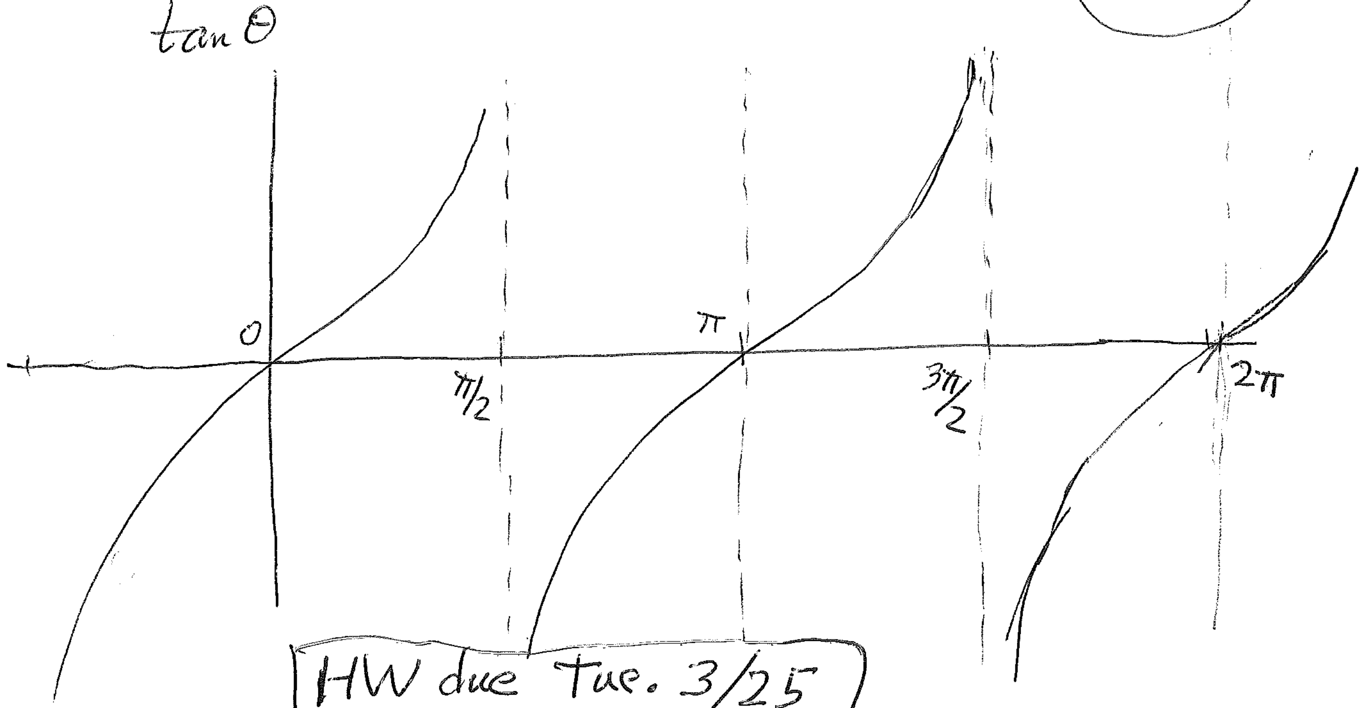
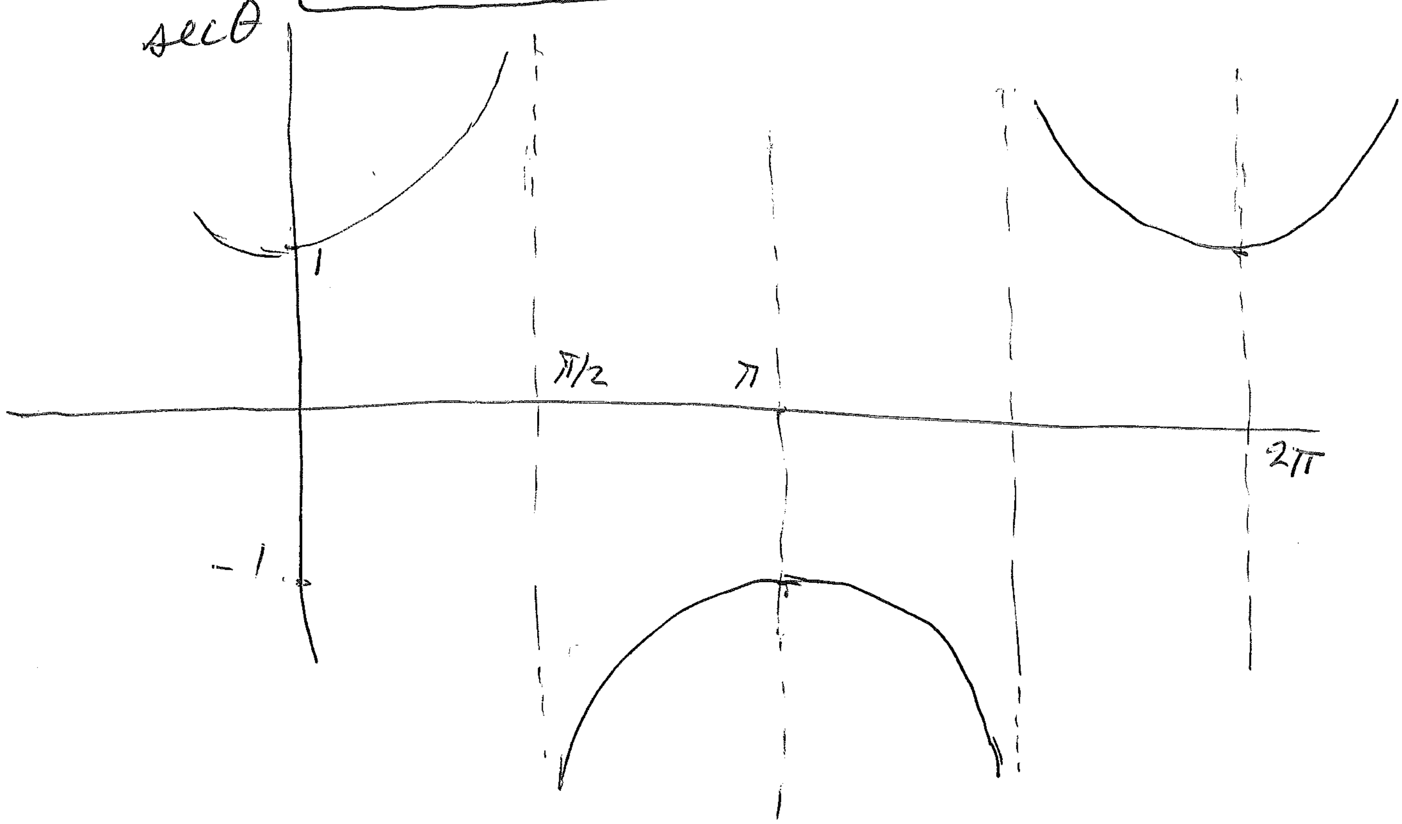


