Section 10.2

Calculus with parametric curves.

First some formulas involving the Chain Rule as stated on p. 645:

We assume that a curve is defined parametrically: $x = f(t)$, $y = g(t)$ although the names of f, g do not enter the formulas below:

Chain Rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

Hence $$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Example. Consider the curve defined parametrically by $x = e^t - 1$, $y = e^{2t}$ as on p. 305 of the Notes.

Then $\dfrac{dx}{dt} = e^t$, $\dfrac{dy}{dt} = 2e^{2t}$,

Hence $\dfrac{dy}{dx} = \dfrac{2e^{2t}}{e^t} = 2e^t$,

where $\dfrac{dy}{dx}$ is calculated using only differantiation with respect to t, not w/r to x.

One could also first eliminate t as on p. 305, obtaining $y = (x+1)^2$ and thus

$$\frac{dy}{dx} = 2(x+1)$$

Since $x = e^t - 1$, substituting into we obtain

$$\frac{dy}{dx} = 2(e^t - 1 + 1) = 2e^t \text{ as before.}$$

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One can also obtain $\frac{d^2y}{dx^2}$ by differentiating with respect to t only:

Let's first restate the chain Rule again: For any w (differentiable with resp. to x, and x diff. with resp. to t) we have

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

Thus if we set $w = \frac{dy}{dx}$,

we obtain $$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt},$$

hence $$\frac{dw}{dx} = \frac{\frac{dw}{dt}}{\frac{dx}{dt}}$$

But $$\boxed{\frac{dw}{dx}} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \boxed{\frac{d^2y}{dx^2}}$$

Also $$\boxed{\frac{dw}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right)}$$ $\frac{dw}{dx} = \frac{d}{dx}(w)$

Hence
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

And, $$\left\{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right\}$$

Hence all differentiations in the box above, on the right hand side, can be done by differentiating w.r. to x only.

Thus Returning to the Example on p. 312, $x = e^t - 1$, $y = e^{2t}$,

we found $\frac{dy}{dx} = 2e^t$, $\frac{dx}{dt} = e^t$,

hence

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(2e^t)}{e^t} =$$

$$= \frac{2e^t}{e^t} = 2;$$

On the other hand, we found by eliminating t that

$$\frac{dy}{dx} = 2(x+1) \quad \text{(top p. 313)}$$

Hence differentiating directly with respect to x, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2(x+1)) = 2$$

which agrees with the result above

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More Complicated Example.

We use Example 2, p. 646 in the Book: The parameter is called θ, instead of t: $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$

Eliminating θ is not a handy thing to do, hence calculating $\frac{dy}{dx}$ "directly" is not a good idea.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\sin\theta}{r - r\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)}{r - r\cos\theta}$$

First calculate $\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right) =$

$$= \frac{\cos\theta\,(1-\cos\theta) - \sin\theta\sin\theta}{(1-\cos\theta)^2} =$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2}$$

$$= \frac{1}{\cos\theta - 1}$$

Hence $$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)}{r - r\cos\theta} =$$

$$= \frac{\frac{1}{\cos\theta - 1}}{r - r\cos\theta} = \frac{1}{r(\cos\theta-1)(1-\cos\theta)} =$$

$$= -\frac{1}{r(1-\cos\theta)^2}$$

Since this quantity is < 0 (for all values of θ which are not an integer multiple of 2π), it follows that the curve is concave down (when viewed as the graph of a function of x).

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Example. Find the equation of the tangent line to the curve at the given point:

(a) $x = e^t - 1$, $y = e^{2t}$, point $(0,1)$

(b) $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, at the point corresponding to $\theta = \frac{\pi}{4}$

Solution.

(a) We have $\frac{dy}{dx} = 2e^t$ (p. 312)

Moreover, $x = 0$, i.e. $e^t - 1 = 0$, i.e. $e^t = 1$, hence $t = 0$.

Thus when $x = 0$, we obtain $\frac{dy}{dx} = 2e^t\Big|_{t=0}$, i.e. $\frac{dy}{dx} = 2e^0 = 2$.

