The Exercise 40, p. 727 in the Book gives <u>another form of the Limit Comparison Test</u>:

Suppose that $\sum a_n$, $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

If $\sum b_n$ converges, then $\sum a_n$ converges.

If $\sum a_n$ diverges, then $\sum b_n$ diverges.

<u>Example.</u> Determine if $\sum_{n=1}^{\infty} \frac{n^{10}}{e^{n^4}}$ converges or diverges.

<u>Solution.</u> We will use the form of the Limit Comparison Test stated above. Consider second series $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ which is a geom. series with common ratio $\frac{1}{e}$, hence it is convergent. Moreover both series $\left(\sum_{n=1}^{\infty} \frac{n^{10}}{e^{n^4}}, \sum_{n=1}^{\infty} \frac{1}{e^n} \right)$ are

series of positive terms. Now we observe that the terms $\frac{n^{10}}{e^{n^4}}$, of the given series, converge to 0 much faster than the terms $\frac{1}{e^n}$ since the denominator e^{n^4} goes to ∞ much faster than e^n, and n^{10} is easily "overpowered" by the exponential e^{n^4} (or even e^n).

Hence since $\frac{n^{10}}{e^{n^4}}$ converges to 0 much faster that $\frac{1}{e^n}$, the series $\sum \frac{n^{10}}{e^{n^4}}$ should converge since $\sum \frac{1}{e^n}$ converges.

What we did so far was just "intelligent guessing" – to make it precise, we use the Limit Comparison test in the form near the top of p. 463: So we calculate

$$\lim_{n\to\infty} \frac{\frac{n^{10}}{e^{n^4}}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{n^{10}e^n}{e^{n^4}}$$

Using L'H. Rule directly does not work, but we can use it in conjunction with Squeezing Theorem.

We note that $n^4 \geq 2n$ for all $n \geq 2$ (since $n^4 \geq 2n$ if and only if $n^3 \geq 2$ which is certainly true for all $n \geq 2$). Thus for all $n \geq 2$, $e^{n^4} \geq e^{2n}$,

hence $\frac{n^{10}e^n}{e^{n^4}} \leq \frac{n^{10}e^n}{e^{2n}} = \frac{n^{10}}{e^n}$

But $\lim_{n\to\infty} \frac{n^{10}}{e^n} = 0$ by using L'H. Rule,

hence $\lim_{n\to\infty} \frac{n^{10}e^n}{e^{n^4}} = 0$ by Squeezing Thm.

We thus conclude that $\sum_{n=1}^{\infty} \frac{n^{10}}{e^{n^4}}$ converges.

Comment. There are other ways to prove the fact in the preceding Example.

E.g. using the "ordinary" Comparison Test for series stated on p. 456, or bringing in the Integral Test.

Also Look at other similar Exercises in the Book, including the Improper Integrals section 7.8 (Assigned Homeworks, as well as others.)

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Section 11.5: Alternating Series.

HW due Th. 4/10:

11.4: As on the syllabus.

The assigned exercises for section 11.5 are changed to:

7, 11, 24, 25, 27, 29; Moreover, change the Question #27, 29 in the Book to make it similar to #24, 25: I.e. How many terms do we need to add so that |error| < 0.0001.

11.6: 3, 9, 13, 19 (half of what's on the syllabus).

Alt. Series: It is a series whose signs alternate between +, −:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots + (-1)^n \frac{n}{n+1} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

There are two important properties of alternating series stated in the two Boxes on pages 727 and 730:

Alternating Series Test (Box p. 727):

Suppose $b_n \geq 0$, and $b_{n+1} \leq b_n$ for all n, and $\lim_{n \to \infty} b_n = 0$.

Then the alternating series created from b_n, which is

$$b_1 - b_2 + b_3 - \cdots + (-1)^{n-1} b_n + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

is convergent.

Error Estimation Theorem for Alternating Series (Box p. 730):

Suppose $b_n \geq 0$, $b_{n+1} \leq b_n$ for all n, and $\lim_{n\to\infty} b_n = 0$. (Like for Alter. Series Test)

Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$ be the n-th partial sum of

$$b_1 - b_2 + b_3 - \cdots + (-1)^{n-1} b_n + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad \left(s = \lim_{n\to\infty} s_n \right)$$

and let $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ (which is finite by the Alternating Series Test).

Let $R_n = s - s_n$, i.e. R_n is the error we make if we replace the sum of the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ the n-th partial sum s_n.

Then $|R_n| \leq b_{n+1}$, i.e. at most the first term that we have omitted.

Examples.

(1) $1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$$

Is this series convergent, and if so, estimate $R_{100} = S - S_{100}$

Solution. We consider the sequence

$1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots$: all ≥ 0;

$$1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \cdots \geq \frac{1}{n} \geq \frac{1}{n+1} \geq \cdots$$

decreasing;

$\lim_{n\to\infty} \frac{1}{n} = 0$;

Hence we have verified all assumptions of the Altern. Series Test. Hence the series at the top of the page is convergent.

Now consider

$$S_{100} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots - \frac{1}{100}$$

Then the first omitted term is $\frac{1}{101}$,

hence the absol. value of the error R_n is at most $\frac{1}{101}$,

hence certainly $|R_n| < 0.01$ $\left(\frac{1}{101} < \frac{1}{100}\right)$.

(2) Consider the series

$$1 - \frac{1}{2^2} \cdot \frac{1}{2!} + \frac{1}{2^4} \cdot \frac{1}{4!} - \frac{1}{2^6} \cdot \frac{1}{6!} + \cdots$$

$$+ (-1)^n \frac{1}{2^{2n}} \cdot \frac{1}{(2n)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}} \cdot \frac{1}{(2n)!}$$

[Note that $0! = 1$, hence

$$(-1)^0 \cdot \frac{1}{2^0} \cdot \frac{1}{0!} = 1 .$$]

Is the series convergent? How many terms do we need to add so that the error would be less than 0.0001?

Solution. The given series is the alternating series created from positive terms $b_n = \frac{1}{2^{2n}} \cdot \frac{1}{(2n)!}$,

Also $b_n \geq b_{n+1}$ for all n,

and $\lim_{n\to\infty} b_n = 0$ (which follows from

$\lim_{n\to\infty} \frac{1}{2^n} = 0$, $\lim_{n\to\infty} \frac{1}{(2n)!} = 0$, lim of the

product $\frac{1}{2^n} \cdot \frac{1}{(2n)!}$ also $= 0$).

Hence the series converges by the alternating series test.

The Error: If we keep only

$1 - \frac{1}{2^2} \cdot \frac{1}{2!}$ then we can only

say that the error is $\leq \frac{1}{2^4} \cdot \frac{1}{4!}$

which $= \frac{1}{16} \cdot \frac{1}{24} = \frac{1}{388}$,

which is not < 0.0001.

If we keep

$$1 - \frac{1}{2^2} \cdot \frac{1}{2!} + \frac{1}{2^4} \cdot \frac{1}{4!},$$

then the error (in absol. value)

is $\leq \frac{1}{2^6} \cdot \frac{1}{6!} = \frac{1}{64} \cdot \frac{1}{720} = \frac{1}{46080} < 0.0001$

Hence we need to add the first three terms for the error to be < 0.0001.

$\ast$