HW due Tue. 3/25
10.5, 11.1

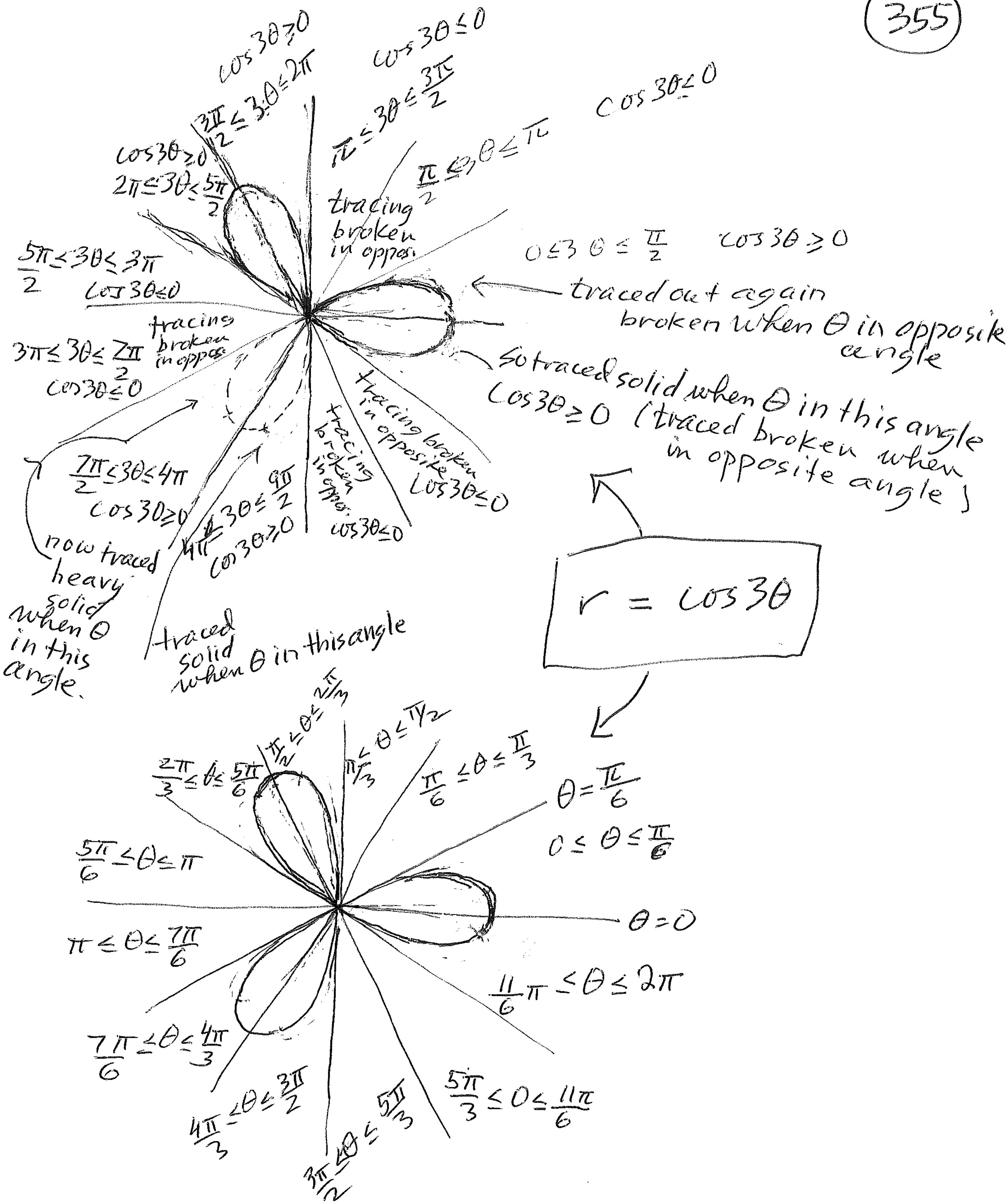


Useful for sketching the curve
$r = 4\tan\theta\sec\theta$ (p. 349, 350, 351).

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\sec\theta$	$\tan\theta\sec\theta$
$(0, \frac{\pi}{2})$	>0	>0	>0	>0	>0
$(\frac{\pi}{2}, \pi)$	>0	<0	<0	<0	>0
$(\pi, \frac{3\pi}{2})$	<0	<0	>0	<0	<0
$(\frac{3\pi}{2}, 2\pi)$	<0	>0	<0	>0	<0

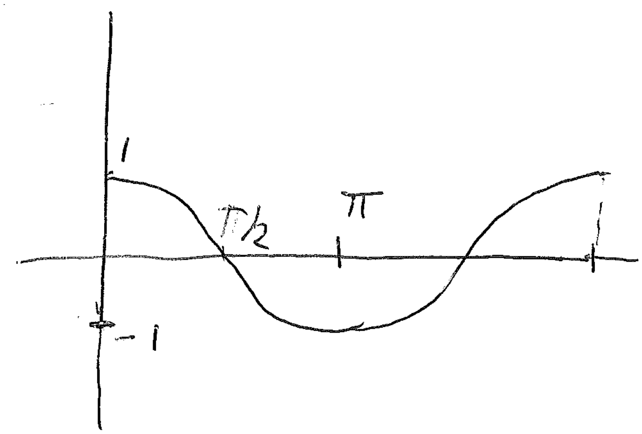
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \sec\theta = \frac{1}{\cos\theta}$$

Useful for sketching the curve
$r = 4\tan\theta\sec\theta$ (p. 349, 350, 351)


$\cos 3\theta \geq 0$
$\frac{3\pi}{2} \leq 3\theta \leq 2\pi$
$\cos 3\theta \leq 0$
$\pi \leq 3\theta \leq \frac{3\pi}{2}$
$\cos 3\theta \leq 0$
$\frac{\pi}{2} \leq 3\theta \leq \pi$
$\cos 3\theta \geq 0$
$2\pi \leq 3\theta \leq \frac{5\pi}{2}$
tracing broken in oppos.
$0 \leq 3\theta \leq \frac{\pi}{2}$
$\cos 3\theta \geq 0$
$\frac{5\pi}{2} \leq 3\theta \leq 3\pi$
$\cos 3\theta \leq 0$
traced out again broken when θ in opposite angle
tracing broken in oppos.
$3\pi \leq 3\theta \leq \frac{7\pi}{2}$
$\cos 3\theta \leq 0$
So traced solid when θ in this angle $\cos 3\theta \geq 0$ (traced broken when in opposite angle)
tracing broken in opposite
tracing broken in oppos.
$\frac{7\pi}{2} \leq 3\theta \leq 4\pi$
$\cos 3\theta \geq 0$
$4\pi \leq 3\theta \leq \frac{9\pi}{2}$
$\cos 3\theta \geq 0$
$\cos 3\theta \leq 0$
$\cos 3\theta \leq 0$
now traced heavy solid when θ in this angle.
traced solid when θ in this angle
$r = \cos 3\theta$
$\frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$
$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$
$\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$
$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$
$\theta = \frac{\pi}{6}$
$0 \leq \theta \leq \frac{\pi}{6}$
$\frac{5\pi}{6} \leq \theta \leq \pi$
$\theta = 0$
$\pi \leq \theta \leq \frac{7\pi}{6}$
$\frac{11}{6}\pi \leq \theta \leq 2\pi$
$\frac{7\pi}{6} \leq \theta \leq \frac{4\pi}{3}$
$\frac{5\pi}{3} \leq \theta \leq \frac{11\pi}{6}$
$\frac{4\pi}{3} \leq \theta \leq \frac{3\pi}{2}$
$\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{3}$

