substitution:

$\int \frac{(\ln x)^5}{x} dx$: since $(\ln x)' = \frac{1}{x}$,

we substitute $u = \ln x$,

$du = \frac{1}{x} dx$, hence

$$\int u^5 du = \frac{1}{6} u^6 + C$$

$$= \frac{1}{6} (\ln x)^6 + C$$

Similarly $\int \frac{(\arctan x)^5}{1+x^2} dx$:

$(\arctan x)' = \frac{1}{1+x^2}$,

hence $\arctan x = u$,

$du = \frac{1}{1+x^2} dx$

$$= \int (\arctan x)^5 \cdot \frac{1}{1+x^2} dx$$

$$= \int u^5 du = \frac{u^6}{6} + C = \frac{(\arctan x)^6}{6} + C$$

etc.

If the functions $\ln x$, $\tan^{-1}x$, $\sin^{-1}x$, $\cos^{-1}x$ in the integral are replaced by functions e^x, $\sin x$, $\cos x$ (in particular), i.e. we consider integrals of the form

$$\int f(x)e^x dx, \quad \int f(x)\sin x\, dx, \quad \int f(x)\cos x\, dx$$

then we often do the integration by parts where we set

$$dv = e^x dx, \quad dv = \sin x\, dx,$$

$$dv = \cos x\, dx$$

The point is, it is easy to find v, and, of course this is especially advantageous if $f'(x)$ is simpler than $f(x)$.

In fact the Main Examples are

$\int x^n e^x dx$, $\int x^n \sin x\, dx$, $\int x^n \cos x\, dx$

Of course it can be e^{3x} instead of e^x, $\sin 5x$, $\cos 2x$, etc. instead of $\sin x$, $\cos x$. Relevant Examples worked in the text are

Example 1, p. 464, Example 3, p. 465;

Calculate: $\int \underbrace{x^2}_{u} \underbrace{\cos 3x\, dx}_{dv}$

thus $\longrightarrow$

Hence $v = \frac{1}{3}\sin 3x$, $du = 2x\,dx$

Thus

$$\int x^2 \cos 3x\, dx = \underbrace{x^2}_{u} \cdot \underbrace{\frac{1}{3}\sin x}_{v} - \int \left(\frac{1}{3}\sin 3x\right) 2x\, dx$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\, dx$$

So need to calculate

$$\int x \sin 3x\, dx$$

$$\int \underbrace{x}_{u} \underbrace{\sin 3x \, dx}_{dv} \quad , \quad v = -\frac{1}{3}\cos 3x$$

$$du = dx$$

$$= \underbrace{x}_{u} \underbrace{\left(-\frac{1}{3}\cos 3x\right)}_{v} - \int \underbrace{\left(-\frac{1}{3}\cos 3x\right)}_{v} \underbrace{dx}_{du}$$

$$= -\frac{1}{3} x \cos 3x + \int \frac{1}{3} \cos 3x \, dx$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C$$

Hence $\int x^2 \cos 3x \, dx =$

$$= \frac{1}{3} x^2 \sin 3x - \frac{2}{3}\left(-\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x\right) + C$$

$$= -\frac{1}{3} x^2 \sin 3x + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + C$$

Another Important Integral.

$$\int e^{ax}\sin bx\,dx, \quad \int e^{ax}\cos bx\,dx$$

Calculate $\int \underbrace{e^{2x}}_{u} \underbrace{\sin 3x\,dx}_{dv}$

$v = \int \sin 3x\,dx =$

$= -\frac{1}{3}\cos 3x \; (+C)$

$du = 2e^{2x}dx$

$$= \underbrace{e^{2x}}_{u}\underbrace{\left(-\frac{1}{3}\cos 3x\right)}_{v} - \int \underbrace{\left(-\frac{1}{3}\cos 3x\right)}_{v} \underbrace{2e^{2x}dx}_{du}$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int \underbrace{e^{2x}}_{u} \underbrace{\cos 3x\,dx}_{dv}$$

Repeat the step:

$v = \frac{1}{3}\sin 3x,$

$du = 2e^{2x}dx$

$$= -\frac{1}{3}e^{2x}\cos 3x$$

$$+\frac{2}{3}\left(\underbrace{\frac{1}{3}(\sin 3x)}_{v} \underbrace{e^{2x}}_{u} - \int \left(\frac{1}{3}\sin 3x\right) 2e^{2x}dx \right)$$

Thus simplifying, we obtain

$$\boxed{\int e^{2x} \sin 3x \, dx} =$$

$$-\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \boxed{\int e^{2x} \sin 3x \, dx}$$

We have the same integral on both sides. Thus

$$\frac{13}{9} \int e^{2x} \sin 3x \, dx = e^{2x} \left(-\frac{1}{3} \cos 3x + \frac{2}{9} \sin 3x \right)$$

$$\int e^{2x} \sin 3x \, dx = \frac{1}{13} e^{2x} (2 \sin 3x - 3 \cos 3x) + C$$

We need to add a constant of integration.

*

We can also solve the problem by setting $u = \sin 3x$, $dv = e^{2x} dx$:

$$\int \underbrace{(\sin 3x)}_{u} \underbrace{e^{2x} dx}_{dv} \quad \text{(ON YOUR OWN)}$$

However, there is an IMPORTANT THING to pay attention to:

While there two equally good ways to solve the problems, within each of the two ways you must remain CONSISTENT:

I.e. if you use the method worked out in these Notes, the dv has to have the trig. function in both steps, i.e. on p. 28 $dv = \sin 3x\, dx$ (STEP I), and

$dv = \cos 3x\, dx$ (STEP II).

If in STEP II, you would have "switched", and set

$dv = e^{2x}dx$, the method would not work — and you would waste time (on the Exam!).

Similarly, if you start out by setting $dv = e^{2x}dx$ (see bottom p. 29), then in the STEP II you must again have $dv = e^{2x}dx$ (you cannot switch to dv involving a trig. function). *

Example 3, p. 465 is $\int \underset{u}{t^2} \underbrace{e^t dt}_{dv}$

and the Step has to be repeated, dv always being $= e^t dt$.

Likewise Example 4 p. 466 is similar to the one worked out on pages 28, 29 in these Notes.

<u>Similar Exercises to the ones worked out on pages 26 – 31:</u>

p. 468: #2, 3, 5, 7, 4, 6, 8, 14, 16, 17, 18, 19, 23, 24, 25, 28,

p. 469: # 29, 36, 38, 39 ("basically" similar to the ones mentioned above)

Up to this point we have covered all the "<u>basic</u>" types of Integrals from section 7.1, i.e. all the most basic types of INTEGRATION BY PARTS.

Now we need to consider Problems where we have to start with a preparatory step which is often a substitution or applying a trigonometric identity.

One Substitution which one really ought to memorize involves integrals of the form

$$\int f(\sqrt{x})\,dx$$

we substitute $t = \sqrt{x}$,

so $dt = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\,dx = \frac{1}{2}\cdot\frac{1}{t}\,dx$,

hence $dx = 2t\,dt$, hence

$$\int f(\sqrt{x})\,dx = \int f(t)\,2t\,dt$$

$$= 2\int t f(t)\,dt$$

Thus for Example

$$\int e^{\sqrt{x}}\,dx \underset{\substack{t=\sqrt{x},\\ dx=2t\,dt}}{=} 2\int \underset{u}{t}\,\underbrace{e^t dt}_{dv} =$$

see also p. 466 near top of the page

$$= 2\Big(\underset{u}{t}\underset{v}{e^t} - \int \underset{v}{e^t}\underset{du}{dt}\Big) =$$

$$= 2(te^t - e^t) + C$$

$$= 2(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}) + C \quad *$$

We handle $\int \cos\sqrt{x}\,dx$, $\int \sin\sqrt{x}\,dx$ the same way, and likewise

$$\int \arctan\sqrt{x}\,dx \quad ; \quad t=\sqrt{x}, \quad dx = 2t\,dt$$

$$= 2\underbrace{\int t \arctan t\,dt}$$

worked out on p. 22 in these Notes

$$= 2\Big[\tfrac{1}{2}(\arctan t)(t^2+1) - \tfrac{1}{2}t\Big] + C \;=$$

$= (t^2+1)\arctan t - t + C$

$= (x+1)\arctan\sqrt{x} - \sqrt{x} + C$ *

The square root substitution is very common and you have studied it in 1-st Semester Calc.

