(This quiz turned out mostly disastrous due to part (b), and was ignored in the class grading.)

Math 3592H Honors Math I
Quiz 4, Thursday Dec. 1, 2016

Instructions:
20 minutes, closed book and notes, no electronic devices. There is one problem worth 20 points, with four parts each worth 5 points.

1. For any scalar c in $\mathbb{R}$, consider the symmetric matrix $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$.

(a) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A. What is its eigenvalue?

$$\begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c+2 \\ c+2 \\ c+2 \end{bmatrix} = \underbrace{(c+2)}_{\text{the eigenvalue is } \lambda = c+2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) What are *all* of the eigenvalues of A?

(1) The saner (but cleverer) approach was to note $A^T = A$

so A acts on $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^\perp = \{\bar{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

which has basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, both of which one can check

are eigenvectors with eigenvalue $c-1$:

$$\begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} c-1 \\ 1-c \\ 0 \end{bmatrix} = (c-1)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ c-1 \\ 1-c \end{bmatrix} = (c-1)\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

So the eigenvalues of A are $(c+2, c-1, c-1)$, or $\{c+2, c-1\}$ as a set.

(2) The less sane, but still do-able if one is really careful approach is

to compute $\chi_A(t) = \det\begin{bmatrix} t-c & -1 & -1 \\ -1 & t-c & -1 \\ -1 & -1 & t-c \end{bmatrix} = (t-c)\det\begin{bmatrix} t-c & -1 \\ -1 & t-c \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ -1 & t-c \end{bmatrix} + (-1)\det\begin{bmatrix} -1 & t-c \\ -1 & -1 \end{bmatrix}$

nasty algebra omitted! $= \ldots = t^3 - 3ct^2 + (3c^2-3)t - (c^3+3c-2)$

$= (t-(c+2))(t-(c-1))^2$

I had hoped that knowing $t-(c+2)$ was a factor, from part (a), would make this easier, but that was naive!

(c) Find an explicit orthonormal basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for $\mathbb{R}^3$ consisting of eigenvectors for A.

$\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector $\rightsquigarrow$ $\boxed{\bar{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}}$ is a unit length eigenvector

For $\lambda = c-1$, $\ker(\lambda I - A) = \ker\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}^\perp$ has basis $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$

normalize

$$\boxed{\bar{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}}$$

To get $\bar{v}_3$, solve $\begin{bmatrix}1&1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&1&1\\0&-2&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \bar{x} = \begin{bmatrix}-x_3/2\\-x_3/2\\x_3\end{bmatrix} = x_3\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$

normalize

$$\boxed{\bar{v}_3 = \frac{1}{\sqrt{3/2}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}}$$

(d) Find an explicit 3×3 matrix P which is orthogonal ($P^{-1} = P^\top$) and for which $P^T AP$ is diagonal.

$$P = \begin{bmatrix}|&|&|\\\bar{v}_1&\bar{v}_2&\bar{v}_3\\|&|&|\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{3/2}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{2\sqrt{3/2}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3/2}}\end{bmatrix}$$ will work.