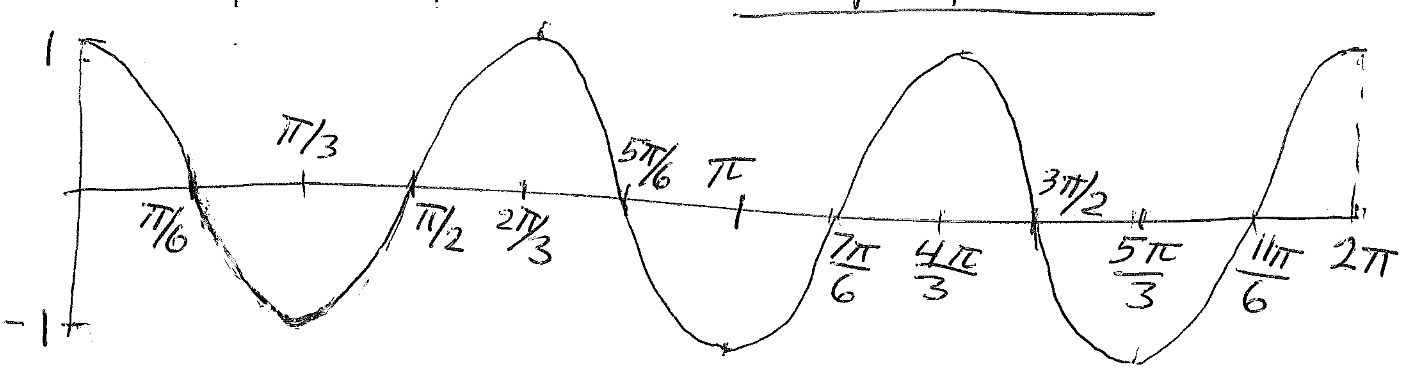
θ	3θ	$\cos 3\theta$
$(0, \frac{\pi}{6})$	$(0, \frac{\pi}{2})$	> 0
$(\frac{\pi}{6}, \frac{\pi}{3})$	$(\frac{\pi}{2}, \pi)$	< 0
$(\frac{\pi}{3}, \frac{\pi}{2})$	$(\pi, \frac{3\pi}{2})$	< 0
$(\frac{\pi}{2}, \frac{2\pi}{3})$	$(\frac{3\pi}{2}, 2\pi)$	> 0
$(\frac{2\pi}{3}, \frac{5\pi}{6})$	$(2\pi, \frac{5\pi}{2})$	> 0
$(\frac{5\pi}{6}, \pi)$	$(\frac{5\pi}{2}, 3\pi)$	< 0
$(\pi, \frac{7\pi}{6})$	$(3\pi, \frac{7\pi}{2})$	< 0
$(\frac{7\pi}{6}, \frac{4\pi}{3})$	$(\frac{7\pi}{2}, 4\pi)$	> 0
$(\frac{4\pi}{3}, \frac{3\pi}{2})$	$(4\pi, \frac{9\pi}{2})$	> 0
$(\frac{3\pi}{2}, \frac{5\pi}{3})$	$(\frac{9\pi}{2}, 5\pi)$	< 0
$(\frac{5\pi}{3}, \frac{11\pi}{6})$	$(5\pi, \frac{11\pi}{2})$	< 0
$(\frac{11\pi}{6}, 2\pi)$	$(\frac{11\pi}{2}, 6\pi)$	< 0

$\cos\varphi$



φ	$\cos\varphi$
$(0, \frac{\pi}{2})$	> 0
$(\frac{\pi}{2}, \pi)$	< 0
$(\pi, \frac{3\pi}{2})$	< 0
$(\frac{3\pi}{2}, 2\pi)$	> 0

Graph of $\cos 3\theta$.



<u>Example</u>. Find the area inside the curve $r = \cos 3\theta$.

<u>Solution</u>. The curve is drawn on p. 355 and consists of three <u>leaves of the same shape</u>. So it suffices to find the area inside one half of a leaf and multiply by 6.

So we will find the area inside the half leaf which lies in the angle $0 \le \theta \le \frac{\pi}{6}$:

$$\frac{1}{2}\int_0^{\pi/6} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/6} \cos^2 3\theta\,d\theta =$$

$$= \frac{1}{2}\int_0^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta =$$

$$= \frac{1}{2}\left(\frac{1}{2}\theta + \frac{1}{12}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{12}\underbrace{\sin\pi}_{=0}\right) - \frac{1}{2}\underbrace{\left(\frac{1}{2}0 + \frac{1}{12}\sin 0\right)}_{=0}$$

$= \frac{1}{24}\pi$. Hence the total area inside all 6 leaves is $\frac{\pi}{4}$. ✱

Example. Exercise #4, p. 676 in the Book: Find the vertex, focus and directrix of the parabola and sketch its graph: $3x^2 + 8y = 0$

Use the fact in Box [1], p. 671 in the Book:

Eq. of parabola with focus $(0, p)$ and directrix $y = -p$ is

p can be > 0 or < 0

$$x^2 = 4py \quad \text{(vertex at } (0,0))$$

$$3x^2 + 8y = 0 \implies x^2 = -\frac{8}{3}y$$

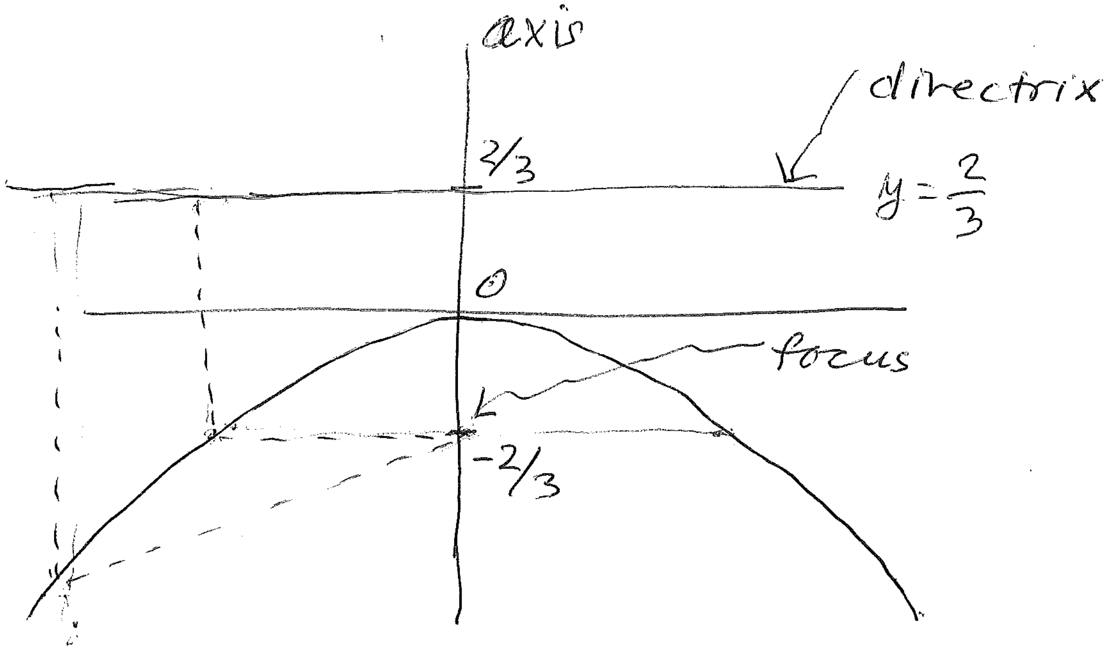
So $4p = -\frac{8}{3}$

$p = -\frac{2}{3}$

Focus at $\left(0, -\frac{2}{3}\right)$, Directrix $y = \frac{2}{3}$

For any parabola of the form $y = ax^2$ or $x = ay^2$, where $a \neq 0$, vertex is at $(0,0)$.

