Another Example on L'Hospital's Rule :

$$\boxed{a=b=1 \therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$

Example. A basic Example on L'Hospital's Rule from 1-st Semester Calculus is

$$\boxed{\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab} \quad (a, b \neq 0)}$$

That is, we write

$$\left(1+\frac{a}{x}\right)^{bx} = e^{bx \ln\left(1+\frac{a}{x}\right)},$$

so we need to evaluate

$$\lim_{x\to\infty} bx \ln\left(1+\frac{a}{x}\right)$$

which is an indeterminate product since as $x \to \infty$, $1 + \frac{a}{x} \to 1$ and thus $\lim_{x \to \infty} \ln\left(1 + \frac{a}{x}\right) = \ln 1 = 0.$

Thus $bx \ln\left(1 + \frac{a}{x}\right)$ is an indeterminate form $\infty \cdot 0$, as $x \to \infty$.

We can turn it into indeterminate form $\frac{0}{0}$ by writing it as

$$b \cdot \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}}$$

Thus we want to calculate

$$\lim_{x \to \infty} \frac{b \ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}} \underset{\text{L'H. Rule}}{=} \lim_{x \to \infty} \left(\frac{\frac{b}{1 + \frac{a}{x}} \cdot \left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}} \right)$$

$$= \lim_{x \to \infty} \frac{ab}{1 + \frac{a}{x}} = ab.$$

Hence $\lim_{x \to \infty} bx \ln\left(1 + \frac{a}{x}\right) = ab$,

and thus $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x \to \infty} e^{bx \ln\left(1 + \frac{a}{x}\right)} = e^{ab}$

We can use the preceding result to calculate limits of sequences:

Example: Evaluate

$$\lim_{n\to\infty}\left(1+\frac{3}{5n}\right)^{\frac{7n}{2}}$$

Solution:

$$\underbrace{\lim_{n\to\infty}\left(1+\frac{3}{5n}\right)^{\frac{7n}{2}}} = \lim_{n\to\infty}\left(1+\frac{\frac{3}{5}}{n}\right)^{\frac{7}{2}n}$$

However

$$\lim_{x\to\infty}\left(1+\frac{\frac{3}{5}}{x}\right)^{\frac{7}{2}x} = e^{\frac{3}{5}\cdot\frac{7}{2}}$$

$$= e^{\frac{21}{10}}$$

Hence by the theorem stated at the bottom of p. 392 of these Notes, we obtain

$$\lim_{n\to\infty}\left(1+\frac{3}{5n}\right)^{\frac{7n}{2}} = e^{\frac{21}{10}}$$

$\times$

The Squeeze Theorem for Sequences.

(For functions done in first semester, p. 105 in the Book.)

If $a_n \le b_n \le c_n$ for $n \ge n_0$

(i.e. the inequality does not have to be true for all n, only for all sufficiently large n) and both $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} c_n = L$, then

$\lim\limits_{n\to\infty} b_n = L$.

Example. Evaluate $\lim\limits_{n\to\infty} \frac{\sin n}{\sqrt{n}}$

Solution.

$$-1 \le \sin n \le 1$$

hence
$$-\frac{1}{\sqrt{n}} \le \frac{\sin n}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

Also $\lim\limits_{n\to\infty} \left(-\frac{1}{\sqrt{n}}\right) = 0$ and $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$,

hence by the Squeeze Thm.,

$$\lim_{n\to\infty} \frac{\sin n}{\sqrt{n}} = 0$$

*

The Subsequence Theorem.

A subsequence is a sequence created from a given sequence by omitting some of the terms of the given sequence.

Example. Let $a_n = \frac{1}{n}$, $n \geq 1$,

i.e. the sequence is

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

We can obtain a subsequence by keeping only the terms corresponding to even n:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots$$

Hence the new sequence is

$$b_n = \frac{1}{2n}, \quad n \geq 1,$$

which is a subsequence of the original given sequence $a_n = \frac{1}{n}$.

The Subsequence Theorem states that:
If $\lim_{n \to \infty} a_n = L$, and b_n is a subsequence of the sequence a_n, then $\lim_{n \to \infty} b_n = L$.

I.e. every subsequence has the same limit as the original sequence, provided the original sequence has a limit.

An Important Application of the Subsequence Theorem is to proving that a given sequence is divergent:

Example. Determine whether the sequence

$$\cos n\pi + \frac{n}{2n+1}$$

is convergent or divergent.

Solution. For n odd, $\cos n\pi = -1$, for n even, $\cos n\pi = 1$.