Hence the tangent line passes through $(0,1)$ and has slope $= 2$:

$$y - 1 = 2(x - 0), \text{ i.e.}$$

$$\underline{\underline{y = 2x + 1}}$$

(b) $x = \theta - \sin\theta$, $y = 1 - \cos\theta$,

point corresp. to $\theta = \frac{\pi}{4}$:

$$x = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2},$$

$$y = 1 - \cos\frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2}$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\frac{\pi}{4}}{1-\cos\frac{\pi}{4}} =$$

$$= \frac{\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{2-\sqrt{2}}{2}} = \frac{\sqrt{2}}{2-\sqrt{2}} =$$

$$= \frac{\sqrt{2}(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})} = \frac{\sqrt{2}(2+\sqrt{2})}{2} =$$

$$= \frac{2\sqrt{2}+2}{2} = \underline{\sqrt{2}+1}$$

$$y - \left(1 - \frac{\sqrt{2}}{2}\right) = (\sqrt{2}+1)\left(x - \left(\frac{\pi}{4} - \frac{\sqrt{2}}{2}\right)\right)$$

Not simplify further

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If $x = f(t)$, $y = g(t)$ are the (x,y)-coord. of a particle at time t, then the distance traveled by the particle between $t = \alpha$ and $t = \beta$, i.e. $\alpha \le t \le \beta$, is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

This integral also equals the length of the curve defined by

$x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$

provided the curve is traced out only once as t increases.

<u>Example.</u> Find the length of the curve defined by $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$, $\alpha \le \theta \le \beta$.

Solution. We have $\frac{dx}{dt} = r(1 - \cos\theta)$, hence $\frac{dx}{dt} \ge 0$ and $= 0$ only when θ is an integer multiple of 2π. Hence x is strictly increasing as θ increases, which means that

every point on the curve is visited only once. Thus we can use the formula
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

We have $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$

$$(r(1-\cos\theta))^2 + (r\sin\theta)^2 =$$

$$r^2(1-2\cos\theta + \cos^2\theta) + r^2\sin^2\theta =$$

$$= r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) =$$

$$= r^2(2 - 2\cos\theta) = 4r^2\left(\tfrac{1}{2}(1-\cos\theta)\right)$$

$$= 4r^2 \sin^2\left(\frac{\theta}{2}\right)$$

Hence $L = \int_\alpha^\beta \sqrt{4r^2 \sin^2\left(\frac{\theta}{2}\right)}\, d\theta =$

$$= \int_\alpha^\beta 2r \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

Thus if, for example $0 \le \theta \le 2\pi$, then $0 \le \frac{\theta}{2} \le \pi$, hence $\sin\frac{\theta}{2} \geq 0$

and $|\sin\frac{\theta}{2}| = \sin\frac{\theta}{2}$, hence

$$\int_0^{2\pi} 2r\,|\sin\tfrac{\theta}{2}|\,d\theta = \int_0^{2\pi} 2r\sin\tfrac{\theta}{2}\,d\theta$$

$$= \left(2r\cdot 2(-\cos\tfrac{\theta}{2})\right)\Big|_0^{2\pi} =$$

$$4r\left(-\cos\pi - (-\cos 0)\right) =$$

$$4r\left(-(-1)-(-1)\right) = \underline{\underline{8r}}$$

On the other hand, if $2\pi \le \theta \le 3\pi$, then $\pi \le \frac{\theta}{2} \le \frac{3}{2}\pi$, hence $\sin(\frac{\theta}{2}) \le 0$ and thus $|\sin(\frac{\theta}{2})| = -\sin\frac{\theta}{2}$, and

thus $$L = \int_{2\pi}^{3\pi} 2r\,|\sin(\tfrac{\theta}{2})|\,d\theta =$$

$$= \int_{2\pi}^{3\pi} 2r(-\sin\tfrac{\theta}{2})\,d\theta = \left(4r\cos\tfrac{\theta}{2}\right)\Big|_{2\pi}^{3\pi}$$

$$= 4r\left(\cos\tfrac{3\pi}{2} - \cos\pi\right) = 4r(0-(-1))$$

$$= \underline{\underline{4r}}$$

The Area of the Surface Obtained by Rotating a parametrically defined curve about the x-axis or about the y-axis:

This is similar to what we did in the section 8.2, pages 235-239 in the Notes.