Moreover the parabola $y = ax^2$ opens upwards when $a > 0$, and opens downwards when $a < 0$. Similarly parabola $x = ay^2$ opens to the right if $a > 0$, and opens to the left when $a < 0$.



The points on the parabola have equal distance from the focus and from the directrix

*

It works similarly for Equations of the type $y^2 = ax$:

<u>Example.</u> Find the vertex, focus and directrix of the parabola

$$5x - 2y^2 = 0$$

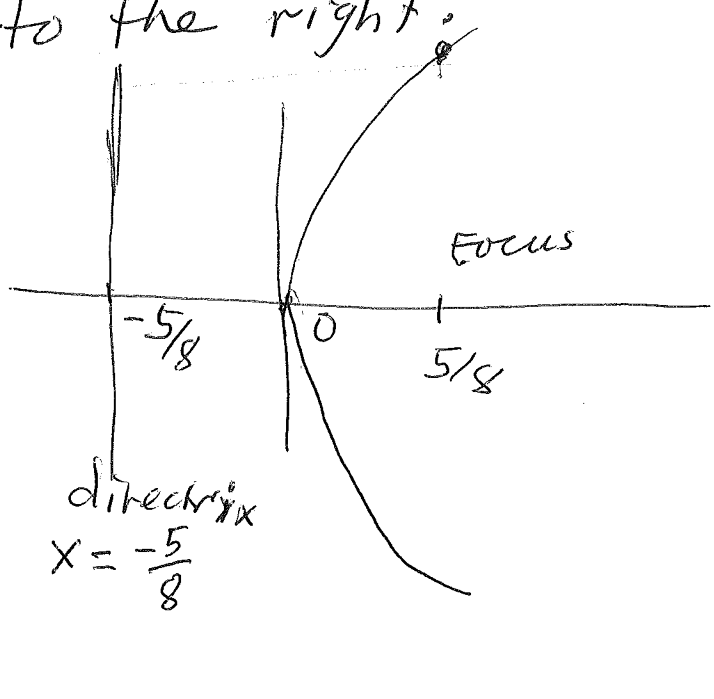
<u>Solution.</u> $y^2 = \frac{5}{2}x$

$$4p = \frac{5}{2}, \quad p = \frac{5}{8}$$

Thus focus is at $(\frac{5}{8}, 0)$,

directrix is $x = -\frac{5}{8}$.

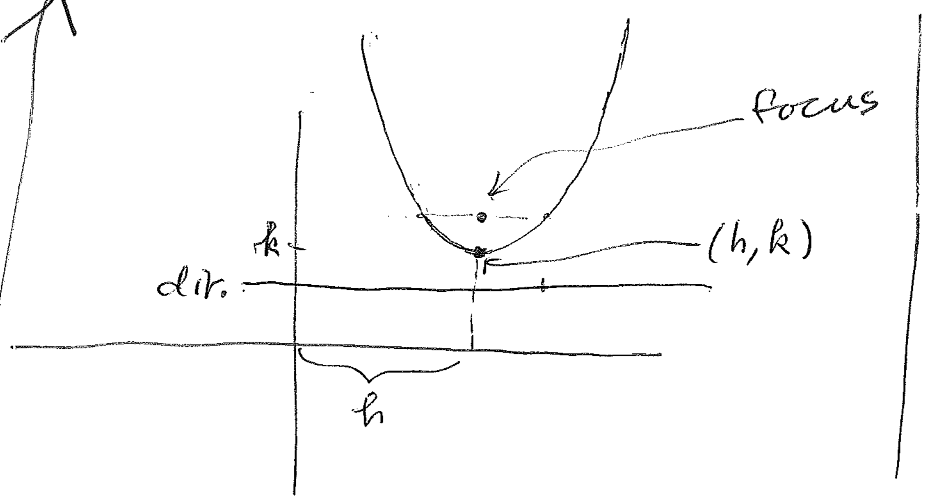
Vertex at $(0,0)$, opens to the right.



Shifting the parabola.

We modify the statement in the box on p. 358: Shifting $x^2 = 4py$ from $(0,0)$ vertex to (h,k) vertex

> Eq. of parabola with focus at $(h, p+k)$, directrix $y = k-p$, vertex at (h,k) is
> $$(x-h)^2 = 4p(y-k)$$



Example. Find the vertex, focus and directrix of the parabola

$$3y = 2x^2 - 12x$$

Solution on next page

So compared to the Box on p. 358, since we are adding h, k to the coordinates $(0,0)$ of the vertex, we do the same for the focus: $(0,p) \rightsquigarrow (h, p+k)$. We proceed similarly with the directrix:

$$y = -p \rightsquigarrow y = -p+k, \text{ i.e. } y = k-p.$$

Solution:

$$3y = 2x^2 - 12x$$

$$3y = 2(x^2 - 6x)$$

$$3y = 2(x^2 - 6x + 9) - 18$$

$$3y + 18 = 2(x-3)^2 \Rightarrow 3(y+6) = 2(x-3)^2$$

~~$y + 6 = \frac{2}{3}(x-3)^2$~~

$$(x-3)^2 = \frac{3}{2}(y+6)$$

axis vertical since y has degree 1, x has degree 2.

$$\frac{3}{2} = 4p, \quad p = \frac{3}{8}$$

Vertex at $(3, -6)$

Focus at $(3, -6 + \frac{3}{8}) = (3, -\frac{45}{8})$

Directrix

$$y = k - p = -6 - \frac{3}{8} = -\frac{51}{8}$$

$$y = -\frac{51}{8}$$

*

Ellipses.

I am not going to copy everything from the Book, so you should definitely use the Book in addition to the Notes when learning this material.

The first and simplest comment is that if we are given an equation of the form

$$\alpha x^2 + \beta y^2 = d$$

where α, β, d are POSITIVE numbers, i.e. > 0 (not ≥ 0), then it is an equation of a circle centered at the origin when $\boxed{\alpha = \beta}$;

and it is an equation of an ellipse centered at the origin, when $\boxed{\alpha \neq \beta}$.

Also, at the bottom of the preceding page, the word <u>ellipse</u> is used in the colloquial sense, i.e. an ellipse which is not a circle: circle is usually considered a special case of ellipse with equal semiaxes.

<u>When $\alpha = \beta$:</u> i.e.

$$\alpha x^2 + \alpha y^2 = d$$

then we usually put the equation in the form

$$x^2 + y^2 = \frac{d}{\alpha} = r^2,$$

where $r = \sqrt{\frac{d}{\alpha}}$ is the radius of the circle.