Thus we can consider two subsequence of the original sequence, the subsequence with n odd, which is

$$(-1) + \frac{n}{2n+1}$$

and has limit $(-1) + \lim_{n\to\infty} \frac{n}{2n+1} =$

$$= (-1) + \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = -1 + \frac{1}{2} = -\frac{1}{2},$$

and the subsequence with n even, which is

$$1 + \frac{n}{2n+1}$$

and has limit

$$1 + \lim_{n\to\infty} \frac{n}{2n+1} = 1 + \frac{1}{2} = \frac{3}{2}$$

Thus the given sequence

$$\cos n\pi + \frac{n}{2n+1}$$

has subsequences that have different limits. Thus the given sequence cannot be convergent because if it were, all subsequences would have to have the same limit.

Section 11.2. SERIES.

$\boxed{\text{Homework}}$ for sections 10.5 and 11.1 will be due Thursday March 27 and includes the Exercises listed on the syllabus.

A Series is obtained from a sequence, and yields a new sequence (related to the original one). Namely, if we start with a sequence a_n, $n \geq 1$, the corresponding series is written as (the sequence a_n is called the sequence of the terms of the series)

$$a_1 + a_2 + \dots + a_n + \dots,$$

or, a shorter notation,

$$\sum_{n=1}^{\infty} a_n$$

Examples.

Sequence: $\frac{1}{n}$, $n \geq 1$

The Corresponding Series:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

$$= 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$$

Short Notation: $\sum_{n=1}^{\infty} \frac{1}{n}$

Sequence: $\frac{1}{2^n}$, $n \geq 1$

The Corresp. Series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

Short Notation: $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Sequence: $\frac{1}{n+1}$, $n \geq 1$

Corresp. Series:

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots$$

Short Notation:

$$\sum_{n=1}^{\infty} \frac{1}{n+1}$$

Recall (page 383 in these Notes) that a sequence does not have to start with $n = 1$. Thus for

Example:

Sequence: $\frac{1}{n+1}$, $n \geq 0$

Corresp. Series:

$$\frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \cdots + \frac{1}{n+1} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots$$

Short Notation: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$

You should compare the two Examples on p. 417 (top half of the page, bottom half of the page).

Normally, the series is specified without writing down the sequence a_n itself:

<u>Examples:</u>

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

or, using the short Notation:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad (\text{same as} \qquad)$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+5} \qquad \text{etc.}$$

Then, we can, (if we wish) to write down the sequence that gave rise to the given series. This sequence is called <u>the sequence of the terms of the given series</u>:

See also p. 415

Thus, for example, for the series

$$1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}+\cdots,$$

the sequence of the terms of the series is $\frac{1}{n^2}$, $n \geq 1$.

For the series $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+5}$, the sequence of the terms is

$$\frac{1}{n^2+3n+5}, \quad n \geq 0.$$

There is, however, another important sequence associated with a given series, called <u>the sequence of partial sums</u>:

So if the series is

$$a_1 + a_2 + \cdots + a_n + \cdots$$

then the sequence of partial sums, denoted by s_n, $n \geq 1$, is

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n$$
$$= \sum_{i=1}^{n} a_i$$

Likewise if the series is

$$\sum_{n=1}^{\infty} a_n$$

then the sequence of partial sums s_n is exactly as on p. 420, since $\sum_{n=1}^{\infty} a_n$ is just short notation for the series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

Example. Recall from college algebra

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

(when $r \neq 1$),

and more generally,

$$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^n = a_1 \frac{1 - r^{n+1}}{1 - r}$$

Thus if the series is

$$1 + r + r^2 + \cdots + r^n + \cdots,$$

or

$$\sum_{n=0}^{\infty} r^n,$$

then the sequence s_n, $n \geq 0$, of partial sums is

$$s_0 = 1,\ s_1 = 1 + r,\ \cdots,\ s_n = 1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

For Example, if $r = \frac{1}{2}$, we obtain the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots,$$

i.e.