E.g. Calculate $\displaystyle\int \frac{\sqrt{x}}{x+1}\,dx$

$t = \sqrt{x}$, $dx = 2t\,dt$

$$= \int \frac{t}{t^2+1}\,2t\,dt =$$

$$= 2\int \frac{t^2}{t^2+1}\,dt = 2\int \frac{t^2+1-1}{t^2+1}\,dt =$$

$$= 2\int \left(\frac{t^2+1}{t^2+1} - \frac{1}{t^2+1}\right)dt =$$

$$= 2\int \left(1 - \frac{1}{t^2+1}\right)dt = 2t - 2\arctan t + C$$

$$= 2\sqrt{x} - 2\arctan\sqrt{x} + C$$ *

Another Substitution from 1-st Semester: $t = ax + b$

At the bottom of p. 6 and on p. 7 of these Notes we found $\int \ln(2x+5)\,dx$

This can be done nicer by starting with the substitution

$t = 2x+5$, $dt = 2\,dx$, $dx = \frac{1}{2}dt$

Hence $\int \ln(2x+5)\,dx =$

$$= \int (\ln t)\,\tfrac{1}{2}dt = \tfrac{1}{2}\int \ln t\,dt =$$

$$= \tfrac{1}{2}\left(t\ln t - t \right) + C$$

$$= \tfrac{1}{2}\left((2x+5)\ln(2x+5) - 2x - 5 \right) + C$$

$$= \left(x + \tfrac{5}{2}\right)\ln(2x+5) - x - \tfrac{5}{2} + C$$

*

Similarly:

Calculate $\int x \ln(2x+5)\,dx$

Again $t = 2x+5$, $dx = \frac{1}{2}dt$, $x = \frac{1}{2}(t-5)$

Hence $\int x \ln(2x+5)\,dx =$

$$= \int \frac{1}{2}(t-5)(\ln t)\frac{1}{2}dt =$$

$$= \frac{1}{4}\int (t\ln t - 5\ln t)\,dt$$

NOW YOU SHOULD BE ABLE TO FINISH THIS IF YOU STUDIED CAREFULLY UP TO THIS POINT,

Of course, basic Examples from 1st Semester are

e.g. $\int \frac{x}{\sqrt{2x+5}}\,dx$: $t = 2x+5$

$dx = \frac{1}{2}dt$

$$= \int \frac{\frac{1}{2}(t-5)}{\sqrt{t}} \cdot \frac{1}{2}dt$$

$x = \frac{1}{2}(t-5)$

cont. on Next page

$$= \frac{1}{4} \int \frac{t-5}{\sqrt{t}}\, dt = \frac{1}{4} \int \left(\frac{t}{\sqrt{t}} - \frac{5}{\sqrt{t}} \right) dt$$

$$= \frac{1}{4} \int \left(\sqrt{t} - \frac{5}{\sqrt{t}} \right) dt \quad \text{etc.}$$

$*$

Sometimes one has to be alert:

Calculate

$$\int x \ln(3x^2+5)\, dx \; :$$

We see that $(3x^2+5)' = 6x$,

hence $x = \frac{1}{6}(3x^2+5)'$, hence

the substitution $t = 3x^2+5$

gives $dt = 6x\,dx$, hence

$x\,dx = \frac{1}{6}\,dt$, hence

$$= \int (\ln(3x^2+5))\, x\,dx =$$

$$= \int (\ln t)\, \frac{1}{6}\, dt \quad \text{etc.}$$

$*$

The Exercises 37 – 42, p. 469 are specifically suggested to first use a Substitution before doing Integr. by Parts, i.e. as discussed on pages 33 – 38 of these Notes.