<u>When $\alpha \neq \beta$:</u> Then we put the equation in the form

$$\frac{\alpha x^2}{d} + \frac{\beta y^2}{d} = 1,$$

and

$$\frac{x^2}{\frac{d}{\alpha}} + \frac{y^2}{\frac{d}{\beta}} = 1$$

$$\frac{x^2}{\left(\sqrt{\frac{d}{\alpha}}\right)^2} + \frac{y^2}{\left(\sqrt{\frac{d}{\beta}}\right)^2} = 1$$

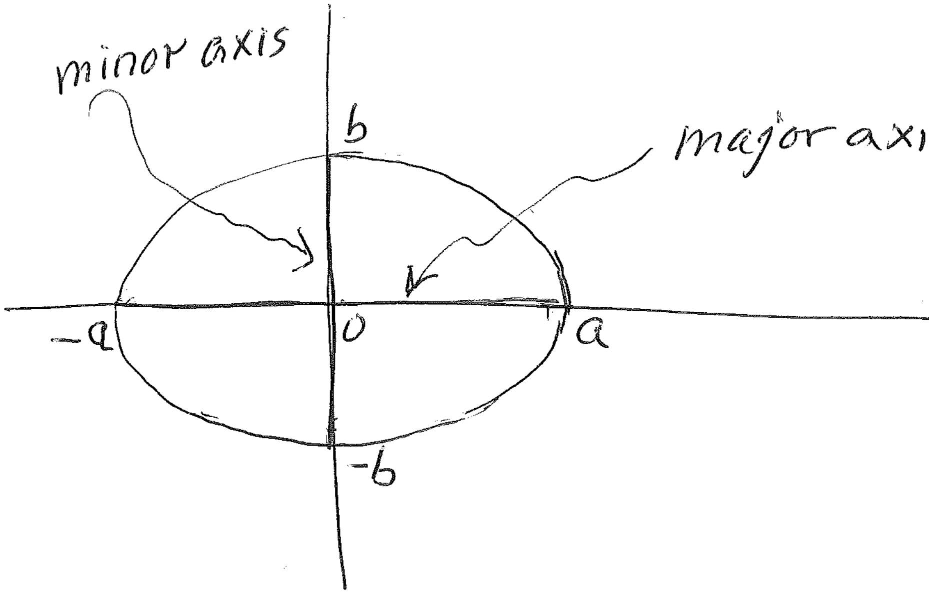
Then we set

$$a = \sqrt{\frac{d}{\alpha}}, \quad b = \sqrt{\frac{d}{\beta}},$$

hence

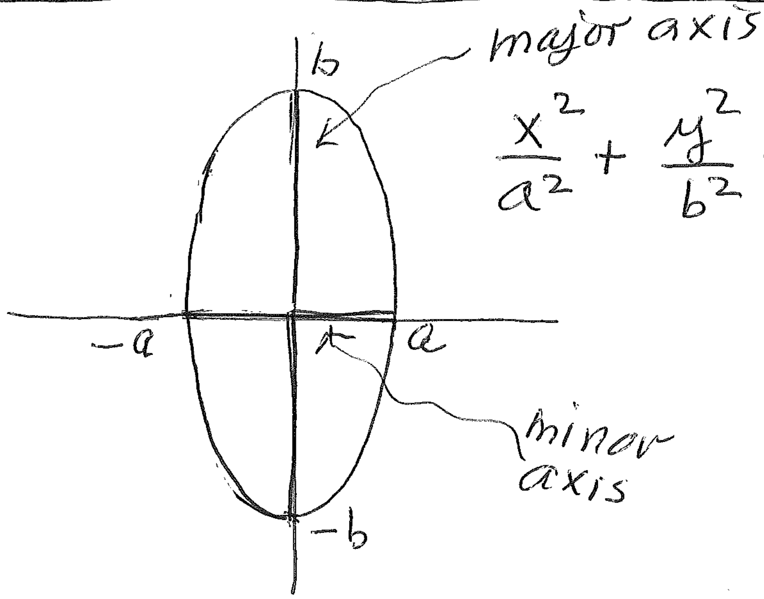
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hence the ellipse passes through $(\pm a, 0)$ and $(0, \pm b)$ which are called the vertices of the ellipse:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \underline{a > b}$$

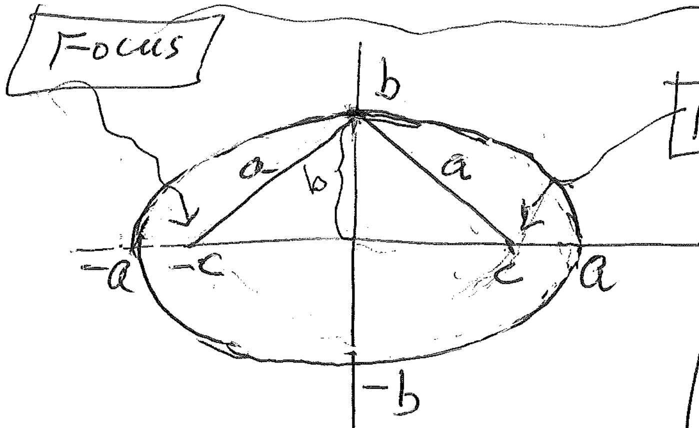
The segment joining $(-a, 0), (0, a)$ is called the major axis.
The segment joining $(0, -b)$ and $(0, b)$ is called the minor axis.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \underline{a < b}$$

The Foci of an Ellipse:

(Foci — plural of "Focus".)



$a > b$ picture

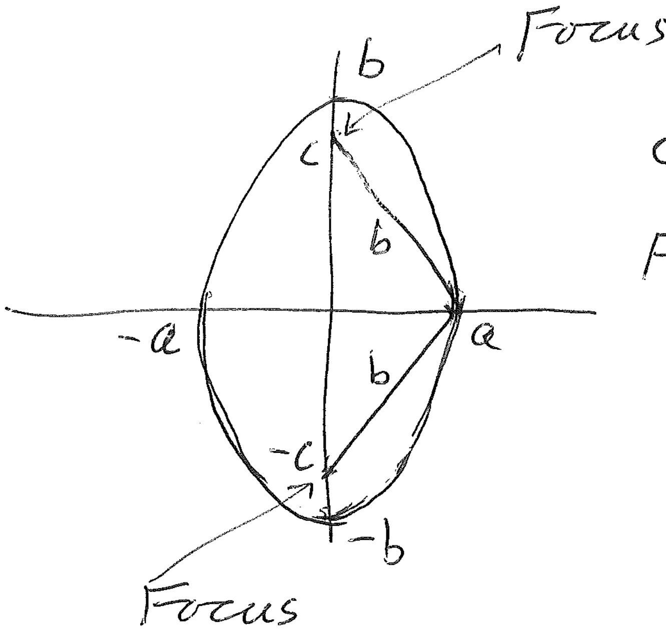
$c = \sqrt{a^2 - b^2}$

The points $(\pm c, 0)$ are the two foci

Foci lie on the major axis, on the inside of the ellipse, and have the property that the sum of the distances, of any point on the ellipse, from the two foci, remains constant, and $= 2a$ (when $a > b$), so, equals the length of the major axis.

this sum of the distances

$a < b$
picture



$c = \sqrt{b^2 - a^2}$

Foci at $(0, \pm c)$

Example. Sketch the graph of $15x^2 + 4y^2 = 36$ and locate the foci.