$$\sum_{n=0}^{\infty} \frac{1}{2^n},$$

and the sequence of partial sums is

$$s_0 = 1,\ s_1 = 1 + \frac{1}{2},\ \cdots\ s_n = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} =$$

$$= 2\left(1 - \frac{1}{2^{n+1}}\right) = 2 - \frac{1}{2^n}$$

Changing slightly the series at the top of p. 422, we can have

$$r + r^2 + \cdots + r^n + \cdots$$

or $\displaystyle\sum_{n=1}^{\infty} r^n$

The the partial sums are

$s_1 = r,\ s_2 = r^2, \cdots$

$$s_n = r + r^2 + \cdots + r^n = \sum_{i=1}^{n} r^i =$$

$$= r(1 + r + \cdots + r^{n-1}) =$$

$$= r\,\frac{1 - r^n}{1 - r}$$

Thus if $r = \frac{1}{2}$, the series is

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n}$$

and the partial sums are

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}, \ \cdots$$

$$s_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \sum_{i=1}^{n} \frac{1}{2^i} =$$

$$= \frac{1}{2} \cdot \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

A series is called <u>convergent</u> if its sequence of partial sums s_n is convergent, i.e. if $\lim_{n\to\infty} s_n$ exists and is finite, i.e. $\lim_{n\to\infty} s_n = s$ is a real number (not ∞ or $-\infty$).

COMPARE with the DEFINITION at the top of p. 399.

Thus for Example, the series

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n}$$

is convergent, since by the calculation at the top of p. 424 we have obtained

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$

We then also write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

or in general $\sum\limits_{n=1}^{\infty} a_n = s$

if $\lim\limits_{n\to\infty} s_n = s$.

This is similar to what we have done with improper integrals:

If the improper integral

$$\int_1^\infty f(x)\,dx$$

is convergent, i.e. if

$$\lim_{t\to\infty} \int_1^t f(x)\,dx \text{ exists, is}$$

finite, and equals a real number L, then we define the value of the improper integral $\int_1^\infty f(x)\,dx$ to be L,

and write $\int_1^\infty f(x)\,dx = L$

*

Returning to the Examples on pages 422, 423, the series of the form

$$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^n + \cdots,$$

or using short notation,

$$\sum_{n=0}^{\infty} a_1 r^n,$$

such series are called <u>geometric series</u>, with common ratio r.

Assuming $a_1 \neq 0$, the geom. series above is convergent for r such that $|r| < 1$, i.e. $-1 < r < 1$

and has sum

$$s = \frac{a}{1-r}$$

Thus for Example, the series

$$\sum_{n=0}^{\infty} 2\left(\frac{2}{3}\right)^n$$ is convergent and

has sum

$$s = 2 \cdot \frac{1}{1-\frac{2}{3}} = 2 \cdot \frac{1}{\frac{1}{3}} = 6$$

Note that the same series can also be written as

$$\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1}$$

since if we write out the first few terms of each series we see that they are identical:

$$2 + 2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right)^2 + \cdots$$

*

Example. Express the repeating decimal expansion as an ordinary fraction:

$$0.23747474\cdots$$
$$= 0.23\overline{74}$$

Solution.

$$0.123\overline{74} =$$

$$= 0.123 + \frac{74}{100000} + \frac{74}{10000000} + \cdots + \frac{74}{100000}\cdot\left(\frac{1}{100}\right)^n + \cdots$$

$$= \frac{123}{1000} + \frac{74}{100000}\cdot\frac{1}{1-\frac{1}{100}} =$$

$$= \frac{123}{1000} + \frac{74}{100000}\cdot\frac{1}{\frac{99}{100}} =$$

$$= \frac{123}{1000} + \frac{74}{100000}\cdot\frac{100}{99} =$$

$$= \frac{123}{1000} + \frac{74}{99}\cdot\frac{1}{1000} =$$

$$= \frac{1}{1000}\left(\frac{123(99)+74}{99}\right) =$$

$$= \frac{1}{1000}\cdot\frac{12251}{99} = \frac{12251}{99000}$$

Telescoping Series.

Example. Determine whether the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$$ is convergent or divergent
and if convergent, find its sum.

Solution.

We find a partial fraction decomposition of $\frac{1}{n^2+3n}$ exactly like for $\frac{1}{x^3+3x}$

So $$\frac{1}{n^2+3n} = \frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$= \frac{An+3A+Bn}{n(n+3)} = \frac{(A+B)n+3A}{n(n+3)}$$

