Math 3592H Honors Math I
Midterm exam 1, Thursday October 6, 2016

Instructions:
50 minutes, closed book and notes, no electronic devices.
There are four problems, worth a total of 100 points.

1. (48 points total; 8 points each part)
For these vectors in $\mathbb{R}^3$,

$$\overline{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \overline{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \overline{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

compute the following via dot and cross products. Your answers are allowed to contain unevaluated inverse trigonometric functions.

(i) The length of $\overline{u}$.

$$|\bar{u}| = \sqrt{1^2+2^2+0^2} = \sqrt{5}$$

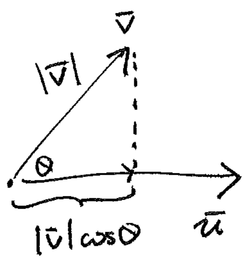
(ii) The angle between $\overline{u}, \overline{v}$.

$$\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|}\right) = \cos^{-1}\left(\frac{1\cdot 0+2\cdot 1+0\cdot 2}{\sqrt{1^2+2^2+0^2}\sqrt{0^2+1^2+2^2}}\right) = \cos^{-1}\left(\frac{2}{\sqrt{5}\sqrt{5}}\right) = \cos^{-1}\left(\frac{2}{5}\right)$$

(iii) The length of the projection of $\overline{v}$ orthogonally (perpendicularly) onto the line spanned by $\overline{u}$.

$$|\bar{v}|\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}|} = \frac{2}{\sqrt{5}}$$



(iv) The area of the parallelogram in $\mathbb{R}^3$ spanned by $\overline{u}$ and $\overline{v}$, that is, having vertices $\{\overline{0}, \overline{u}, \overline{v}, \overline{u}+\overline{v}\}$.

$$\text{area} = |\overline{u} \times \overline{v}| = \left| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right| = \left| \begin{bmatrix} +\det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \\ -\det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ +\det \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{bmatrix} \right| = \left| \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right|$$

$$= \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

(v) Some vector in $\mathbb{R}^3$ orthogonal (perpendicular) to both $\overline{u}$ and $\overline{v}$.

$$\overline{u} \times \overline{v} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

(vi) The volume of the parallelepiped (slanted box) in $\mathbb{R}^3$ spanned by $\overline{u}, \overline{v}, \overline{w}$, that is, having vertices $\{\overline{0}, \overline{u}, \overline{v}, \overline{w}, \overline{u}+\overline{v}, \overline{u}+\overline{w}, \overline{v}+\overline{w}, \overline{u}+\overline{v}+\overline{w}\}$.

$$\text{volume} = \det \begin{bmatrix} | & | & | \\ \overline{w} & \overline{u} & \overline{v} \\ | & | & | \end{bmatrix} = \overline{w} \cdot (\overline{u} \times \overline{v}) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = 9$$

2. (21 points total; 7 points each part)

Assuming that $f(x) = \sin(x)$ is continuous, and $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, compute with proof and/or explanations the values of the following limits of functions $g : \mathbb{R}^3 \to \mathbb{R}$ as $\overline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ approaches $\overline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(i)

$$\lim_{\overline{\mathbf{x}}\to\overline{0}} 3x^2 + 5y + z = 3\cdot 0^2 + 5\cdot 0 + 0 = 0$$

polynomial functions are continuous on all of $\mathbb{R}^3$ (Cor 1.5.30)

(ii)

$$\lim_{\overline{\mathbf{x}}\to\overline{0}} \sin(3x^2 + 5y + z) = \sin\left(\lim_{\overline{x}\to\overline{0}} 3x^2+5y+z\right) = \sin(0) = 0$$

$f(x) = \sin(x)$ was assumed continuous on all of $\mathbb{R}$

by (i)

(iii)

$$\lim_{\overline{\mathbf{x}}\to\overline{0}} \frac{\sin(3x^2 + 5y + z)}{3x^2 + 5y + z}$$

$$= \lim_{\overline{x}\to\overline{0}} \frac{\sin(h(\overline{x}))}{h(\overline{x})}$$ where $\lim_{\overline{x}\to\overline{0}} h(\overline{x}) = 0$ by (i)

$$= \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

limit of a composite function is the composition of the limits (Thm 1.5.24)

by assumption

3. (15 points total) Consider the matrix $A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$.

(i) (5 points) Compute A^2 and A^3.

$A^2 = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(ii) (10 points) Compute $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

Since $A^3 = 0$, $A^4 = A^5 = \dots = 0$,

and hence $e^A = I + A + \frac{A^2}{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{a^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & \frac{a^2}{2}+b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$

4. (16 points total; 8 points each part) Prove or disprove:

(a) An arbitrary union of (possibly infinitely many) closed sets is closed.

No, e.g. $[0,1) \subset \mathbb{R}$ is not closed since it has 1 as a limit point,

but $[0,1) = \bigcup_{x \in [0,1)} \{x\}$

every singleton set is closed in $\mathbb{R}$

(b) If $\lim_{k\to\infty} a_k = L$ in $\mathbb{R}$, and $a_k \le M$ for all k, then $L \le M$.

Yes, and one can prove this by contradiction.

Assume $\lim_{k\to\infty} a_k = L$ and $a_k \le M \ \forall k$, but $L > M$.

Then picking any ϵ with $0 < \epsilon < L - M$, such as $\epsilon = \frac{L-M}{2}$,

there exists some K for which $k > K$ implies $|a_k - L| < \epsilon$

$\Rightarrow a_k > L - \epsilon$

$> L - (L-M)$ since $\epsilon < L-M$

$> M$

Contradiction.