Solutions for Midterm 3.

1.(11 pts) Partial fraction decomposition of

$$\frac{3x^7 - x^5 + 2}{(x^4 - 1)^2}$$

should be sought in the form

(a) $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x^2+1}$

(b) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$

✓ (c) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}$

Since (x^2+1) can not be factored any further (using only real numbers, i.e. there are only complex roots.

(d) $\frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)^2}$

(e) $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$

(f) $\frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{(x^2+1)^2}$

$$(x^4-1)^2 = \left[(x^2-1)(x^2+1)\right]^2 =$$

$$\left[(x+1)(x-1)(x^2+1)\right]^2 = (x+1)^2(x-1)^2(x^2+1)^2$$

2.(11 pts) A curve is specified by parametric equations

$$x = t^3 - 4t, \ y = t^2 + t$$

$t = 2 \Rightarrow \frac{dy}{dx} = \frac{2(2)+1}{3(4)-4} = \boxed{\frac{5}{8}}$

The equation of the tangent line to the curve at the point $\boxed{(0, 6)}$ is

(a) $y = \frac{5}{8}x + 6$

(b) $y = \frac{8}{5}x + 6$

(c) $y = 6$

(d) $x = 0$

(e) $y = x + 6$

(f) $y = \frac{3t^2 - 4}{2t+1}x + 6$

$\frac{dx}{dt} = 3t^2 - 4t, \quad \frac{dy}{dt} = 2t + 1$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+1}{3t^2-4}$

$\boxed{y - 6 = \frac{5}{8}(x - 0)}$

that yields (0,6)

find t: $0 = t^3 - 4t, \quad 6 = t^2 + t$

$t(t+2)(t-2) = 0 \qquad t^2 + t - 6 = 0$

$(t+3)(t-2) = 0$

$t = 2$

OR:

$t = 0, t = -2, t = 2$

$t = 0 \Rightarrow y = 0 \neq 6$

$t = -2 \Rightarrow y = (-2)^2 - 2 = 2 \neq 6$

$t = 2 \Rightarrow y = 2^2 + 2 = \underline{6}$ So $\underline{t = 2}$

3.(11 pts) if the cartesian coordinates of a point are $(-1, \sqrt{3})$ then all possible pairs of polar coordinates of this point are

(a) $(2, \frac{2\pi}{3} + (2n+1)\pi)$ and $(-2, \frac{2\pi}{3} + 2n\pi)$

(b) $(2, \frac{4\pi}{3} + 2n\pi)$ and $(-2, \frac{4\pi}{3} + (2n+1)\pi)$

(c) $(2, \frac{2\pi}{3} + 2n\pi)$ and $(-2, \frac{2\pi}{3} + (2n+1)\pi)$

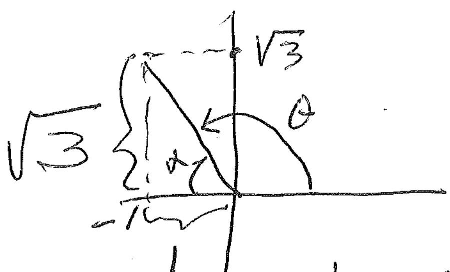
(d) $(\sqrt{2}, \frac{2\pi}{3} + 2n\pi)$ and $(-\sqrt{2}, \frac{2\pi}{3} + (2n+1)\pi)$

(e) $(\sqrt{2}, \frac{2\pi}{3} + (2n+1)\pi)$ and $(-\sqrt{2}, \frac{2\pi}{3} + 2n\pi)$

(f) $(\sqrt{2}, \frac{4\pi}{3} + 2n\pi)$ and $(-\sqrt{2}, \frac{4\pi}{3} + (2n+1)\pi)$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = \pm 2$

with $r = +2$; the Quadrant of $(-1, \sqrt{3})$ is $\underline{II}$;



$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3}$ ($60°$)

So $\underline{\underline{\theta}} = \pi - \frac{\pi}{3} = \underline{\underline{\frac{2\pi}{3}}}$,

Hence with $r = 2$ (> 0) and $0 \le \theta < 2\pi$,

we get $(2, \frac{2\pi}{3}) \Rightarrow (2, \frac{2\pi}{3} + 2n\pi)$

and $(-2, \frac{2\pi}{3} + (2n+1)\pi)$

4.(17 pts) Determine if the improper integral

$$\int_4^\infty \frac{9x+2}{2x(x-2)(2x+1)}dx$$

is convergent or divergent. If it is convergent, evaluate it.