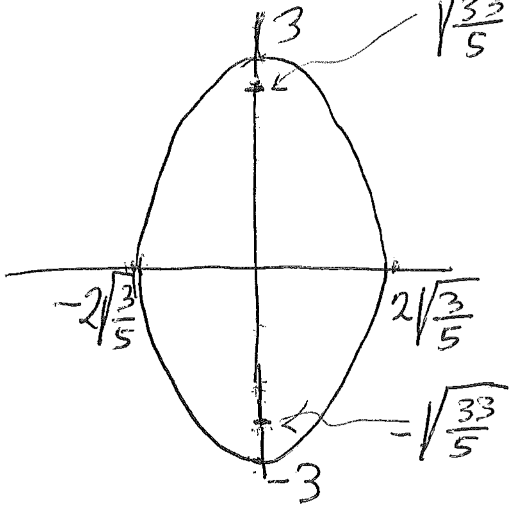
Solution. $\frac{15}{36}x^2 + \frac{4}{36}y^2 = 1$

$$\frac{5}{12}x^2 + \frac{1}{9}y^2 = 1$$

$$\frac{x^2}{\frac{12}{5}} + \frac{y^2}{9} = 1$$

So $a^2 = \frac{12}{5}$, $a = \sqrt{\frac{12}{5}} = 2\sqrt{\frac{3}{5}} \simeq 1.55$,

$b^2 = 9$, $b = 3$



$$c = \sqrt{b^2 - a^2}$$

$$= \sqrt{9 - \frac{12}{5}}$$

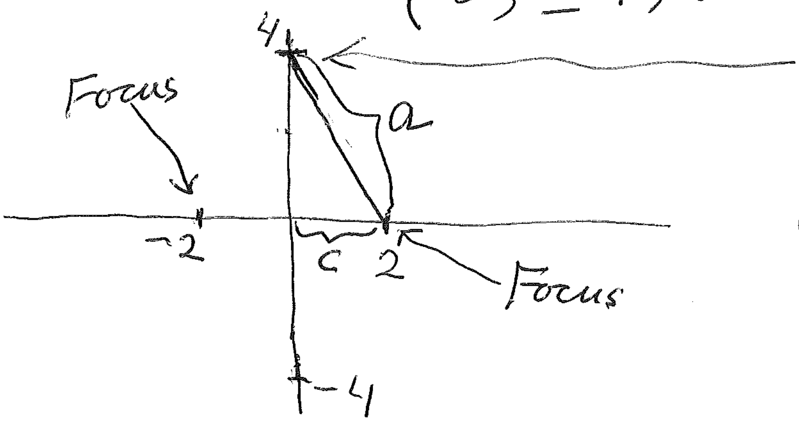
$$= \sqrt{\frac{45-12}{5}} = \sqrt{\frac{33}{5}} \approx 2.57$$

So the foci are on the y-axis at the points $\left(0, \pm\sqrt{\frac{33}{5}}\right)$

$\approx (0, \pm 2.57)$

*

Find equation of the ellipse with foci at $(\pm 2, 0)$ and two of its vertices $(0, \pm 4)$.



So $b = 4$,

$c = 2$

$a = \sqrt{b^2 + c^2}$

$= \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

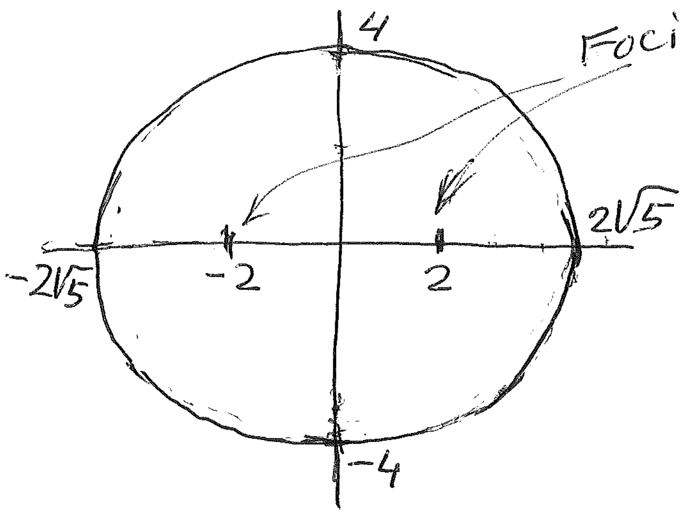
≈ 4.47

Hence an equation of the ellipse

is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, i.e.

$$\frac{x^2}{20} + \frac{y^2}{16} = 1$$

i.e. $4x^2 + 5y^2 = 80$



Hence the length of the major axis is $2(2\sqrt{5}) = 4\sqrt{5}$. The length of the minor axis is $2(4) = 8$.

<u>Comment.</u> In these notes I am using a to mean the length of the x-semiaxis, regardless whether it is the major or the minor semiaxis, whereas the Book uses a to always denote the major semiaxis.

Shifted Circles / Ellipses.

Consider the Eq.

$$3x^2+5x+3y^2-2y-1=0$$

We complete, separately, the x-terms to a square and the y-terms to a square:

$$3\left(x^2+\frac{5}{3}x\right)+3\left(y^2-\frac{2}{3}y\right)-1=0$$

$$3\left(x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2\right)+3\left(y^2-\frac{2}{3}y+\left(\frac{1}{3}\right)^2\right)$$

$$-3\left(\frac{5}{6}\right)^2-3\left(\frac{1}{3}\right)^2-1=0$$

$$-3\left(\frac{5}{6}\right)^2-3\left(\frac{1}{3}\right)^2-1=-3\cdot\frac{25}{36}-3\cdot\frac{1}{9}-1$$

$$=-\frac{25}{12}-\frac{1}{3}-1=-\frac{25}{12}-\frac{4}{12}-\frac{12}{12}$$

$$=-\frac{41}{12}$$

$$3\left(x+\frac{5}{6}\right)^2+3\left(y-\frac{1}{3}\right)^2-\frac{41}{12}=0$$

$$\left(x+\frac{5}{6}\right)^2+\left(y-\frac{1}{3}\right)^2=\frac{41}{36}$$

This is an equation of the circle centered at $\left(-\frac{5}{6}, \frac{1}{3}\right)$ and having radius $= \sqrt{\frac{41}{36}} = \frac{\sqrt{41}}{6}$ *

A key point is that in the Equation near the top of p. 371, both x^2; y^2 have equal coefficients, both > 0.

Another issue is that after we completed the squares and simplified, the number we ended up with on the Right Hand side is > 0.

Consider instead the eq.

$$3x^2 + 5x + 3y^2 - 2y + 4 = 0,$$

i.e. we only changed "-1" to "4". Now after completing the squares as before, we end up with

$$-\frac{25}{12}-\frac{4}{12}+4=\frac{19}{12}$$

on the left hand side, and thus finally

$$3\left(x+\frac{5}{6}\right)^2+3\left(y-\frac{1}{3}\right)^2=-\frac{19}{12}$$

which is not satisfied by any point.