Preparatory Step Using an Identity:

Calculate $\int x \tan^2 x \, dx$:

$\tan^2 x = \sec^2 x - 1$,

$$= \int x(\sec^2 x - 1)\,dx = \int \underset{u}{x} \underbrace{\sec^2 x\,dx}_{dv} - \int x\,dx$$

$$= \underset{u}{x}\,\underset{v}{\tan x} - \int \underset{v}{\tan x}\,\underset{du}{dx} - \frac{1}{2}x^2$$

$v = \tan x$,
$du = dx$

$$= x\tan x - \ln|\sec x| - \frac{1}{2}x^2 + C$$

*

Calculate $\int \arctan(1/x)\, dx$

We have $\arctan x + \arctan(1/x) = \frac{\pi}{2}$,

hence $\arctan 1/x = \frac{\pi}{2} - \arctan x$,

hence

$$= \int \left(\frac{\pi}{2} - \arctan x\right) dx$$

$$= \frac{\pi}{2} x - x \arctan x + \frac{1}{2} \ln(1+x^2) + C$$

The Integral $\int \arctan x\, dx$ is worked out in Example 5, p. 467 in the Book.

$\ast$

Calculate $\int x \cos^2 2x\, dx$:

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$= \int x \left(\frac{1}{2} + \frac{1}{2} \cos 4x\right) dx \quad \text{etc.}$$

$\ast$

Calculate $\int x\sqrt{1+\cos 3x}\,dx$

Using the formula $\cos^2 x = \dfrac{1+\cos 2x}{2}$,

we have $\cos^2 \frac{3}{2}x = \dfrac{1+\cos 2(\frac{3}{2}x)}{2}$

i.e. $\cos^2 \frac{3}{2}x = \dfrac{1+\cos 3x}{2}$,

hence $1+\cos 3x = 2\cos^2 \frac{3}{2}x$,

hence $\sqrt{1+\cos 3x} = \sqrt{2}\cos\frac{3}{2}x$

(e.g. when $0 \le \frac{3}{2}x \le \frac{\pi}{2}$)

Thus $\int x\sqrt{1+\cos 3x}\,dx =$

$= \int x\sqrt{2}\cos\frac{3}{2}x\,dx = \sqrt{2}\int x\cos\frac{3}{2}x\,dx$

which is one of our standard problems on Integr. by Parts

*.

TRIG. INTEGRALS.

Of the form $\int \sin^m x \cos^n x \, dx$:

Easiest if at least one of m, n odd:
calculate
$\int \sin^4 x \cos^5 x \, dx$:=

$$\int \sin^4 x \cos^4 x \cos x \, dx =$$

$$= \int \sin^4 x (\cos^2 x)^2 \cos x \, dx =$$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x \, (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

Now we make subst. $u = \sin x$,
hence $du = \cos x \, dx$

$$\rightsquigarrow \int u^4 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^4 - 2u^6 + u^8) \, du = \frac{u^5}{5} - \frac{2}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C$$

*

If both powers of $\sin x$, $\cos x$ are even, we use the formulas

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \quad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

to decrease the powers, and repeat until we get some odd powers:

$$\int \sin^4 x \cos^6 x\, dx = \int (\sin^2 x)^2 (\cos^2 x)^3 dx$$

$$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \left[\frac{1}{2}(1+\cos 2x)\right]^3 dx$$

$$= \frac{1}{32}\int \left[(1-\cos 2x)(1+\cos 2x)\right]^2 (1+\cos 2x)\, dx$$

$$= \frac{1}{32}\int (1-\cos^2 2x)^2 (1+\cos 2x)\, dx$$

$$= \frac{1}{32}\underbrace{\int (1-\cos^2 2x)^2 dx}_{\text{only even powers of } \cos 2x} + \frac{1}{32}\underbrace{\int (1-\cos^2 2x)^2 \cos 2x\, dx}_{\text{only odd powers of } \cos 2x}$$

only even powers of $\cos 2x$

so use the formula

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

etc.

only odd powers of $\cos 2x$, so can be done as explained on p. 42

Read Examples 1 – 4 in the Book, p. 471, 472.

