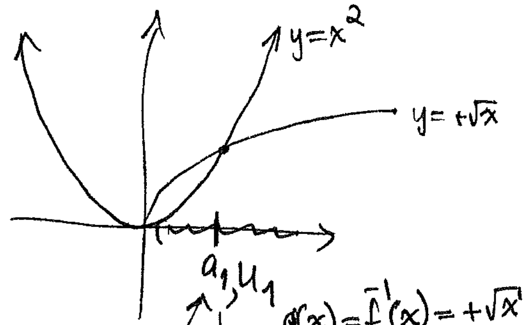
Note: 1. For a given function $\bar{f}(\bar{x})$, the W and $\bar{g}=\bar{f}^{-1}$ may depend on $\bar{a}$ (and U)

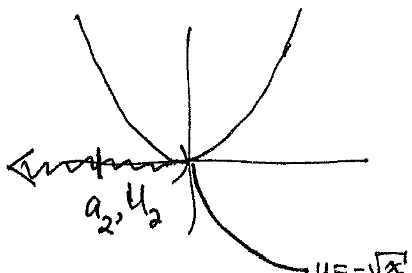
2. We may have no nice formula for $\bar{g}=\bar{f}^{-1}$

EXAMPLES

① $f(x)=x^2$
$\mathbb{R}^1 \longrightarrow \mathbb{R}^1$



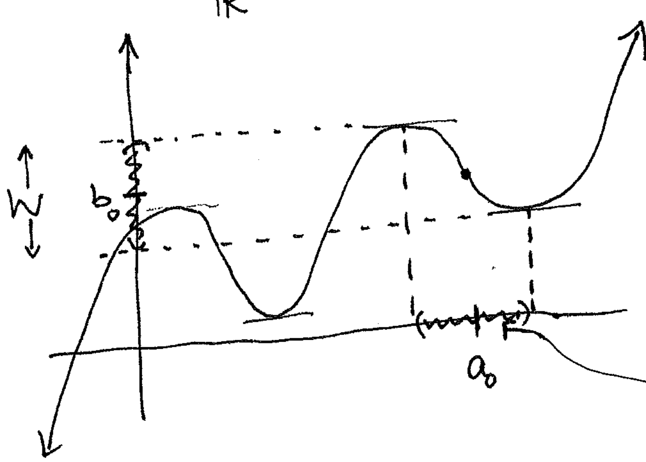
has $g(x)=f^{-1}(x)=+\sqrt{x}$



② A quintic polynomial

$$f(x)= x^5+ax^4+bx^3+cx^2+dx+e$$

$$\mathbb{R}^1 \longrightarrow \mathbb{R}^1$$



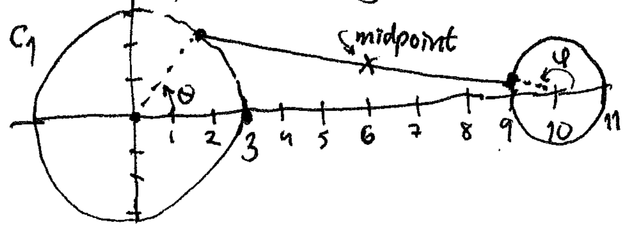
has no simple radical formula for its 5 roots to $f(x)=a_0$

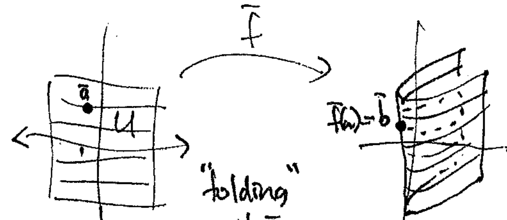
Nevertheless, for a_0 in here, we can (approximately) solve for $x=f^{-1}(y)$ with y lying in the region W shown, via Newton's method.

12/1/2016 ➤

③ (EXAMPLE 2.10.6) One can use the fact that $D\bar{f}(\bar{a})$ not invertible gives a clue to where $\bar{f}: U \longrightarrow \mathbb{R}^n$ ($U \subset \mathbb{R}^n$) may have done some "folding" near $\bar{a}$, to guess where the boundary of $\mathrm{img}(\bar{f})$ lies:

If C_1, C_2 are the circles shown here, where do the midpoints of line segments between their points actually trace out in $\mathbb{R}^2$?





(113) Model it as the image of this map $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix}\theta\\ \varphi\end{pmatrix} \longmapsto F\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}10+\cos\varphi\\ \sin\varphi\end{pmatrix} + \begin{pmatrix}3\cos\theta\\ 3\sin\theta\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix}3\cos\theta + \cos\varphi + 10\\ 3\sin\theta + \sin\varphi\end{pmatrix}$$

(which has some obvious periodicity in θ, φ)

and see where $DF\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$ is not invertible, to help detect the image boundary:

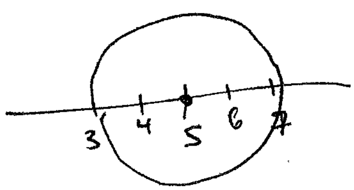
$$\left[JF\begin{pmatrix}\theta\\ \varphi\end{pmatrix}\right] = \frac{1}{2}\begin{bmatrix}-3\sin\theta & -\sin\varphi\\ 3\cos\theta & \cos\varphi\end{bmatrix} \overset{\det}{\rightsquigarrow} \det\left[JF\begin{pmatrix}\theta\\ \varphi\end{pmatrix}\right] = -\left(\frac{1}{2}\right)^2 3(\sin\theta\cos\varphi - \cos\theta\sin\varphi)$$

$$= -\frac{3}{4}\sin(\theta - \varphi)$$

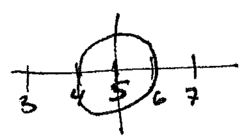
$$\Rightarrow \det\left[JF\begin{pmatrix}\theta\\ \varphi\end{pmatrix}\right] = 0 \text{ when } \boxed{\begin{matrix}\theta - \varphi = 0\\ \theta - \varphi = \pi\end{matrix}}$$

