Math 3592H Honors Math I
Quiz 3, Thursday Nov. 3, 2016

Instructions:
15 minutes, closed book and notes, no electronic devices.
There are three problems, worth a total of 20 points.

1. (8 points) Parametrize/describe all solutions to the system

$$\begin{array}{ccccccccc} x & + & y & + & z & + & w & = & 0 \\ x & + & 2y & + & 3z & + & 4w & = & 1. \end{array}$$

matrix form $\rightsquigarrow \begin{pmatrix}1&1&1&1\\1&2&3&4\end{pmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \xrightarrow{\text{augmented matrix}} \left[\begin{array}{cccc|c}1&1&1&1&0\\1&2&3&4&1\end{array}\right] \xrightarrow{\text{row-reduce}} \left[\begin{array}{cccc|c}1&1&1&1&0\\0&1&2&3&1\end{array}\right] \xrightarrow{\text{row reduce}} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c}1&0&-1&-2&-1\\0&1&2&3&1\end{array}\right] \end{array}$

$$\Rightarrow \begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}z+2w-1\\-2z-3w+1\\z\\w\end{bmatrix}=z\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+w\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}+\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$$

2. (4 points) Prove or disprove:
Assume two functions $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ are both differentiable on $\mathbb{R}^n$, and satisfy $\mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{x}$ and $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
Then for every $\mathbf{a}$ in $\mathbb{R}^n$, the Jacobian matrix $J\mathbf{f}(\mathbf{a})$ is invertible.

True: $\bar{g}(\bar{f}(\bar{x}))=\bar{x} \ \forall \bar{x}\in\mathbb{R}^n$

$\Rightarrow \bar{g}\circ\bar{f} = 1_{\mathbb{R}^n}$

$\Rightarrow \underbrace{D(\bar{g}\circ\bar{f})(\bar{a})}_{[D\bar{g}(\bar{f}(\bar{a}))][D\bar{f}(\bar{a})]} = \underbrace{D(1_{\mathbb{R}^n})}_{I_n}$

i.e. $[J\bar{g}(\bar{f}(\bar{a}))]\,[J\bar{f}(\bar{a})] = I_n$

so $[J\bar{f}(\bar{a})]$ is $n\times n$ with a left-inverse, hence invertible.

Although we don't really need it, if $\bar{b}=\bar{f}(\bar{a})$, so $\bar{a}=\bar{g}(\bar{b})$, then

$\underbrace{D(\bar{f}\circ\bar{g})(\bar{b})}_{[D\bar{f}(\bar{g}(\bar{b}))][D\bar{g}(\bar{b})]} = \underbrace{D(1_{\mathbb{R}^n})}_{I_n}$

i.e. $[J\bar{f}(\underbrace{\bar{g}(\bar{b})}_{\bar{a}})]\underbrace{[J\bar{g}(\bar{b})]}_{\text{also a right-inverse for } [J\bar{f}(\bar{a})]} = I_n$

3. (8 points total)
Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 3 & c \end{bmatrix}.$$

(a) (4 points) Find all values of c that make A invertible

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 3 & c \end{bmatrix} \overset{\text{row-reduce}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & c \end{bmatrix} \overset{\text{row-reduce}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & c-6 \end{bmatrix}$$

(row-reduce further (if you like)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & c-6 \end{bmatrix}$$)

Invertible $\Longleftrightarrow c-6 \neq 0$
i.e. $c \neq 6$

(b) (2 points) For each of the values of c found in part (a), how many solutions are there to the matrix system $A\mathbf{x} = \mathbf{0}$?

Exactly one: $A\bar{x} = \bar{0} \iff \bar{x} = A^{-1}\bar{0} = \bar{0}$

(c) (2 points) For each of the values of c found in part (a), how many solutions are there to the matrix system $A\mathbf{x} = \mathbf{e}_1$?

Exactly one: $A\bar{x} = \bar{0} \iff \bar{x} = A^{-1}\bar{e}_1$