Example. Find all possible polar coordinates of the point with Cartesian coordinates $(-2, 2\sqrt{3})$.

HW will be due Thursday 3/13: sections 10.3, 10.4.

Solution. On page 342 we found polar coordinates $(4, \frac{2}{3}\pi)$.

As explained in a paragraph near the top of p. 330, all possible polar coordinates then are

$$(4, \tfrac{2}{3}\pi + 2n\pi) \text{ and } (-4, \underbrace{\tfrac{2}{3}\pi + (2n+1)\pi})$$

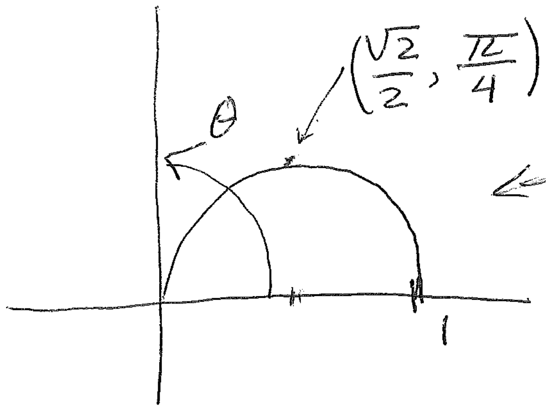
where n is any integer (pos., neg. or 0).

For example, we can obtain a pair of polar coordinates with $r<0$ if we set $n=0$, obtaining $(-4, \frac{2}{3}\pi + \pi) = (-4, \frac{5}{3}\pi)$.

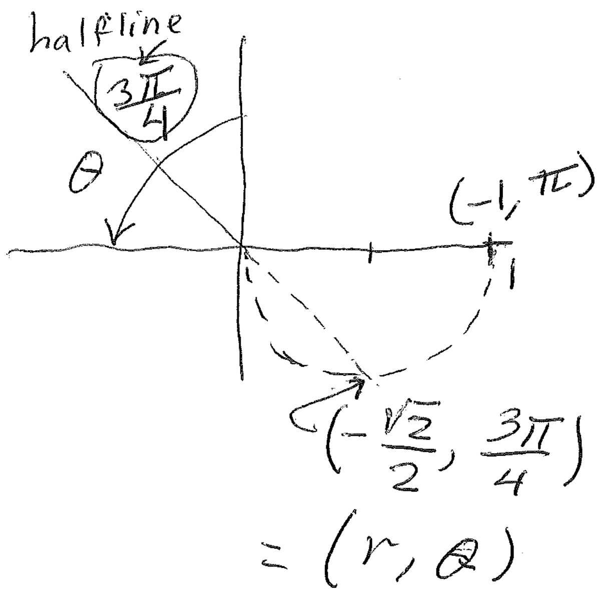
ADD at the BOTTOM of p. 339. ✱

If P lies on the negative y-axis, i.e. $P=(0,y)$ with $y<0$, then $\theta = \frac{3\pi}{2}$, $r = |y| = -y$, i.e. $(|y|, 3\pi/2)$ is a pair of polar coordinates of P. ✱

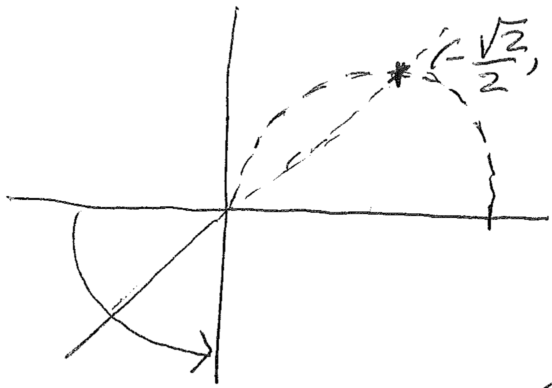
Sketch the curve $r = \cos\theta$:



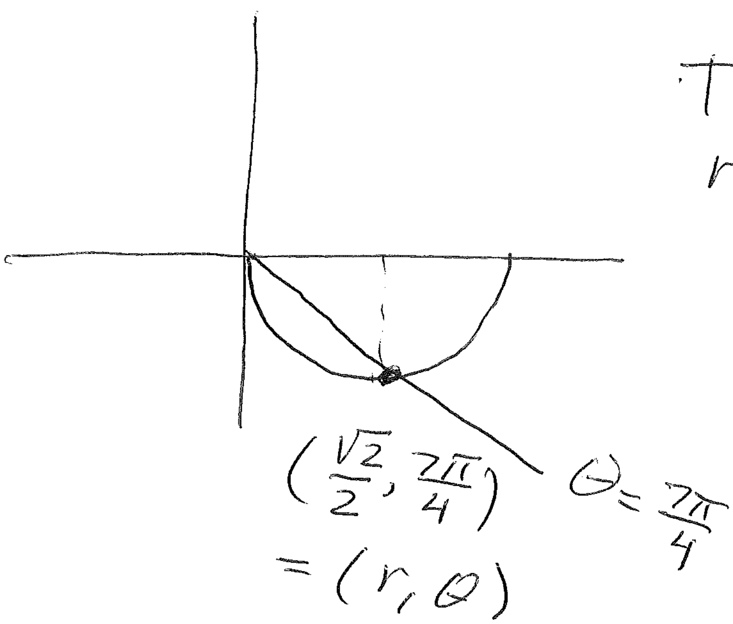
This is the part of the curve $r = \cos\theta$, $0 \le \theta \le \frac{\pi}{2}$



The broken line semicircle is the part of the curve $r = \cos\theta$, for $\frac{\pi}{2} < \theta \le \pi$; r is negative, hence as θ varies over the 2nd quadrant, the corresponding point varies over the "opposite" quadrant, i.e. the 4-th Quadrant.

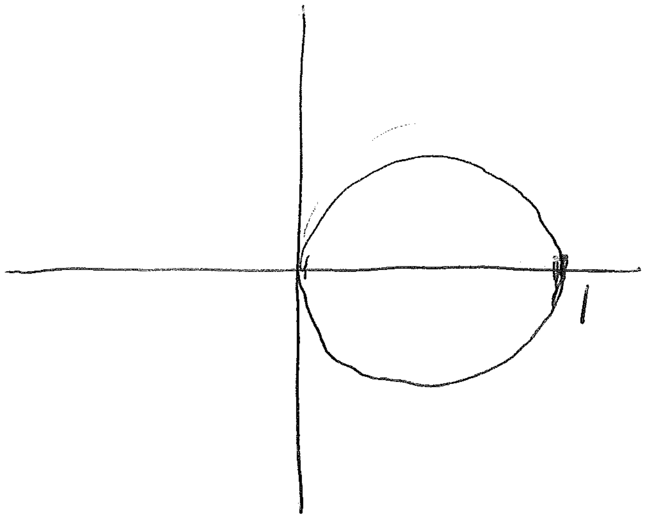


The part of the curve $r = \cos\theta$ for $\pi < \theta \leq \frac{3\pi}{2}$ r is negative, so as θ varies over Quadr. III, the corresponding part of the curve is being traced out in Quadrant I.



The part of the curve $r = \cos\theta$ for $\frac{3\pi}{2} \leq \theta < 2\pi$ r is positive, so as θ varies over the quadrant 4, the corresponding part of the curve is likewise being traced out in Quadrant 4.

Sketch of the entire curve $r = \cos\theta$.



The curve is traced out twice, as the sketches on pages 344, 345 show. Once for r positive, when $0 \le \theta \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le \theta \le 2\pi$;

and once for r negative, when $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

Similarly as one can convert polar coordinates of a point into cartesian coordinates and vice versa, one can also convert equations of curves from one of the two coordinate systems into the other.

Example. Find a Cartesian equation of the curve $r = \sin\theta$.

Solution. We will use the equation $y = r\sin\theta$ (see near top of p. 339 of the Notes). So $\frac{y}{r} = \sin\theta$, hence we obtain $r = \frac{y}{r}$, and $\boxed{r^2 = y}$. (We can obtain the equation $r^2 = y$ more easily by multiplying the eq. $r = \sin\theta$ by r, obtaining $r^2 = r\sin\theta$, hence $r^2 = y$. This second method also has the advantage that we don't need to divide by r which is not permissible for $r = 0$.)