Hence $A+B=0$, $3A=1$,

hence $A=\frac{1}{3}$, $B=-\frac{1}{3}$

$$\frac{1}{n^2+3n} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

We write out a few of the partial sums:

$$s_1 = a_1 = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right)$$

$$s_2 = a_1 + a_2 = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right)$$

$$s_4 = a_1 + a_2 + a_3 + a_4 =$$

$$= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$$

$$= \frac{1}{3}\left[\frac{1}{1} + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{7}\right].$$

$$s_5 = a_1 + a_2 + \cdots + a_5 =$$

$$= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$= \frac{1}{3}\left[\frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{7} - \frac{1}{8}\right]$$

$$s_6 = a_1 + a_2 + \cdots + a_5 + a_6$$

$$= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right)$$

$$+ \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{6} - \frac{1}{9}\right)$$

$$= \frac{1}{3}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9}\right]$$

$$s_{10} = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{3} - \frac{1}{6}\right) +$$

$$+ \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{6} - \frac{1}{9}\right)$$

$$+ \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right) + \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right) + \frac{1}{3}\left(\frac{1}{9} - \frac{1}{12}\right)$$

$$+ \frac{1}{3}\left(\frac{1}{10} - \frac{1}{13}\right)$$

$$= \frac{1}{3}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{11} - \frac{1}{12} - \frac{1}{13}\right]$$

$$s_n = \frac{1}{3}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right]$$

Hence $\lim_{n\to\infty} s_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18}$

Thus the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$$

converges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n} = \frac{11}{18}$

$\ast$

An Important Method for Proving Divergence of a Series:

Theorem. If the series

$\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-n-2}{5n^2+4n+1}$ converges or diverges.

Solution. It is shown on p. 386 that $\displaystyle\lim_{n\to\infty} \frac{3n^2-n-2}{5n^2+4n+1} = \frac{3}{5}$

Hence by the Theorem in the Box above, the given series must diverge.

$\ast$

Midterm 2 Solutions.

1. (6 pts) Write out the form of the partial fraction decomposition. Do not evaluate the coefficients:

$$\frac{x^2+1}{(x^2+2x+1)(x^2+2x+2)(x^2+3x+2)} =$$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} + \frac{Ex+F}{x^2+2x+2}$$

$$(x^2+2x+1)(x^2+2x+2)(x^2+3x+2)$$

$$= (x+1)^2 (x+1)(x+2) (x^2+2x+2)$$

$$= (x+1)^3 (x+2) \underbrace{(x^2+2x+2)}_{\text{No Real Roots}}$$

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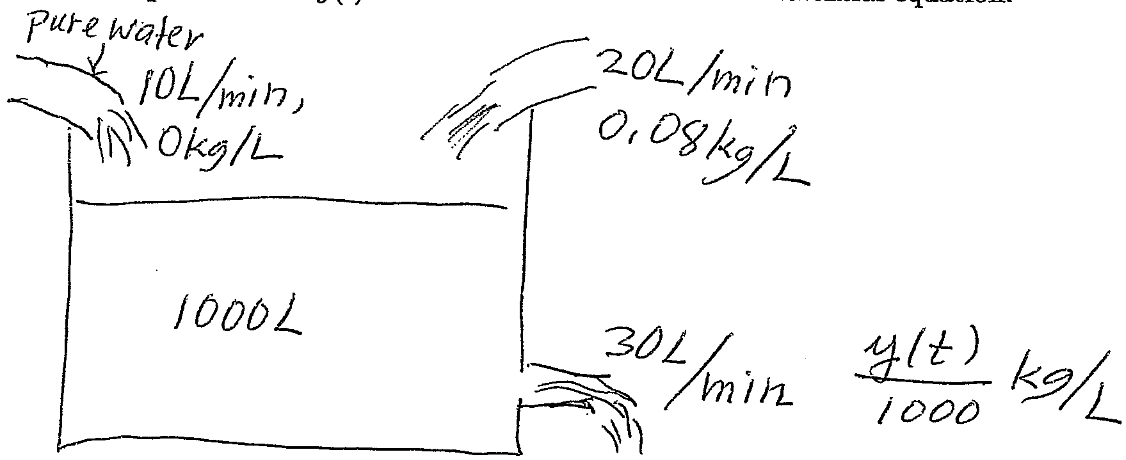
2. (6 pts) A tank contains 1000L of brine with 25 kg of dissolved salt. Pure water enters the tank at the rate of 10L/min. Brine that contains 0.08 kg of salt per liter enters the tank at the rate of 20L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 30L/min.

Let $y(t)$ denote the amount of salt (in kgs) in the tank at time t. Write down the differential equation that $y(t)$ satisfies. Do not solve this differential equation.



