We could understand this if we knew more about the powers A^n

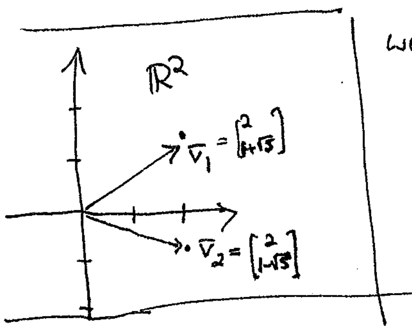
Luckily, certain vectors in $\mathbb{R}^2$ are scaled by A:

or is it luck? (Nope!)

$$A\begin{bmatrix}2\\1+\sqrt5\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1+\sqrt5\end{bmatrix} = \begin{bmatrix}1+\sqrt5\\3+\sqrt5\end{bmatrix} = \frac{1+\sqrt5}{2}\begin{bmatrix}2\\1+\sqrt5\end{bmatrix}$$

$$A\begin{bmatrix}2\\1-\sqrt5\end{bmatrix} = \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}2\\1-\sqrt5\end{bmatrix} = \begin{bmatrix}1-\sqrt5\\3-\sqrt5\end{bmatrix} = \frac{1-\sqrt5}{2}\begin{bmatrix}2\\1-\sqrt5\end{bmatrix}$$

DEF'N 2.7.2: If A is square $n\times n$ and $\bar v \in \mathbb{R}^n \setminus \{\bar 0\}$ satisfies $A\bar v = \lambda \bar v$, we say that $\bar v$ is an eigenvector for A, with eigenvalue λ.



e.g. A above has $\bar v_1 = \begin{bmatrix}2\\1+\sqrt5\end{bmatrix}$ as an eigenvector, with eigenvalue $\lambda_1 = \frac{1+\sqrt5}{2} (= \varphi)$

and $\bar v_2 = \begin{bmatrix}2\\1-\sqrt5\end{bmatrix}$ as an eigenvector, with eigenvalue $\lambda_2 = \frac{1-\sqrt5}{2}$

How did this help us with A^n?

$$A\bar v_1 = \lambda_1 \bar v_1 \Rightarrow A^n \bar v_1 = \lambda_1^n \bar v_1$$
$$A\bar v_2 = \lambda_2 \bar v_2 \Rightarrow A^n \bar v_2 = \lambda_2^n \bar v_2$$

so if we form the matrix $P = \begin{bmatrix}|&|\\\bar v_1&\bar v_2\\|&|\end{bmatrix} = \begin{bmatrix}2&2\\1+\sqrt5&1-\sqrt5\end{bmatrix}$, which is invertible (Why?)

then $AP = \begin{bmatrix}|&|\\A\bar v_1&A\bar v_2\\|&|\end{bmatrix} = \begin{bmatrix}|&|\\\lambda_1\bar v_1&\lambda_2\bar v_2\\|&|\end{bmatrix} = \begin{bmatrix}|&|\\\bar v_1&\bar v_2\\|&|\end{bmatrix}\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$

$$AP = P\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$$

mult. on left by P^{-1}

$$P^{-1}AP = \begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$$ a diagonal matrix!

Similarly $P^{-1}A^nP = \underbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}_{n \text{ times}} = (P^{-1}AP)^n = \begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}^n = \begin{bmatrix}\lambda_1^n&0\\0&\lambda_2^n\end{bmatrix}$

(It's easy to take powers of diagonal matrices: $\begin{bmatrix}\lambda_1&&0\\&\lambda_2&\\0&&\ddots\ \lambda_m\end{bmatrix}^n = \begin{bmatrix}\lambda_1^n&&0\\&\lambda_2^n&\\0&&\ddots\ \lambda_m^n\end{bmatrix}$.)