For Example, on p. 235, if the surface is obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, then the area equals

$$A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

or

$$\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is defined parametrically, where x, y are functions of t, $\alpha \le t \le \beta$,

then $$A = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For Example, # 62, p. 652 in the Book: Find the exact area of the surface obtained by rotating the curve

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 1$$

about the x-axis:

$$A = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

First $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2$

$$= 9 - 18t^2 + 9t^4 + 36t^2 =$$

$$= 9t^4 + 18t^2 + 9 = 9(t^4 + 2t^2 + 1)$$

$$= 9(t^2+1)^2,$$

Hence $A = \int_0^1 2\pi y \sqrt{9(t^2+1)^2}\, dt$

$$= \int_0^1 2\pi y \cdot 3(t^2+1)\, dt =$$

$$= 6\pi \int_0^1 3t^2 (t^2+1)\, dt = 18\pi \left(\frac{t^5}{5} + \frac{t^3}{3}\right)\Bigg|_0^1$$

$$= 18\pi\left(\frac{1}{5} + \frac{1}{3}\right) = 18\pi \cdot \frac{8}{15} = \underline{\underline{\frac{48}{5}\pi}} \quad \ast$$

Rotating about the y-axis works similarly. The Area is given by

$$\int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt,$$

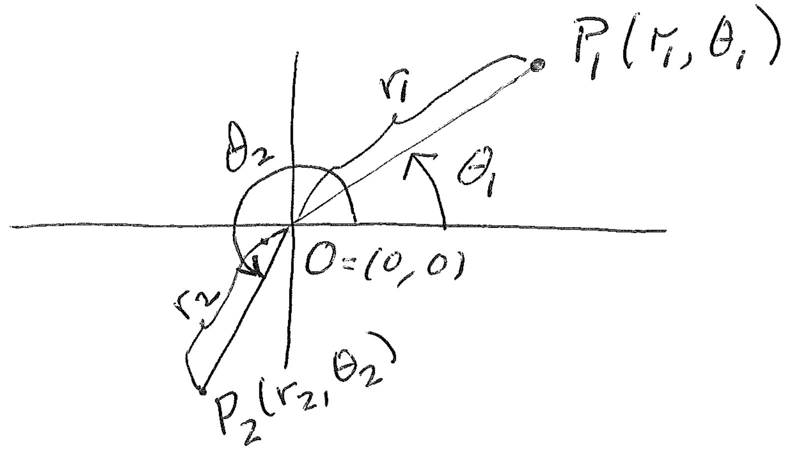
where x is substituted by its expression in terms of the parameter t.

For Example for the surface from the Exercise #62, p. 652 in the Book (which we just did for rotation about the y-axis), the integral is

$$\int_0^1 2\pi \underbrace{(3t-t^3)}_{x} \cdot \underbrace{3(t^2+1)}_{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}} dt \qquad \ast$$

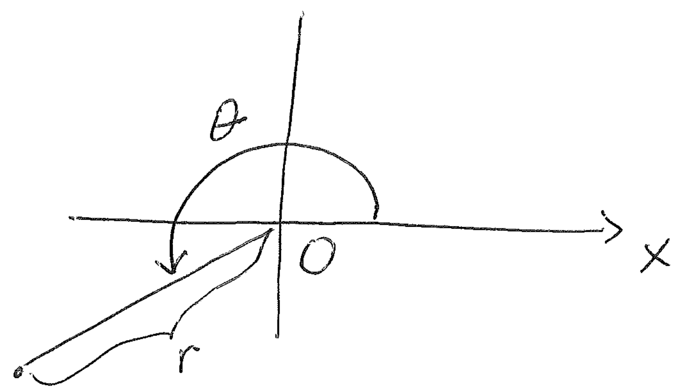
Section 10.3, Polar Coordinates.

Every point in the plane can be assigned exactly one pair of x, y - coordinates but many different pairs of polar coordinates. Firstly, every point $P(x, y)$ which is not the origin $(0,0)$, can be assigned exactly one pair of "basic" polar coordinates (r, θ) which are obtained as follows:



The first coordinate, denoted by r, is the distance of the point from the origin. The 2nd coordinate, denoted by θ, is the angle between the segment PO joining

P to the origin and the positive direction of the x-axis, measured from the positive direction of the x-axis to the segment PO, in counterclockwise direction.

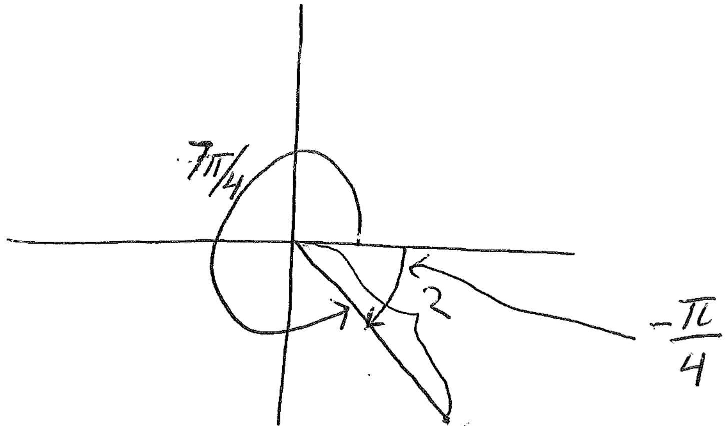


$P = P(r, \theta)$

Furthermore, the point P with "basic" polar coordinates (r, θ) is also assigned additional polar coordinates $(r, \theta + 2n\pi)$ where n is any integer — positive, negative (or zero — we then obtain the original coordinates (r, θ)).