The other cases we study are

$$\int \tan^m x \sec^n x \, dx$$

Recall $(\tan x)' = \sec^2 x$, $(\sec x)' = \sec x \tan x$

We also use the identity

$$\sec^2 x = 1 + \tan^2 x, \quad \tan^2 x = \sec^2 x - 1$$

Easiest case: n even (i.e. even power of $\sec x$)

Example. $\int \tan^5 x \sec^6 x \, dx$:

$$= \int \tan^5 x \sec^4 x \sec^2 x \, dx$$

$$= \int \tan^5 x (\sec^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^5 x (1 + \tan^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^5 x (1 + 2\tan^2 x + \tan^4 x) \sec^2 x \, dx$$

$\tan x = u$, $du = \sec^2 x \, dx$

$$\int u^5 (1 + 2u^2 + u^4) \, du, \text{ etc.}$$

*

Also easy case is when m is odd: i.e. odd power of $\tan x$

$$\int \tan^3 x \sec^5 x\,dx :$$

$$= \int \tan^2 x \sec^4 x \,(\sec x \tan x)\,dx$$

$$= \int (\sec^2 x - 1)\sec^4 x\,(\sec x \tan x)\,dx$$

$u = \sec x$, $du = \sec x \tan x\,dx$

$$= \int (u^2 - 1)u^4\,du = \int (u^6 - u^4)\,du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C \quad *$$

Also Need to Memorize the Formula

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Note that $\int \tan^3 x\,dx$, $\int \tan^5 x\,dx$, $\int \tan^7 x\,dx$, etc. can be done by both the method on p.44, as well as the one on p.45:

Method on p.44:

Let's do $\int \tan^5 x\,dx$:

$$= \int (\tan^2 x)^2 \tan x\,dx = \int (\sec^2 x - 1)^2 \tan x\,dx$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \tan x\,dx =$$

$$= \underbrace{\int \sec^4 x \tan x\,dx}\; - 2 \underbrace{\int \sec^2 x \tan x\,dx} + \underbrace{\int \tan x\,dx}$$

both have even power of $\sec x$, so OK for the method of p.44

And the third integral, of course, $= \ln|\sec x| + C$

Let's do the same integral, $\int \tan^5 x\,dx$, using the method on p.45, which in fact works faster:

$\int \tan^5 x\,dx$, method on p. 45:

At first we proceed exactly as on p. 46, obtaining the expression with the three integrals marked by arrows:

$$= \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx$$

continue:

$$= \int \sec^3 x\,(\sec x \tan x)\,dx - 2\int \sec x(\sec x \tan x)\,dx + \int \tan x\,dx$$

But $(\sec x)' = \sec x \tan x$, hence the first two integrals can be evaluated by substitution $u = \sec x$, $du = \sec x \tan x\,dx$:

$$\int u^3\,du - 2\int u\,du + \ln|\sec x| + C$$

$$= \frac{1}{4}u^4 - u^2 + \ln|\sec x| + C$$

$$= \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C$$

$*$

Even Power of $\tan x$:

E.g. $\int \tan^6 x \, dx$:

$$= \int \tan^4 x \tan^2 x \, dx = \int \tan^4 x (\sec^2 x - 1) \, dx$$

$$= \underbrace{\int \tan^4 x \sec^2 x \, dx} - \underbrace{\int \tan^4 x \, dx}$$

$$= \frac{1}{5} \tan^5 x + C$$

(using subst. $u = \tan x$, $du = \sec^2 x \, dx$)

Now, that integral has a lower power of $\tan x$, so we continue the same way as we began:

$$\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx =$$

$$= \int \tan^2 x (\sec^2 x - 1) \, dx =$$

$$= \underbrace{\int \tan^2 x \sec^2 x \, dx} - \int \tan^2 x \, dx$$

$$= \frac{1}{3} \tan^3 x + C$$

Finally $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$= \tan x - x + C$, hence

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

Thus $\int \tan^6 x\, dx =$

$= \frac{1}{5}\tan^5 x - \int \tan^4 x\, dx =$

$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$

*

So the remaining Case is

$$\int \tan^m x \sec^n x\, dx$$

where m is even, n odd.