Going back to Fibonacci numbers $\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

we have $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

$$A^n = \underbrace{\left(P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\right) \cdots \left(P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\right)}_{n \text{ times}}$$

$$= P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1}$$

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So $\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = P \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P^{-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

some algebra! (not worth us doing by hand)

$$= \begin{bmatrix} \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ * \end{bmatrix}$$

an entry we don't care about! (but it must be same, replacing n by $n+1$)

$\Rightarrow$ Fibonacci (exact) formula: $\boxed{a_n = \frac{5+\sqrt{5}}{10}\varphi^n + \frac{5-\sqrt{5}}{10}\alpha^n}$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, $\alpha = \frac{1-\sqrt{5}}{2} \approx -0.618$

$\Rightarrow$ $a_n \approx \frac{5+\sqrt{5}}{10}\varphi^n$ for large n, since $\alpha^n \to 0$.

PROP-DEF'N (2.7.3) Given an $n \times n$ matrix A, one can find a basis $\{\bar v_1, \dots, \bar v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors for A (called an eigenbasis for A), say with eigenvalues $\lambda_1, \dots, \lambda_n$ (i.e. $A\bar v_i = \lambda_i \bar v_i$)

$\Longleftrightarrow$ one can diagonalize A, that is $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} =: \Lambda$ is diagonal,

where $P = \begin{bmatrix} | & | & & | \\ \bar v_1 & \bar v_2 & \cdots & \bar v_n \\ | & | & & | \end{bmatrix}$ is invertible

Terminology: $A \mapsto P^{-1}AP$ is called conjugating A by P or a similarity transformation; $A, P^{-1}AP$ are similar

RMK: ~~Note that~~ the diagonal matrix Λ is merely expressing $\bar x \mapsto A\bar x$ in the $(\bar v_i)$ basis:

$$\begin{tikzcd} \text{basis } (\bar e_1,\dots,\bar e_n)\ \mathbb{R}^n \arrow[r, "A", "\bar x \mapsto A\bar x"'] & \mathbb{R}^n \text{ basis } (\bar e_1,\dots,\bar e_n) \arrow[d, shift left, "P^{-1} = \Phi_{\{\bar v\}}"] \\ \text{basis } (\bar v_1,\dots,\bar v_n)\ \mathbb{R}^n \arrow[u, "P = \Phi_{\{\bar v\}}^{-1}"] \arrow[r, "P^{-1}AP"', "\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}"] & \mathbb{R}^n \text{ basis } (v_1,\dots,v_n) \arrow[u, shift left] \end{tikzcd}$$

proof: Same calculation we just did with Fibonaccis ...

$A\vec{v}_i = \lambda_i \vec{v}_i$ for $= 1, 2, \ldots n \iff$

$$AP = A\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{P\text{, invertible iff } \{\vec{v}_i\} \text{ give a basis}} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}}_{P}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = P\Lambda$$

$\iff P^{-1}AP = \Lambda$. All reversible! ∎

Q: So how do we find eigenvectors, eigenvalues?

Q: Does every A have an eigenbasis? We'll see, NO.

But every A has at least one eigenvalue, eigenvector

and in many cases we can insure that A does have an eigenbasis.

It will help us to dip into §4.8 and define determinants $\det M$ for all square $n\times n$ matrices M, not just $n = 1, 2, 3$, as some polynomial in the entries (m_{ij}) with the following magic property (to be proven in a bit):

THM 4.8.3: M $n\times n$ is not invertible $\iff \det M = 0$
(so invertible $\iff \det M \neq 0$)

How will this help?

Note $\vec{v}$ is an eigenvector for A with eigenvalue λ

$\iff A\vec{v} = \lambda\vec{v}$ has (nonzero) soln $\vec{v} \neq \vec{0}$

$\Updownarrow$

$\lambda\vec{v} - A\vec{v} = \vec{0}$

$\Updownarrow$

$(\lambda I_n - A)\vec{v} = \vec{0}$, i.e. $\underset{\neq \vec{0}}{\vec{v}} \in \ker(\lambda I_n - A)$

$\Updownarrow$

$\ker(\lambda I_n - A) \neq \vec{0}$

$\Updownarrow$

$\lambda I_n - A$ is not invertible

$\Updownarrow$ THM 4.8.3

$\det(\lambda I_n - A) = 0$

$\Updownarrow$

λ is a root of the polynomial $\det(tI_n - A) = \chi_A(t)$ in the variable t, called (DEF'N 4.8.17) the characteristic polynomial of A.

$$\det\begin{bmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & & \\ \vdots & & \ddots & \\ -a_{n1} & & & t-a_{nn} \end{bmatrix} = \det(tI_n - A)$$

always at least one by fund. Thm. of Algebra!

