Section 11.4. The Comparison Test.

Stated in the Box, bottom p. 722 in the Book. I will state it in an amplified form which incorporates the comment in Note 1 near the bottom of p. 723.

The Comparison Test.

Suppose that $\sum a_n$, $\sum b_n$ are series with positive terms.

Suppose that there is some positive integer N such that $a_n \le b_n$ for all $n \geq N$.

(i) If the series $\sum b_n$ is convergent then the series $\sum a_n$ is convergent.

(ii) If the series $\sum a_n$ is divergent, then the series $\sum b_n$ is divergent.

Comment. Yet further amplification can be obtained by assuming only that a_n, b_n are positive only for $n \geq M$ for some positive integer M.

Both amplifications are based on the fact stated in Note 1 (near bottom of page 723) that convergence, respectively divergence, of a series cannot be affected by altering finitely many terms.

It is best to incorporate one more amplification — this has been also done with the comparison test(s) for Improper Integrals. Namely, there is the following <u>Theorem</u>:

Let $\sum a_n$ be a series, and c a constant $\neq 0$. Then the series $\sum c a_n$ is convergent, respectively divergent, if and only if the series $\sum a_n$ is convergent, resp. divergent.

We can the restate the Comparison Test as follows, including also the comment at the bottom of page 454 of these Notes.

page 454 of these Notes.

<u>Amplified Comparison Test.</u>

Suppose that $\sum a_n$, $\sum b_n$ are series, M, N positive integers and c a positive constant (i.e. $c > 0$). Also suppose that a_n, b_n are ≥ 0 for all $n \geq M$, and $a_n \leq c b_n$ or $a_n c \leq b_n$ for all $n \geq N$.

(i) If the series $\sum b_n$ is convergent, then the series $\sum a_n$ is convergent.

(ii) If the series $\sum a_n$ is divergent, then the series $\sum b_n$ is divergent.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{3n^2+n+100}{2n^4+3n-1}$ is convergent or divergent.

Solution. First we need to make an "intelligent guess" whether convergent or divergent. We do that by counting the highest power in the entire fraction — as we did with improper integrals — isolating only the highest powers in both numer. & denom., we have

$\frac{n^2}{n^4} = \frac{1}{n^2}$, hence the given series should behave similarly as $\sum \frac{1}{n^2}$.

Thus we would like to prove that the given series is convergent — since the series $\sum \frac{1}{n^2}$ is convergent by the p-test.

Both the given series and $\sum \frac{1}{n^2}$ have positive terms for all $n \geq 1$.

Thus we would like to prove that

$$\frac{3n^2+n+100}{2n^4+3n-1} \leq \frac{c}{n^2}$$

for some positive constant c and for $n \geq N$ for a suitably large positive integer N.

We first work on the numerator $3n^2+n+100$. We know $n \leq n^2$ for all pos. integers n, and $100 \leq n^2$ when $n \geq 10$. Hence for all $n \geq N = 10$ we obtain $3n^2+n+100 \leq 3n^2+n^2+n^2 = 5n^2$. Thus for all $n \geq N=10$, we have

$$\frac{3n^2+n+100}{2n^4+3n-1} \leq \frac{5n^2}{2n^4+3n-1}$$

We now work on the denominator:

We need to make the fraction

$\frac{5n^2}{2n^4+3n-1}$ larger, hence the denom. smaller, which we can do by subtracting n instead of 1, $\underbrace{}_{}$since $n \geq 1$ hence

$$\frac{5n^2}{2n^4+3n-1} \leq \frac{5n^2}{2n^4+3n-n} = \frac{5n^2}{2n^4+2n}$$

which is $\leq \frac{5n^2}{2n^4} = \frac{5}{2} \cdot \frac{1}{n^2}$

Thus combining with the inequality at the bottom of p. 458, we obtain

$$\frac{3n^2+n+100}{2n^4+3n-1} \leq \frac{5}{2} \cdot \frac{1}{n^2}$$

Thus since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can apply the Comparison Test to conclude that the given series is convergent. $\ast$

The Limit Comparison Test.

Suppose that $\sum a_n$, $\sum b_n$ are such that $a_n > 0$, $b_n > 0$ for all $n \geq N$ for some pos. integer N.

Suppose further that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ where c is some finite positive, i.e. > 0, number.

Then either both series converge, or both diverge.

<u>Example.</u> Determine whether the series $\sum\limits_{n=0}^{\infty} \frac{n^3 - 5n + 2}{\sqrt[3]{(5n^6+2)^2}}$ converges or diverges.

<u>Solution.</u> Counting the highest powers, $\frac{n^3}{\sqrt[3]{(n^6)^2}} = \frac{n^3}{\sqrt[3]{n^{12}}} = \frac{n^3}{n^4} = \frac{1}{n}$

Since the series $\sum \frac{1}{n}$ diverges, we guess that the $\underbrace{\text{series}}_{\text{given}}$ diverges likewise.

Clearly the terms of the given series are positive for n sufficiently large. We would like to apply the Limit Comparison Test, which is easier to implement than the Comparison test itself:

$$\lim_{n\to\infty} \frac{\dfrac{n^3-5n+2}{\sqrt[3]{(5n^6+2)^2}}}{\dfrac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{n(n^3-5n+2)}{\sqrt[3]{(5n^6+2)^2}} =$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n^3(n^3-5n+2)^3}{(5n^6+2)^2}}$$

cont. on next page

Since both the numerator and the denominator have degree $=12$, we will divide by n^{12} in both numer. and denom.:

From near bottom preceding page

$$= \lim_{n\to\infty} \sqrt[3]{\frac{\frac{n^3(n^3-5n+2)^3}{n^{12}}}{\frac{(5n^6+2)^2}{n^{12}}}} \quad \left(\text{cancel } \frac{n^3}{n^{12}} = \frac{1}{n^9}\right)$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{\left(\frac{n^3-5n+2}{n^3}\right)^3}{\left(\frac{5n^6+2}{n^6}\right)^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{\left(\frac{n^3}{n^3}-\frac{5n}{n^3}+\frac{2}{n^3}\right)^3}{\left(\frac{5n^6}{n^6}+\frac{2}{n^6}\right)^2}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{\left(1-\frac{5}{n^2}+\frac{2}{n^3}\right)^3}{\left(5+\frac{2}{n^6}\right)^2}} =$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{1}{5^2}} = \frac{1}{5^{2/3}} > 0$$

Hence the given series diverges by the Limit Comparison Test. *