$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\frac{dy}{dt} = (0\,\text{kg/L})(10\,\text{L/min}) + (0.08\,\text{kg/L})(20\,\text{L/min})$$

$$- \left(\frac{y(t)}{1000}\,\text{kg/L}\right)(30\,\text{L/min})$$

$$\boxed{\frac{dy}{dt} = 1.6 - \frac{3}{100}y}$$

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3. (6 pts) Let $\mathcal{R}$ be a region in the x, y-plane bounded above by the graph $y = f(x)$, below by the graph $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$. Also suppose that the area of the region $\mathcal{R}$ equals 2. Then the coordinates of the centroid of $\mathcal{R}$ are

(a) $\bar{x} = \int_a^b x(f(x) - g(x))\, dx$, $\bar{y} = \int_a^b [(f(x))^2 - (g(x))^2]\, dx$

(b) $\bar{x} = \frac{1}{2}\int_a^b [(f(x))^2 - (g(x))^2]\, dx$, $\bar{y} = \frac{1}{2}\int_a^b x(f(x) - g(x))\, dx$

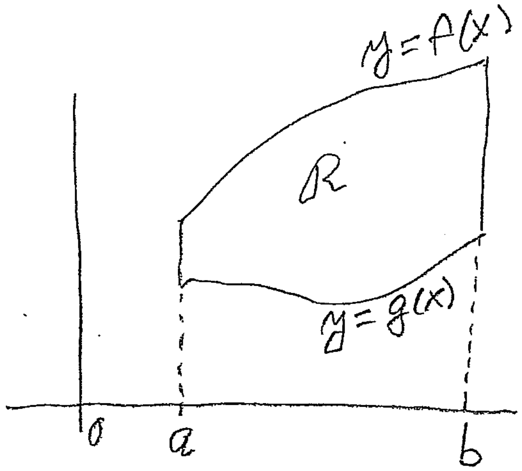
(c) $\bar{x} = \frac{1}{4}\int_a^b [(f(x))^2 - (g(x))^2]\, dx$, $\bar{y} = \frac{1}{4}\int_a^b x(f(x) - g(x))\, dx$

(d) $\bar{x} = \frac{1}{2}\int_a^b x(f(x) - g(x))\, dx$, $\bar{y} = \frac{1}{4}\int_a^b [(f(x))^2 - (g(x))^2]\, dx$

(e) $\bar{x} = 2\int_a^b x[(f(x))^2 - (g(x))^2]\, dx$, $\bar{y} = 2\int_a^b (f(x) - g(x))\, dx$

(f) $\bar{x} = 2\int_a^b x(f(x) - g(x))\, dx$, $\bar{y} = 2\int_a^b [(f(x))^2 - (g(x))^2]\, dx$

Choose the one correct answer above. No work has to be shown.



(Area of R) = 2

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4. (8 pts) Write down the integral for the length of the curve

$$y = \frac{1}{2}\ln x - \frac{1}{4}x^2,\ 1 \le x \le 2$$

and *simplify this integral as much as possible*, but do not evaluate it.

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{1}{2x}-\frac{1}{2}x\right)^2 = 1+\frac{1}{4}\left(\frac{1}{x^2}-2+x^2\right) =$$

$$= 1 - \frac{1}{2} + \frac{1}{4}\frac{1}{x^2} + \frac{1}{4}x^2$$

$$= \frac{1}{4}\frac{1}{x^2} + \frac{1}{2} + \frac{1}{4}x^2 = \frac{1}{4}\left(\frac{1}{x^2}+2+x^2\right) =$$

$$= \frac{1}{4}\left(\frac{1}{x}+x\right)^2$$

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{\frac{1}{4}\left(\frac{1}{x}+x\right)^2}\,dx$$

$$= \int_1^2 \frac{1}{2}\left(\frac{1}{x}+x\right)dx$$

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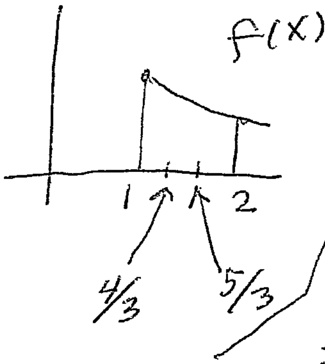
5. (14 pts) (a) Find the exact value of the Trapezoidal Rule approximation T_3 for $\int_1^2 \frac{1}{x}\,dx$. Do not use decimal notation, except (if you wish to use it) in the final answer, and provided your final answer has the form of one number expressed in decimal notation.