(99) EXAMPLE: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ from the Fibonacci example

has $\chi_A(t) = \det(tI_2 - A) = \det \begin{bmatrix} t & -1 \\ -1 & t-1 \end{bmatrix} = t(t-1) - (-1)^2$
$= t^2 - t - 1$

and its roots are $t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} = \underset{\substack{\| \\ \lambda_1 \\ \| \\ \varphi}}{\frac{1+\sqrt{5}}{2}}, \underset{\substack{\| \\ \lambda_2}}{\frac{1-\sqrt{5}}{2}}$

the two eigenvalues of A that we saw earlier.

In fact, to find the eigenvectors $\bar{v}_1, \bar{v}_2$, one simply computes $\ker(\lambda_i I_2 - A)$:

$$\ker(\lambda_1 I_2 - A) = \ker \begin{bmatrix} \lambda_1 & -1 \\ -1 & \lambda_1 - 1 \end{bmatrix} = \ker \begin{bmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} - 1 \end{bmatrix}$$

$$= \left\{ \text{sol'ns to } \begin{bmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix} \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} =$$

row-reduce

$$\begin{bmatrix} 1 & \frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{bmatrix}$$

$\in \mathbb{R}\bar{v}_1$ where $\bar{v}_1 = \begin{bmatrix} 2 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$

Similarly $\ker(\lambda_2 I_2 - A) = \mathbb{R}\bar{v}_2$ where $\bar{v}_2 = \begin{bmatrix} 2 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$

So how to define the general $n \times n$ determinant? Recursively is one way...

$$\det : \left\{ \begin{array}{l} \text{square} \\ \text{matrices} \end{array} A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \right\} \rightarrow \mathbb{R}$$

is defined as $\det[a_{11}] = a_{11}$ for $n=1$

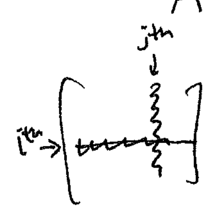
and $\det A := a_{11} \det A_{11} - a_{21} \det A_{21} + a_{31} \det A_{31} - \dots \pm (-1)^{n-1} a_{n1} \det A_{n1}$

$$\det \begin{bmatrix} a_{11} & & \\ a_{21} & & \\ \vdots & & \\ a_{n1} & & \end{bmatrix} = \sum_{i=1}^{n} (-1)^{i-1} \det A_{i1}$$

(= Laplace expansion along first column)

where $A_{ij} = (i,j)$-minor of A

:= A with i^{th} row, j^{th} column removed



(100) We'll take a different approach, using Leibniz's expansion

First we need sign or signature of a permutation

(DEF'N 4.8.10) $\mathrm{sgn} : \mathrm{Perm}_n \longrightarrow \{+1,-1\}$

$$\mathrm{Perm}_n = \left\{ \begin{array}{c} \text{all permutations} \\ (=\text{bijections}) \\ \sigma : \{1,\dots,n\} \to \{1,\dots,n\} \\ \| \\ \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \end{array} \right\}$$

$$\sigma \longmapsto \mathrm{sgn}(\sigma) := (-1)^{\overbrace{\#\{(i,j) : 1 \le i < j \le n,\ \sigma(i) > \sigma(j)\}}^{\text{inversions in } \sigma}}$$

e.g. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 4 & 2 & 3 \end{pmatrix}$ has $4+3+2=9$ inversions

so $\mathrm{sgn}(\sigma) = (-1)^9 = -1$

PROP: For any pair $i < j$, if σ' is obtained from σ by swapping $\sigma(i), \sigma(j)$

i.e. $\sigma'(k) = \begin{cases} k & \text{if } k \ne i,j \\ \sigma(j) & \text{if } k = i \\ \sigma(i) & \text{if } k = j \end{cases}$

then $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$

Proof: It's very easy to check when $j = i+1$:

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & i & \overset{i+1}{j} & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \sigma(j) & \cdots & \sigma(n) \end{pmatrix}$$

only gain exactly one inversion if $\sigma(i) < \sigma(j)$

or lose exactly one inversion if $\sigma(i) > \sigma(j)$

$$\sigma' = \begin{pmatrix} 1 & 2 & \cdots & i & j & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(j) & \sigma(i) & \cdots & \sigma(n) \end{pmatrix}$$

Either way $\mathrm{sgn}(\sigma') = (-1)^{\#\text{inversions in } \sigma'} = (-1)^{\#(\text{inversions in } \sigma) \pm 1} = -\mathrm{sgn}(\sigma)$.

When $j - i > 1$, mimic the swapping by $2(j-i)-1$ adjacent swaps:

e.g.

$$\left.\begin{array}{l} 1\,\overset{i}{2}\,3\,4\,\overset{j}{5}\,6 \\ \left.\begin{array}{l} 1\,2\,3\,5\,4\,6 \\ 1\,2\,5\,3\,4\,6 \end{array}\right\} j-i-1 \text{ swaps} \\ \left.\begin{array}{l} 1\,5\,2\,3\,4\,6 \end{array}\right\} 1 \text{ swap} \\ \left.\begin{array}{l} 1\,5\,3\,2\,4\,6 \\ 1\,5\,3\,4\,2\,6 \end{array}\right\} j-i-1 \text{ swaps} \end{array}\right\} 2(j-i)-1 \text{ swaps}$$

This implies $\mathrm{sgn}(\sigma') = (-1)^{2(j-i)-1}\,\mathrm{sgn}(\sigma)$

$= -\mathrm{sgn}(\sigma)$ ∎

DEF'N: (THM 4.8.11) For A $n\times n$ matrix, define

$$\det A := \sum_{\sigma \in \mathrm{Perm}_n} \mathrm{sgn}(\sigma)\, a_{1,\sigma(1)}\, a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

EXAMPLES: $n=1$ $\quad \det[a_{11}] = a_{11}$

$n=2$ $\quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} = +a_{11}a_{22} - a_{12}a_{21}$

$\sigma_1 = \begin{pmatrix}1&2\\1&2\end{pmatrix} \qquad \sigma_2 = \begin{pmatrix}1&2\\2&1\end{pmatrix}$

$n=3$ $\quad \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = +a_{11}a_{22}a_{23} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$

$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \quad \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \quad \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$

$- a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$

$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \quad \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \quad \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

THM 4.8.1: $\det : \left\{\begin{matrix} n\times n \\ \text{square} \\ \text{matrices} \end{matrix}\right\} \longrightarrow \mathbb{R}$ has these properties, and is the unique such function:

$A = \begin{bmatrix} | & | & & | \\ \vec v_1 & \vec v_2 & \cdots & \vec v_n \\ | & | & & | \end{bmatrix}$

(1) det is linear in each column $\vec v_j$, i.e. $\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & a\vec v_j + b\vec v_j{}' & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix}$
("multilinearity")

$$= a\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & \vec v_j & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix} + b\det\begin{bmatrix} | & & | & & | \\ \vec v_1 & \cdots & \vec v_j{}' & \cdots & \vec v_n \\ | & & | & & | \end{bmatrix}$$

(2) swapping any 2 columns in A negates det
("alternating")

(3) $\det\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{bmatrix} = 1$

("normalization")

Proof: (3) is easy since only $\sigma = \begin{pmatrix}1&2&\cdots&n\\1&2&\cdots&n\end{pmatrix}$ gives a nonzero term $+1\cdot \overset{1}{a_{11}}\,\overset{1}{a_{22}}\cdots\overset{1}{a_{nn}} = 1$

(2) comes from the $\mathrm{sgn}(\sigma') = -\mathrm{sgn}(\sigma)$ property when one swaps $\sigma(i), \sigma(j)$ in σ to get σ'.

(1) is easy since each term $a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$ contains exactly one factor a_{ij} from column j (namely for $i = \sigma^{-1}(j)$), so when it is replaced by $a\cdot a_{ij} + b a_{ij}'$ the whole sum behaves the same ∎