The "basic" polar coordinates of a point $P \neq O$ are then those polar coordinates (r, θ) where $r > 0$ and $0 \leq \theta < 2\pi$. Thus we will not use the word basic officially, instead we can say "that pair of polar coordinates where $r > 0$ and $0 \leq \theta < 2\pi$".

Negative θ: Consider e.g. the point P with polar coordinates $(2, -\frac{\pi}{4})$. Then $(2, -\frac{\pi}{4} + 2n\pi)$ are also polar coordinates of the same point, where n is any integer. In particular $(2, -\frac{\pi}{4} + 2\pi) = (2, \frac{7}{4}\pi)$ is a pair of polar coordinates of this point:



$(2, \frac{7}{4}\pi)$; $(2, -\frac{\pi}{4})$

Thus a negative angle means to measure the angle of its absolute value in counterclockwise direction.

Polar Coordinates of the origin:

Any pair $(0, \theta)$, i.e. the first coordinate $= 0$ (distance from the origin), but the 2nd coordinate can be any angle whatsoever.

Negative r: We agree that the point P with polar coordinates (r, θ) will also be assigned coordinates $(-r, \theta+\pi)$, and thus also $(-r, \theta+(2n+1)\pi)$

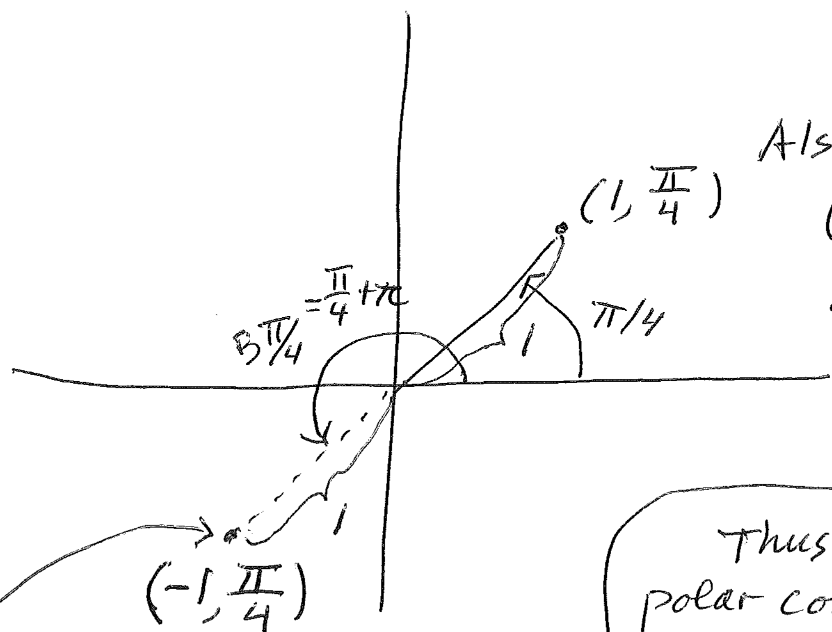
where n is any integer.

Thus if a point P, $\neq O$, has polar coordinates (r, θ), then all pairs of its polar coordinates are

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi),$$

where n is any integer.

A picture for negative r.



Also has pol. coord.

$(-1, \frac{\pi}{4} - \pi) = (-1, -\frac{3\pi}{4})$

and $(-1, \frac{\pi}{4} + \pi) = (-1, \frac{5\pi}{4})$

Also has pol. coordinates

$(1, \frac{\pi}{4} + \pi) = (1, \frac{5\pi}{4})$

and $(1, \frac{\pi}{4} - \pi) = (1, -\frac{3\pi}{4})$

Thus the point with polar coord. $(1, -\frac{3\pi}{4})$ also has polar coord. $(1, \frac{5\pi}{4})$ and $(-1, \frac{\pi}{4})$, which answers 1(b), p. 662