$f(x) = \frac{1}{x}$



$$\Delta x = \frac{b-a}{3} = \frac{2-1}{3} = \frac{1}{3}$$

$$T_3 = \frac{\Delta x}{2}\left(f(1) + 2f\left(\tfrac{4}{3}\right) + 2f\left(\tfrac{5}{3}\right) + f(2)\right)$$

$$= \frac{1}{6}\left(1 + 2\cdot\frac{3}{4} + 2\cdot\frac{3}{5} + \frac{1}{2}\right) =$$

$$= \frac{1}{6}\left(1 + \frac{3}{2} + \frac{1}{2} + \frac{6}{5}\right) = \frac{1}{6}\left(3 + \frac{6}{5}\right) =$$

$$= \frac{1}{6}\cdot\frac{21}{5} = \frac{1}{2}\cdot\frac{7}{5} = \frac{7}{10} = 0.7$$

(b) Is the approximation T_3 in part (a) accurate to within $\frac{1}{50}$?

COMMENT. All required calculations in both part (a) and part (b) can be readily performed without the use of a calculator.

$f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$

$\max|f''(x)|$ on $[1,2]$ occurs at $x = 1$, and $= 2$

Hence $K = 2$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12(3^2)} = \frac{1}{6(9)} = \frac{1}{54}$$

Hence $|E_T| \leq \frac{1}{54}$, thus $|E_T| \leq \frac{1}{50}$

Thus the approximation T_3 is accurate to within $1/50$.

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6.(14 pts) Consider the improper integral

$$\int_1^\infty \frac{\sin^2 x}{x^3+2x^2+5}\,dx$$

(a) State the definition of what it means that the integral above is convergent.

It means that $\lim_{t\to\infty} \int_1^t \frac{\sin^2 x}{x^3+2x^2+5}\,dx$ exists and is finite.

(b) Determine whether the integral above is convergent or divergent. State carefully the two main results (or theorems) that you will use.

The Comparison Test: If $f(x)$, $g(x)$ are continuous on $[a,\infty)$, and both $f(x), g(x) \geq 0$ on $[a,\infty)$ and $f(x) \leq g(x)$ on $[a,\infty)$ and $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is convergent.

The p-test: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p<1$, divergent if $p \geq 1$.

$\frac{\sin^2 x}{x^3+2x^2+5}$ is cont., ≥ 0 on $[1,\infty)$.

$\frac{\sin^2 x}{x^3+2x^2+5} \leq \frac{1}{x^3+2x^2+5} \leq \frac{1}{x^3}$, $\frac{1}{x^3}$ is cont. on $[1,\infty)$, ≥ 0.

since $\sin^2 x \leq 1$, and ≥ 0

$\int_1^\infty \frac{1}{x^3}\,dx$ is conv. by p-Test. Hence the given integral is conv. by the Comparison Test.

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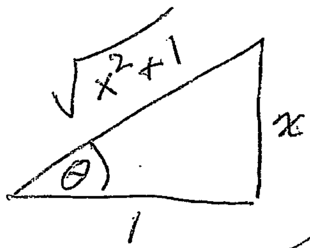
7. (14 pts) Evaluate

$$\int \frac{dx}{x\sqrt{x^2+1}}$$



$$x = \tan\theta$$
$$\sqrt{x^2+1} = \sec\theta$$
$$dx = \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\tan\theta \sec\theta} \sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \int \frac{1}{\tan\theta} \sec\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos\theta}\, d\theta$$

$$= \int \frac{1}{\sin\theta}\, d\theta = \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left| \frac{\sqrt{x^2+1}}{x} - \frac{1}{x} \right| + C$$

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8. (15 pts) Solve the initial value problem

$$\frac{3}{x\ln x}\frac{dy}{dx} = \frac{1}{y^2},\ y(2) = 1$$

$$3y^2\frac{dy}{dx} = x\ln x$$

$$3y^2 dy = x\ln x\, dx$$

$$\int 3y^2 dy = \int x\ln x\, dx$$

$$y^3 = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\, dx =$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

$$y = \sqrt[3]{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$$

$$1 = y(2) = \sqrt[3]{\frac{1}{2}\cdot 2^2\ln 2 - \frac{1}{4}\cdot 2^2 + C}$$

$$= \sqrt[3]{2\ln 2 - 1 + C}$$

$$1^3 = 2\ln 2 - 1 + C \Rightarrow \boxed{C = 2 - 2\ln 2}$$

$$y = \sqrt[3]{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + 2 - 2\ln 2}$$

MATH 1272
Midterm II

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SOME FORMULAS

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin^2(A/2) = (1-\cos A)/2$

$\cos^2(A/2) = (1+\cos A)/2$

$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$\sec^2 x = \tan^2 x + 1$

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$