These have images

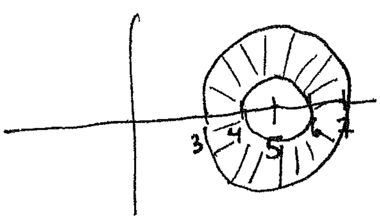
$$F\begin{pmatrix}\theta\\ \theta\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\cos\theta + \cos\theta + 10\\ 3\sin\theta + \sin\theta\end{pmatrix} = \begin{pmatrix}5 + 2\cos\theta\\ 2\sin\theta\end{pmatrix}$$



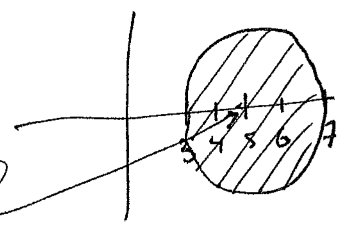
$$F\begin{pmatrix}\theta\\ \theta - \pi\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\cos\theta + \overbrace{\cos(\theta - \pi)}^{-\cos(\theta)} + 10\\ 3\sin\theta + \underbrace{\sin(\theta - \pi)}_{-\sin(\theta)}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\cos\theta + 10\\ 2\sin\theta\end{pmatrix} = \begin{pmatrix}5 + \cos\theta\\ \sin\theta\end{pmatrix}$$



The only bounded sets with boundaries contained in these two circles are these:



or



But one can check that $\begin{pmatrix}5\\ 0\end{pmatrix}$ is not in the image of F, so it must be the picture on the left for $\operatorname{img}(F)$.

Proof of Inverse Function Theorem

(following N. Wallach's expansion of M. Spivak's proof from "Calculus on manifolds")

see syllabus page for PDF

Recall we're given $\bar{f} : \underset{\mathbb{R}^n}{\overset{\cap}{U}}{}^{\text{open}} \to \mathbb{R}^n$, $\bar{f} \in C^1(U)$, $\det J\bar{f}(\bar{a}) \neq 0$ for some $\bar{a} \in U$

and want to exhibit open sets $\bar{a} \in V (\subset U) \subset \mathbb{R}^n$ and $\bar{g} : W \to V$

$f(\bar{a}) =: \bar{b} \in W \subset \mathbb{R}^n$

such that (i) $\bar{f} : V \to W$, $\bar{g} : W \to V$ are inverses

(ii) $\bar{g}$ is differentiable on W.

FIRST a reduction to ease computations: we can assume WLOG that $D\bar{f}(\bar{a}) = 1_{\mathbb{R}^n}$ since if $L := J\bar{f}(\bar{a})$ we can replace $\bar{f}$ with the composite $U^{\text{open}} \xrightarrow{\bar{f}} \mathbb{R}^n \xrightarrow{L^{-1}} \mathbb{R}^n$ having $D\hat{f}(\bar{a}) = L^{-1} \circ L = 1_{\mathbb{R}^n}$

$$\hat{f} = L^{-1} \circ \bar{f}$$

If we then find the inverse $\hat{g}$ for $\hat{f}$ with

$$\begin{tikzcd} \bar{a} \in V \subset \mathbb{R}^n \arrow[r, bend left, "\hat{f}"] & W \subset \mathbb{R}^n \ni \hat{f}(\bar{a}) = L^{-1}(\bar{b}) \arrow[l, bend left, "\hat{g}"] \end{tikzcd}$$

we can check that $\bar{g} := L^{-1} \circ \hat{g}$ does the job:

$$\begin{tikzcd} \bar{a} \in V \arrow[r, shift left, "\bar{f}"] \arrow[rr, bend left=50, "\hat{f}"] & L(W) \ni \bar{b} \arrow[l, shift left, "\bar{g}"] \arrow[r, shift right, "L^{-1}"'] & W \ni L^{-1}(\bar{b}) \arrow[l, shift right, "L"'] \arrow[ll, bend left=50, "\hat{g}"] \end{tikzcd}$$

SECOND, we can shrink U to a ball $B_\delta(\bar{a})$ of small radius $\delta > 0$ about $\bar{a}$ so as to make these things both happen:

- $\left| \frac{\partial f_i}{\partial x_j}(\bar{x}) - \frac{\partial f_i}{\partial x_j}(\bar{a}) \right| < \frac{1}{2n^2}$ $\forall i,j$ $\forall \bar{x} \in U$ (as $\frac{\partial f_i}{\partial x_j}$ are continuous)
- $\det J\bar{f}(\bar{x}) \neq 0$ $\forall \bar{x} \in U$, (as $\det J\bar{f}(\bar{x})$ is continuous as a composite

$$\begin{array}{ccccc} U & \longrightarrow & Mat(n,n) & \xrightarrow{\det} & \mathbb{R} \\ \bar{x} & \longmapsto & \left[\frac{\partial f_i}{\partial x_j}\right] & & \\ & & A & \longmapsto & \det A \end{array}$$

so pick δ small enough that $|\det J\bar{f}(\bar{x}) - \underbrace{\det J\bar{f}(\bar{a})}_{\neq 0}| < \frac{|\det J\bar{f}(\bar{a})|}{2}$)