Into the equation $r^2 = y$ we can now substitute $r^2 = x^2+y^2$; again see p. 339. So we obtain

$x^2+y^2 = y$, hence

$x^2+y^2-y = 0$

We now complete y^2-y to a square:

$x^2+y^2-y+\frac{1}{4} = \frac{1}{4}$

$x^2+(y-\frac{1}{2})^2 = \frac{1}{4}$

Hence this is an equation of the circle centered at $(0, \frac{1}{2})$ and of radius $= \frac{1}{2}$ — See the sketch on p. 331.

✱

Example. Identify the curve by finding a Cartesian equation for the curve:

$r = 4 \tan\theta \sec\theta$

Solution. We know $\sec\theta = \frac{1}{\cos\theta}$, so multiplying both sides by $\cos\theta$, we get $r\cos\theta = 4\tan\theta$

Now we use the equations

$x = r\cos\theta$ and $\tan\theta = \frac{y}{x}$ (see p. 339)

hence $x = 4\frac{y}{x}$ and finally

$y = \frac{1}{4}x^2$ which is a parabola.

We note that the step

$\tan\theta = \frac{y}{x}$ cannot be used for points (x, y) on the y-axis, i.e. $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, in both of which cases, both $\tan\theta$ and $\sec\theta$ are undefined, hence no points on the curve are obtained using $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$.

Likewise the equation $y = \frac{1}{4}x^2$ is not satisfied by any points $(0, y)$ unless $y = 0$. Thus the origin $(0, 0)$ lies on the curve $y = \frac{1}{4}x^2$. But the origin also lies on the curve $r = 4\tan\theta\sec\theta$, since we can set $\theta = 0$, obtaining

$$r = 4\tan 0 \sec 0 = 4(0)(1) = 0,$$

and $(0, 0)$ is a pair of polar coordinates of the origin.

Thus the two equations

$$r = 4\tan\theta\sec\theta, \quad y = \frac{1}{4}x^2$$

define exactly the same sets of points.

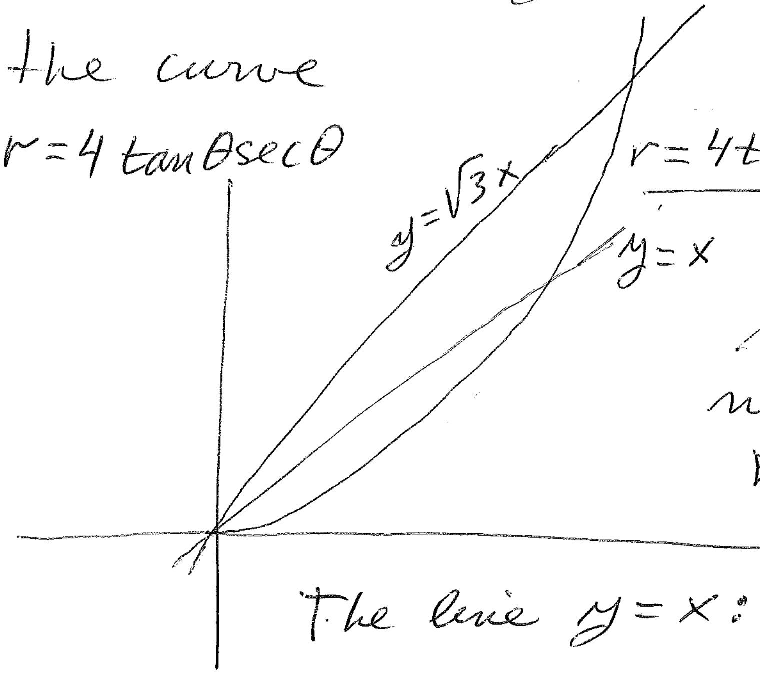
K

Sect. 10.4.

The Area bounded by the half-lines $\theta = a$, $\theta = b$ and the curve $r = f(\theta)$ equals

$$\frac{1}{2}\int_a^b (f(\theta))^2 \, d\theta$$

Example. Find the area of the region bounded by the lines $y = x$, $y = \sqrt{3}x$ and the curve $r = 4\tan\theta\sec\theta$



$r = 4\tan\theta\sec\theta$, i.e. $y = \frac{1}{4}x^2$

where we use the work from the Example on p. 349.