*

If the coefficients of x^2 and y^2 are both positive, but not equal, and provided that any points at all satisfy the equation, we obtain an ellipse with its center shifted away from the origin.

Example. Identify the curve

$$3x^2+5x+y^2-2y-1=0$$

If it is an ellipse, find its vertices and foci.

Solution. As with the Example at the top of p. 371, we complete the squares:

$$3\left(x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2\right)+(y^2-2y+1)$$

$$-3\left(\frac{5}{6}\right)^2-1\quad-1=0$$

$$3\left(x+\frac{5}{6}\right)^2+(y-1)^2-\frac{25}{12}-2=0$$

$$3\left(x+\frac{5}{6}\right)^2+(y-1)^2=\frac{49}{12}$$

$$\frac{3}{\frac{49}{12}}\left(x+\frac{5}{6}\right)^2+\frac{1}{\frac{49}{12}}(y-1)^2=1$$

$$\frac{1}{\frac{49}{36}}\left(x+\frac{5}{6}\right)^2+\frac{1}{\frac{49}{12}}(y-1)^2=1$$

Thus $a^2 = \frac{49}{36} \Rightarrow a = \frac{7}{6} \approx 1.17$

$b^2 = \frac{49}{12} \Rightarrow b = \sqrt{\frac{49}{12}} = \frac{7}{2} \cdot \frac{1}{\sqrt{3}} =$

$= \frac{7}{2\sqrt{3}} \approx 2.02$

Thus the major axis is vertical, and the minor axis is horizontal.

Now the center is shifted to $(-\frac{5}{6}, 1)$ (i.e. look at $(x+\frac{5}{6}), (y-1)$).

The horizontal axis then contains the vertices

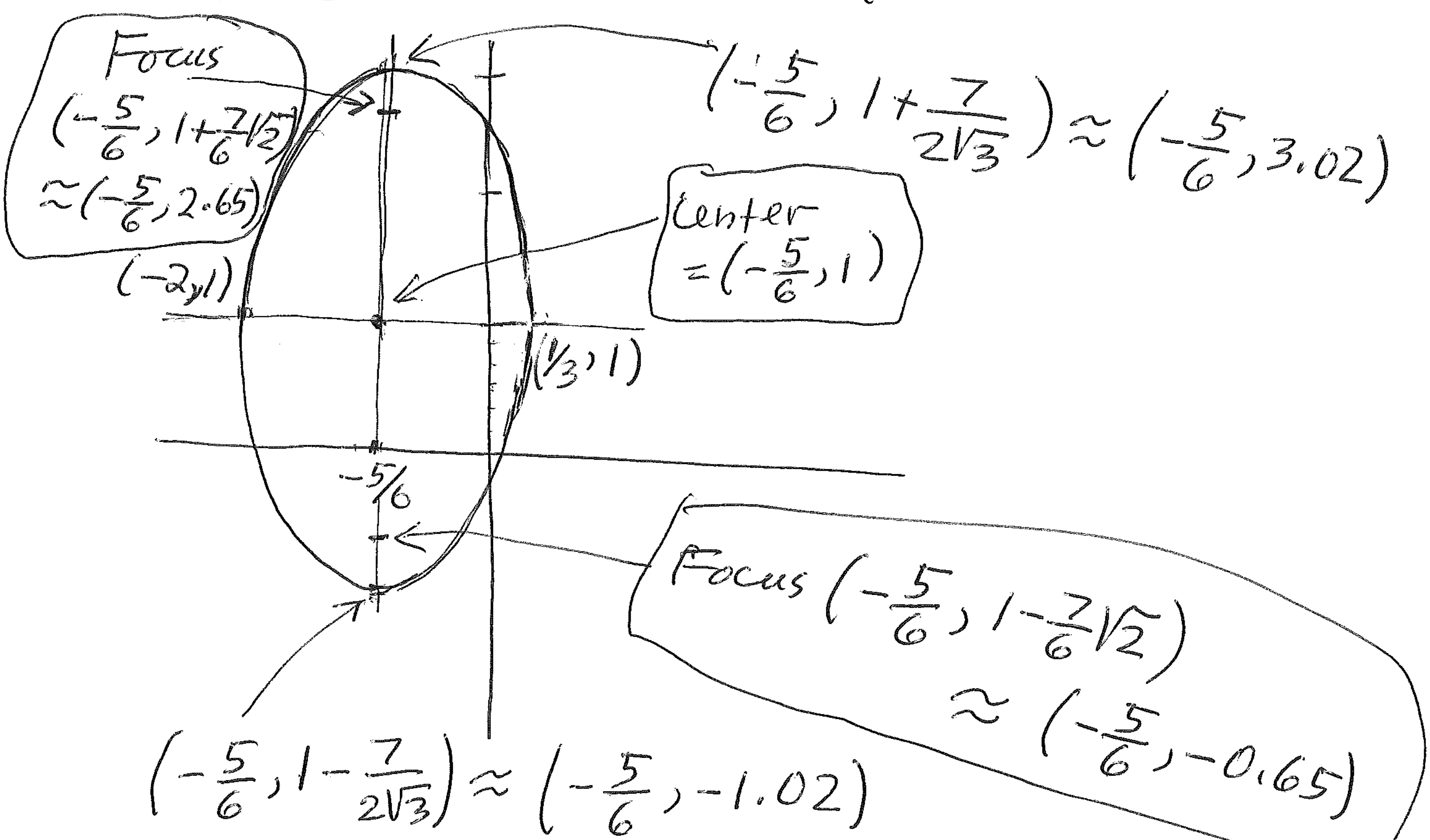
$(-\frac{5}{6} \pm a, 1) = (-\frac{5}{6} \pm \frac{7}{6}, 1) =$

$= (\frac{1}{3}, 1)$ and $(-2, 1)$.

The vertical axis contains the vertices $\left(-\frac{5}{6}, 1 \pm b\right)$ ← letter b (not number 6)

$= \left(-\frac{5}{6}, 1 \pm \frac{7}{2\sqrt{3}}\right)$

$\approx \left(-\frac{5}{6}, 3.02\right)$ and $\left(-\frac{5}{6}, -1.02\right)$



To find the foci, we calculate

$$c = \sqrt{b^2 - a^2} = \sqrt{\frac{49}{12} - \frac{49}{36}} = \sqrt{\frac{98}{36}}$$

$$= \sqrt{\frac{2(49)}{36}} = \frac{7}{6}\sqrt{2} \approx 1.65$$

↑ b is the larger of the two numbers a, b

The two foci are on the major axis which is the vertical axis in this example, so these foci are

$\left(-\frac{5}{6}, 1 \pm c\right)$, i.e. we add and subtract c to/from the y-coordinate of the center, obtaining

$\left(-\frac{5}{6}, 1 \pm \frac{7}{6}\sqrt{2}\right)$

$\approx \left(-\frac{5}{6}, 2.65\right)$ and $\left(-\frac{5}{6}, -0.65\right)$

*

Hyperbolas

Find the vertices, foci and asymptotes of the hyperbola

$$3x^2+5x-y^2+2y-3=0$$

Solution. This is a hyperbola since x^2, y^2 have coefficients of opposite signs.