So let's do

$$\int \tan^4 x \sec^3 x\, dx :$$

$= \int (\tan^2 x)^2 \sec^3 x\, dx =$

$$= \int (\sec^2 x - 1)^2 \sec^3 x \, dx =$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \, dx$$

$$= \int \sec^7 x \, dx - 2\int \sec^5 x \, dx + \int \sec^3 x \, dx,$$

So we see that we need to be able to do $\int \sec^n x \, dx$ where n is odd, which requires integration by parts, as shown in the Book, Example 8, p. 475, 476, which works out $\int \sec^3 x \, dx$:

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$$\int \sec^3 x\,dx = \int \underbrace{\sec x}_{u}\, \underbrace{\sec^2 x\,dx}_{dv}$$

$$= \underbrace{\sec x}_{u}\, \underbrace{\tan x}_{v} - \int \underbrace{\tan x}_{v}\, \underbrace{\sec x \tan x\,dx}_{du}$$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\,(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| \underbrace{- \int \sec^3 x\,dx}_{+C}$$

Thus using the same idea as for $\int e^x \sin x\,dx$, we obtain

$$2 \int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C,$$

hence

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

*

We do $\int \sec^5 x\,dx$ similarly:

$$= \int \underbrace{\sec^3 x}_{u}\, \underbrace{\sec^2 x\,dx}_{dv} =$$

$$= \underbrace{(\sec^3 x)}_{u}\, \underbrace{\tan x}_{v} - \int \underbrace{(\tan x)}_{v}\, \underbrace{3\sec^2 x \sec x \tan x\,dx}_{du}$$

$$= \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\,dx$$

$$= \sec^3 x \tan x - 3\int \sec^3 x\,(\sec^2 x - 1)\,dx =$$

$$= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx$$

Hence we obtain

$$4\int \sec^5 x\,dx = \sec^3 x \tan x + 3\int \sec^3 x\,dx$$

$$= \sec^3 x \tan x + \frac{3}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C,$$

using the result for $\int \sec^3 x\,dx$ at the bottom of p. 51. Thus dividing by 4, we obtain $\int \sec^5 x\,dx$, so now we would be ready to do $\int \sec^7 x\,dx$ *

Similarly as with integration by parts, there are many situations when the methods of trigonometric integrals work, but it may not be immediately obvious. So for example, if there is a substitution that reduces the integral to a trig. integral, or using an identity:

E.g. Calculate $\int \frac{\sin 2x}{\sqrt[3]{\cos x}}\,dx$

We use the identity $\sin 2x = 2\sin x\cos x$, hence

$$\int \frac{2\sin x\cos x}{\sqrt[3]{\cos x}}\,dx = 2\int \sin x(\cos x)^{2/3}\,dx$$

Since $(\cos x)' = -\sin x$, we can do the substitution $\cos x = u$ (So we are proceeding as in the case $\int \sin^m x\cos^n x\,dx$ when m is odd). $du = (-\sin x)dx$,

$$= -2\int u^{2/3}\,du = -2\cdot\frac{3}{5}u^{5/3} + C =$$

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$$= -\frac{6}{5}(\cos x)^{5/3} + C \quad *$$

Calculate $\int x \sin^3 x \cos^2 x \, dx$:

We can calculate the trig. integral $\int \sin^3 x \cos^2 x \, dx$, hence we can apply Integr. by Parts to

$$\int \underset{u}{x} \underbrace{\sin^3 x \cos^2 x \, dx}_{dv} = : \qquad \text{So } du = dx$$

So first $v = \int \sin^3 x \cos^2 x \, dx =$

$$= \int \sin x \sin^2 x \cos^2 x \, dx = \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$$

$$= \int \sin x (\cos^2 x - \cos^4 x) \, dx$$

$\cos x = t$, $dt = -\sin x dx$

$$= \int (t^4 - t^2) \, dt = \frac{1}{5} t^5 - \frac{1}{3} t^3 + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

Thus we use $v = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$,

$$\int \underset{u}{x} \underbrace{\sin^3 x \cos^2 x \, dx}_{dv} =$$

$$= \underset{u}{x} \underbrace{\left(\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x\right)}_{v} - \underbrace{\int \underbrace{\left(\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x\right)}_{v} \underbrace{dx}_{du}}$$

But the last integral can be calculated using the methods of trig. integrals (odd powers of cos x).

WILL NOT FINISH IT. *

We comment that powers of csc x, cot x are handled exactly analogously to powers of sec x, tan x