The line $y = x$: Except for the origin $(x,y) = (0,0)$, we obtain $\frac{y}{x} = 1$, but $\frac{y}{x} = \tan\theta$ (see p. 339), so on the line $y = x$ we have $\tan\theta = 1$, hence $\theta = \frac{\pi}{4}$, which is the equation of the half-line $y = x$ in Quadrant I.

Similarly $y = \sqrt{3}x \Rightarrow \frac{y}{x} = \sqrt{3}$,

hence $\tan\theta = \sqrt{3}$, hence $\theta = \frac{\pi}{3}$ for

x, y in the first quadrant.

Hence the area is

$$\frac{1}{2}\int_{\pi/4}^{\pi/3} (4\tan\theta\sec\theta)^2 d\theta =$$

$$= 8\int_{\pi/4}^{\pi/3} \tan^2\theta \sec^2\theta \, d\theta$$

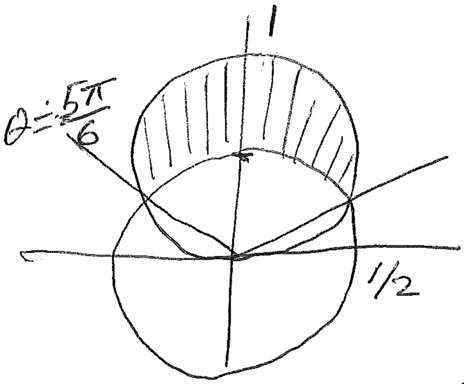
$$= 8 \cdot \frac{1}{3}\tan^3\theta \Big|_{\pi/4}^{\pi/3} =$$

$$= \frac{8}{3}\left(\tan^3\frac{\pi}{3} - \tan^3\frac{\pi}{4}\right)$$

$$= \frac{8}{3}\left((\sqrt{3})^3 - 1\right) = \frac{8}{3}3\sqrt{3} - \frac{8}{3}$$

$$= 8\sqrt{3} - \frac{8}{3}$$

Example. Find the area inside the circle $x^2+(y-\frac{1}{2})^2=\frac{1}{4}$ but outside the circle $x^2+y^2=\frac{1}{4}$



In polar coord.,

$x^2+(y-\frac{1}{2})^2=\frac{1}{4} \Leftrightarrow r=\sin\theta,$

see Example on p. 347, 348.

$x^2+y^2=\frac{1}{4} \Leftrightarrow r=\frac{1}{2}$

We can find the intersection points of $r=\sin\theta$, $r=\frac{1}{2}$ by setting

$\sin\theta=\frac{1}{2}$, i.e. $\theta=\frac{\pi}{6}, \frac{5\pi}{6}$

Then the desired area can be calculated as the Area in the circle $r=\sin\theta$ bounded by the halflines $\theta=\frac{\pi}{6}$ and $\theta=\frac{5\pi}{6}$ minus the area inside the circle $r=\frac{1}{2}$ bounded by the half lines $\theta=\frac{\pi}{6}$, $\theta=\frac{5\pi}{6}$, i.e.

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} \sin^2\theta \, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\frac{1}{2}\right)^2 d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \frac{1}{2}(1-\cos 2\theta)\, d\theta - \frac{1}{8}\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)$$

$$= \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/6}^{5\pi/6} - \frac{1}{8}\cdot\frac{2}{3}\pi$$

$$= \frac{1}{4}\left(\frac{5\pi}{6} - \frac{1}{2}\sin\frac{5\pi}{3}\right) - \frac{1}{4}\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - \frac{1}{12}\pi$$

$$= \frac{5\pi}{24} - \frac{\pi}{24} - \frac{1}{8}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{8}\frac{\sqrt{3}}{2} - \frac{1}{12}\pi$$

$$= \frac{1}{6}\pi - \frac{1}{12}\pi + 2\cdot\frac{1}{8}\frac{\sqrt{3}}{2} =$$

$$= \frac{1}{12}\pi + \frac{\sqrt{3}}{8}$$