We complete to squares:

$$3\left(x^2+\frac{5}{3}x+\left(\frac{5}{6}\right)^2\right)-3\left(\frac{5}{6}\right)^2$$

$$-\left(y^2-2y+1\right)+1-3=0$$

$$3\left(x+\frac{5}{6}\right)^2-(y-1)^2-\frac{25}{12}-2=0$$

$$3\left(x+\frac{5}{6}\right)^2-(y-1)^2=\frac{49}{12}$$

Asymptotes can be found immediately from the last equation by changing the Right Hand side to 0:

$$3\left(x+\frac{5}{6}\right)^2-(y-1)^2=0$$

$$(y-1)^2=3\left(x+\frac{5}{6}\right)^2 \Rightarrow \boxed{y-1=\pm\sqrt{3}\left(x+\frac{5}{6}\right)}$$

As with ellipses, we make the Right Hand side $= 1$ $\therefore$

$$3\left(x+\frac{5}{6}\right)^2 - (y-1)^2 = \frac{49}{12}$$

$$\frac{3\left(x+\frac{5}{6}\right)^2}{\frac{49}{12}} - \frac{(y-1)^2}{\frac{49}{12}} = 1$$

$$\boxed{\frac{\left(x+\frac{5}{6}\right)^2}{\frac{49}{36}} - \frac{(y-1)^2}{\frac{49}{12}} = 1}$$

$$a^2 = \frac{49}{36} \Rightarrow a = \frac{7}{6}$$

$$b^2 = \frac{49}{12} \Rightarrow b = \frac{7}{2\sqrt{3}}$$

We have a sort of a "center" (not an official name)

$\left(-\frac{5}{6}, 1\right)$

where the asymptotes intersect.

We find the vertices by dropping the negative term in the box:

$$\frac{\left(x+\frac{5}{6}\right)^2}{\frac{49}{36}} = 1$$

$$\left(x+\frac{5}{6}\right)^2 = \frac{49}{36}$$

$$x+\frac{5}{6} = \pm\frac{7}{6}$$

$$x = -\frac{5}{6} \pm \frac{7}{6} = \frac{1}{3}, -2$$

Hence the vertices are

$$\left(\frac{1}{3}, 1\right), \left(-2, 1\right)$$

where the y-coordinate comes from $\left(-\frac{5}{6}, 1\right)$ on p. 379.

We set $c = \sqrt{a^2 + b^2} =$

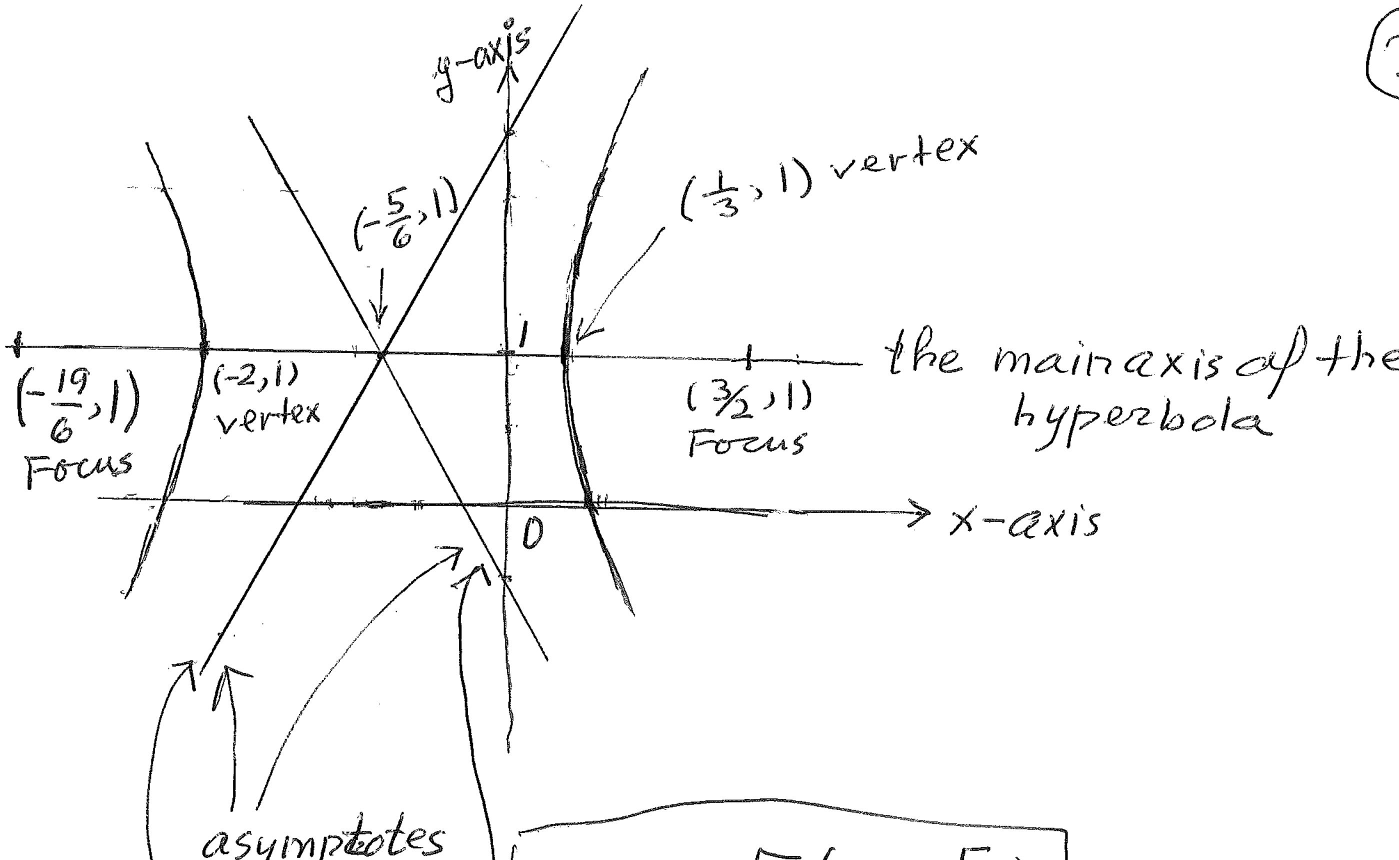
$$\sqrt{\frac{49}{36} + \frac{49}{12}} = \sqrt{\frac{49}{36} + \frac{147}{36}}$$

$$= \sqrt{\frac{196}{36}} = \frac{14}{6} = \frac{7}{}$$

The Foci then are the points $\left(-\frac{5}{6} \pm \frac{7}{3}, 1\right)$

$$= \left(-\frac{5}{6} \pm \frac{14}{6}, 1\right)$$

$$= \left(\frac{3}{2}, 1\right), \left(-\frac{19}{6}, 1\right)$$



$$y - 1 = -\sqrt{3}\left(x + \frac{5}{6}\right)$$

$$y - 1 = \sqrt{3}\left(x + \frac{5}{6}\right)$$

Equations of the asymptotes.

A sketch for the hyperbola

$$3x^2 + 5x - y^2 + 2y - 3 = 0 \quad (p. 378)$$

or

$$\left.\begin{aligned} &\frac{\left(x + \frac{5}{6}\right)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1, \\ &\text{where } a = \frac{7}{6}, \quad b = \frac{7}{2\sqrt{3}} \end{aligned}\right\} (p. 379)$$