. (1) – (4) are of order 2. (5) is of order 3.

An equation is of <u>first order</u> if only the first derivative of the dependent variable y appears; i.e. y' appears, but y'', y''' etc. do not appear.
Of course if the dependent variable is denoted by a different letter, e.g. P, then the differential equation would have P', P'', etc., while a first order one would have only P'.

Moreover writing P', P'', y', y'' etc. does not make it clear by itself what the dependent variable is. So if we want to make it clear, we write $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{dP}{dt}$, $\frac{d^2P}{dt^2}$, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, etc.

So for example

$$\frac{dP}{dt} = 0.02P \quad (\text{p. 580 in the Book})$$

is a first order diff. eq., P dependent, t independent.

Solution of a differential Equation.

A function $f(x)$ is a solution of a diff. Eq. if when substituted for the dependent variable, we obtain a true equation.

Example. Both e^{2x}, e^{-2x} are solutions of the diffential equation

$$y'' - 4y = 0$$

but e^x is not a solution:

$$(e^{2x})'' - 4e^{2x} = (2e^{2x})' - 4e^{2x}$$

$$= 4e^{2x} - 4e^{2x} = 0,$$

hence e^{2x} is a solution.

$$(e^x)'' - 4e^x = e^x - 4e^x = -3e^x \neq 0,$$

hence e^x is not a solution.

Since simply by looking at the eq. $y'' - 4y = 0$ we can not tell what the indep. vrbl. is, we can also say that e^{2t}, e^{-2t} are solutions which means we are specifying that the indep. vrbl. is t. Of course once we specify that the indep. vrbl. is x, then e^{2x}, e^{-2x} are solutions of the eq. $y''-4y=0$, but e^{2t}, e^{-2t} are not. To put it slightly differently, we can consider the eq.

$$\frac{d^2y}{dx^2}-4y=0$$

which is the same eq. as

$$y''-4y=0$$

but the equation makes it clear that the indep. vrbl. is x, so e^{2x}, e^{-2x} are solutions of but e^{2t}, e^{-2t} are not.

Differential Equations typically have infinitely many solutions. For Example all functions $y = c_1e^{2x}+c_2e^{-2x}$

where c_1, c_2 are arbitrary constants, are solutions of the equation

$$y'' - 4y = 0 \quad \left(\text{or } \frac{d^2y}{dx^2} - 4y = 0\right)$$

To verify this:

$$(c_1e^{2x} + c_2e^{-2x})'' - 4(c_1e^{2x} + c_2e^{-2x}) =$$

$$= (2c_1e^{2x} - 2c_2e^{-2x})' - 4(c_1e^{2x} + c_2e^{-2x})$$

$$= (4c_1e^{2x} + 4c_2e^{-2x}) - 4(c_1e^{2x} + c_2e^{-2x}) = 0$$

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Exercise #6, p. 584.

Show that every member of the family of functions $y = (\ln x + C)/x$ is a solution of the differential equation

$$x^2y' + xy = 1.$$

$$y' = \frac{(\ln x + C)'x - (x)'(\ln x + C)}{x^2}$$

$$= \frac{\frac{1}{x} \cdot x - 1 \cdot (\ln x + C)}{x^2}$$

$$= \frac{1 - \ln x - C}{x^2}$$

$$x^2 y' + xy =$$

$$x^2 \cdot \frac{1 - \ln x - C}{x^2} + x \cdot \frac{\ln x + C}{x} =$$

$$= 1 - \ln x - C + \ln x + C = 1,$$

hence $x^2 y' + xy = 1$ if we substitute $y = \frac{\ln x + C}{x}$.

$\ast$

Initial Conditions and Particular Solutions.

As we indicated, a differential equation typically has infinitely many solutions since every value of arbitrary constants yields a specific (different) solution.

Often we want to find a specific solution, which is called a particular solution. A way to single out a particular solution is to specify an initial condition:

An initial condition specifies the value of the solution for some value of the independent variable:

Example. Find a solution of the diff. eq. $x^2y' + xy = 1$ which satisfies the initial condition $y(1) = 2$, and likewise for $y(2) = 1$.

We know that every function of the form $y = \dfrac{\ln x + C}{x}$ is a solution of the equation above. (C arbitr. const.)

For such a function,

$$y(1) = \frac{\ln 1 + C}{1} = C$$

So if $y(1) = 2$, we should choose $C = 2$, hence the desired particular solution satisfying $y(1) = 2$ is given by

$$y = \frac{\ln x + 1}{x}.$$

To find a particular solution which satisfies $y(2)=1$, we proceed similarly:

$$y(2)=\frac{\ln 2 + C}{2} := 1$$

Hence $\ln 2 + C = 2$,

$$C = 2 - \ln 2,$$

and the desired particular solution satisfying the initial condition $y(2)=1$ is

$$y = \frac{\ln x + 2 - \ln 2}{x} \quad *$$

Population Growth.

Exponential: $\frac{dP}{dt} = kP$, $P(0) = P_0$

$$P(t) = P_0 e^{kt}$$

For the world human population, the k has fluctuated between 0.01 and 0.02 over the last 200 years.