You can use the partial fraction decomposition

$$\frac{9x+2}{2x(x-2)(2x+1)} = \frac{1}{x-2} - \frac{1}{2x} - \frac{1}{2x+1}$$

$$\int \frac{9x+2}{2x(x-2)(2x+1)}dx = \ln|x-2| - \frac{1}{2}\ln|x| - \frac{1}{2}\ln|2x+1| + C$$

$$= \ln\left|\frac{x-2}{\sqrt{x(2x+1)}}\right| + C$$

Can omit | | since everything >0 when t large

$$= \lim_{t\to\infty} \int_4^t \frac{9x+2}{2x(x-2)(2x+1)}dx = \lim_{t\to\infty}\Big(\ln\frac{t-2}{\sqrt{t(2t+1)}} - \underbrace{\ln\frac{2}{\sqrt{36}}}_{\ln\frac{1}{3}}\Big)$$

$$\lim_{t\to\infty} \ln\frac{t-2}{\sqrt{t(2t+1)}} = \lim_{t\to\infty}\ln\frac{\frac{t-2}{t}}{\frac{\sqrt{t(2t+1)}}{t}} = \lim_{t\to\infty}\ln\frac{1-\frac{2}{t}}{\sqrt{\frac{t(2t+1)}{t^2}}} =$$

$$= \lim_{t\to\infty}\ln\frac{1-\frac{2}{t}}{\sqrt{2+\frac{1}{t}}} = \ln\frac{1}{\sqrt{2}}$$

$$= \ln\frac{1}{\sqrt{2}} - \ln\frac{1}{3} = \ln\frac{1}{\sqrt{2}} - (-\ln 3) = \ln\frac{1}{\sqrt{2}} + \ln 3$$

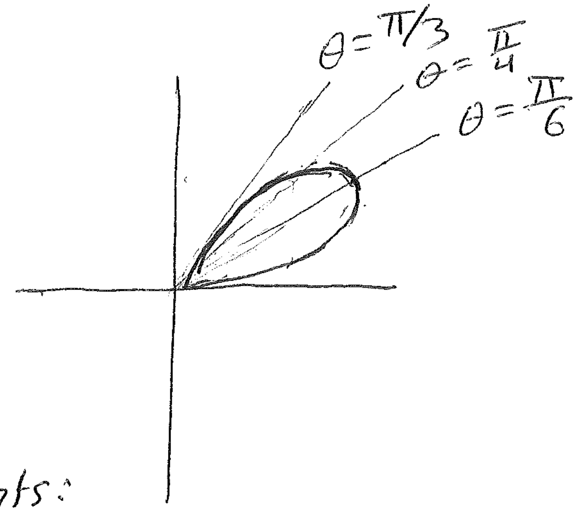
$$= \ln\frac{3}{\sqrt{2}}$$

The Limit exists and is finite,

Hence the given improper integral is convergent and $= \ln\frac{3}{\sqrt{2}}$ *

5.(17 pts) Sketch one leaf of the curve $r = \sin 3\theta$ and calculate the area of the region inside this leaf.

θ	$r = \sin 3\theta$
0	0
$\frac{\pi}{6}$	$\sin 3(\frac{\pi}{6}) = \sin\frac{\pi}{2} = 1$
$\frac{\pi}{4}$	$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\sin 3(\frac{\pi}{3}) = \sin\pi = 0$



One can plot more points: (if one wants to)

$\theta = \frac{\pi}{18} \to r = \sin 3(\frac{\pi}{18}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$\theta = \frac{\pi}{12} \to r = \sin 3(\frac{\pi}{12}) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/3} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/3}(\sin 3\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\frac{1}{2}(1-\cos 6\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/3}$$

$$= \frac{1}{4}\Big(\frac{\pi}{3} - \underbrace{\frac{1}{6}\sin 6(\tfrac{\pi}{3})}_{\sin 2\pi = 0}\Big) - \underbrace{\frac{1}{4}\left(0 - \frac{1}{6}\sin 0\right)}_{=0}$$

$$= \frac{1}{4}\cdot\frac{\pi}{3} = \frac{\pi}{12}$$

6.(16 pts) Evaluate

$$\lim_{n\to\infty} \frac{n^6}{\sqrt{(2n^4+3)^3}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{\sqrt{(2n^4+3)^3}}{n^6}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{(2n^4+3)^3}{n^{12}}}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{(2n^4+3)^3}{(n^4)^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\left(\frac{2n^4+3}{n^4}\right)^3}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{\left(\frac{2n^4}{n^4}+\frac{3}{n^4}\right)^3}} = \lim_{n\to\infty} \frac{1}{\sqrt{\left(2+\underbrace{\frac{3}{n^4}}_{\to 0}\right)^3}}$$

$$= \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

MATH 1272
Midterm III

SOME FORMULAS

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin^2(A/2) = (1 - \cos A)/2$$

$$\cos^2(A/2) = (1 + \cos A)/2$$

$$\sin A \cos B = \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sec^2 x = \tan^2 x + 1$$