Logistic: $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \ P(0) = P_0$$

$P = M$, i.e. $P(t) = M$ remains constant for all t, is a solution:

$$\frac{d}{dt}(M) = 0; \qquad kM\left(1 - \frac{M}{M}\right) = 0,$$

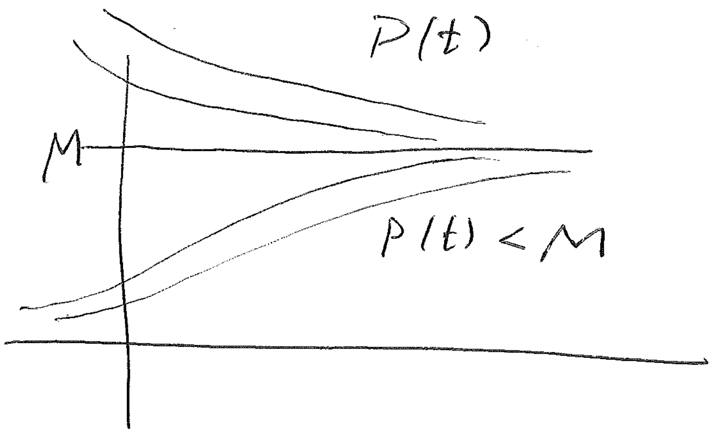
so left hand side = Right hand side; This is called an equilibrium solution. Likewise $P(t) = 0$ (for all t) is an equilibrium solution.

Logistic Eq, Continued:

$\frac{dP}{dt} > 0$ when $P(t) < M$,

hence $P(t)$ is increasing when $P(t) < M$,

and leveling off at M.



$\frac{dP}{dt} < 0$ when $P(t) > M$, so

$P(t)$ is decreasing when $P(t) > M$,

and again leveling off at M.

M is called the carrying capacity.

Solution of Logistic Model:

$$P(t) = \frac{M P_0 e^{kt}}{M + P_0 (e^{kt} - 1)}$$

One website supportive of protecting Yellowstone buffalo says that there are currently 4,200 buffalo there, and that a theoretical food-limited carrying capacity is 6,200.

21 bison introduced in 1902.

About 1,000 bison in 1927, population appears to have grown exponentially 1902 – 1927.

Population fell to about 220 in 1965; again appears to have grown exponentially 1965 – 1995, to about 4,000. There have been removals/killings.

Considering exponential growth

$$P(t) = P_0 e^{kt},$$

for the 1902 – 1927 period, we obtain

$P_0 = 21$, $t = 25$, $P(25) = 1{,}000$, hence

$$P(25) = P_0 e^{25k},$$

$$e^{25k} = \frac{P(25)}{P_0} = \frac{1000}{21} \approx 47.62$$

$$k = \frac{1}{25} \ln(47.62) \approx 0.1545$$

Hence for the period 1902 – 1927 we obtain

$$P(t) = 21 e^{0.1545t}$$

For the period 1965–1995, we have $P_0 = 220$, $P(30) = 4{,}000$, hence

$$P(30) = P_0 e^{30k}$$

i.e.

$$4{,}000 = 220 e^{30k},$$

$$e^{30k} = \frac{4{,}000}{220} \approx 18.18$$

$$k = \frac{1}{30} \ln(18.18) \approx 0.097$$

Taking the average of $k_1 = 0.1545$, and $k_2 = 0.097$, we obtain $k = 0.126$. So we use this for the logistic growth model:

$$P(t) = \frac{M P_0 e^{kt}}{M + P_0(e^{kt} - 1)}$$

where we set $M = 6{,}200$ (the estimated carrying capacity), and $P_0 = 4200$ (the current population), obtaining

$$P(t) = \frac{(6{,}200)4200\, e^{0.126t}}{6{,}200 + 4{,}200(e^{0.126t} - 1)}$$

i.e.

$$P(t) \approx \frac{(2.6)10^7 e^{0.126t}}{6{,}200 + 4{,}200(e^{0.126t} - 1)}$$

$$\simeq \frac{(2.6)10^5 e^{0.126t}}{62 + 42(e^{0.126t} - 1)}$$

We can ask by what year will the population be close to the carrying capacity, let say by what year will the population reach 6,100?

We put $P(t) = 6,100$, hence

$$6,100 = \frac{(2.6)\,10^5\, e^{0.126t}}{62 + 42\,(e^{0.126t} - 1)},$$

i.e.
$$61 = \frac{(2.6)\,10^3\, e^{0.126t}}{62 + 42\,(e^{0.126t} - 1)},$$

i.e.
$$(61).62 + (61).42\,(e^{0.126t} - 1) = (2.6)\,10^3\, e^{0.126t}$$

$$(61).62 - (61)42 = \left[(2.6)\,10^3 - (61)42\right] e^{0.126t}$$

$$1220 = 38\, e^{0.126t}$$

$$0.126t = \ln \frac{1220}{38} \simeq 3.47$$

$$t \approx \frac{3.47}{0.126} \simeq 27.53$$

Hence in approx. 27.5 years, the bison population would be within 100 individuals of the carrying capacity of the Yellowstone Park. *