(103)

11/21/2016 Lots of properties of det now follow.

COR $\begin{pmatrix}4.8.4\\4.8.5\\4.8.6\\4.8.16\end{pmatrix}$

(i) $\det(AB) = \det(A)\det(B)$

(ii) $\det(P^{-1}) = \dfrac{1}{\det P}$

(iii) $\det(P^{-1}AP) = \det A$

(iv) $\chi_{P^{-1}AP}(t) = \chi_A(t)$

proof: (i): If $\det B = 0$ then $\exists \bar{v} \neq \bar{0}$ in $\ker B$ i.e. $B\bar{v} = \bar{0}$

so $\underbrace{AB\bar{v}}_{A\bar{0}} = \bar{0}$ also $\Rightarrow \det(AB) = 0$ (and (i) holds)

Otherwise if $\det B \neq 0$, then use column operations $B \xrightarrow{\text{colops}} I_n$ and write $B = E_1 E_2 \cdots E_N$ for some elementary matrices E_i.

We know $\det B = c_1 c_2 \cdots c_N$ where

$$c_i = \begin{cases} 1 & \text{if } E_i \text{ adds a multiple of a column to another} \\ -1 & \text{if } E_i \text{ swaps two columns} \\ c_i & \text{if } E_i \text{ scales some column by } c_i \end{cases}$$

But then we also know

$$\det(AB) = \det(AE_1E_2 \cdots E_N)$$
$$= c_1 c_2 \cdots c_N \det A$$
$$= \det B \cdot \det A,$$

(ii): $P^{-1} \cdot P = I_n$

$\Rightarrow \underbrace{\det(P^{-1} \cdot P)}_{\text{by (i)}\ \det(P^{-1}) \cdot \det(P)} = \underbrace{\det I_n}_{1}$

(iii): $\det(P^{-1}AP) = \underbrace{\det(P^{-1})}_{\stackrel{(ii)}{=} 1/\det P} \cdot \det A \cdot \det P = \det A$

(iv): $\chi_{P^{-1}AP}(t) = \det(tI_n - P^{-1}AP) = \det(tP^{-1}I_nP - P^{-1}AP)$

$= \det(P^{-1} \cdot (tI_n - A)P)$

$= \det(tI_n - A)$

$= \chi_A(t)$

via (iii), although this is slightly tricky: (iii) applies whenever we choose $t \in \mathbb{R}$, not as a polynomial variable. But this shows the polynomial in t of degree n $p(t) := \det(P^{-1}(tI_n - A)P) - \det(tI_n - A)$ has more than n roots, so it is zero! ∎

(104)

There is a lot more to say about det, but let's get back to eigenvectors!

COR: Every $n \times n$ (real) matrix A has at least one eigenvalue $\lambda \in \mathbb{C}$ and accompanying eigenvector $\bar{v} \in \mathbb{C}^n$ with $A\bar{v} = \lambda \bar{v}$

proof: $\chi_A(t) = \det(tI_n - A) = \det \begin{bmatrix} t-a_{11} & -a_{12} & \cdots \\ -a_{21} & \ddots & \\ \vdots & & t-a_{nn} \end{bmatrix} = t^n - (a_{11}+a_{22}+\ldots+a_{nn})t^{n-1} + \ldots \pm \det A$

is a (non-constant) polynomial in t of degree n,

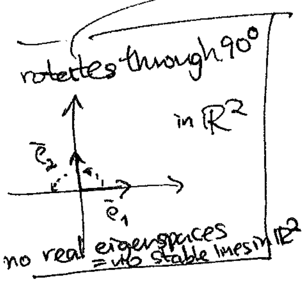
so it has at least one root $t = \lambda$ in $\mathbb{C}$ by Fundamental Theorem of Algebra.

But then one can find $\bar{v} \in \mathbb{C}^n$ solving $(A - \lambda I_n)\bar{v} = \bar{0}$ using row reduction over $\mathbb{C}$

(we only need the fact ~~that~~ every $\underset{x+iy}{\overset{\shortparallel}{z}} \in \mathbb{C} - \{\bar{0}\}$ has a multiplicative inverse $\frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ for scaling pivot entries to 1) ∎

EXAMPLE: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\chi_A(t) = \det \begin{bmatrix} t & -1 \\ +1 & t \end{bmatrix} = t^2 - (-1) = t^2 + 1$ ← no real roots!

$= (t+i)(t-i)$



$\lambda_1 = +i$: $\begin{bmatrix} +i & -1 \\ +1 & +i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \bar{v}_1 \in \mathbb{C} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 = -i$: $\begin{bmatrix} -i & -1 \\ +1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \bar{v}_2 \in \mathbb{C} \begin{bmatrix} i \\ 1 \end{bmatrix}$

We may not have an eigenbasis for A (even using $\mathbb{C}$), e.g. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\chi_A(t) = (t-4)(t-1)^2$

was discussed in section - $\bar{e}_1$ is an eigenvector for $\lambda_1 = 4$

but $\bar{e}_2$ and its multiples are the only eigenvectors for $\lambda_2 = 1$; can't find a third lin. indep. eigenvector!

(105)

The double root at $t=1$ was part of the problem.

THM. 2.7.4: If A has eigenvectors $\bar{v}_1, \dots, \bar{v}_k$ with distinct eigenvalues $\lambda_1, \dots, \lambda_k$ (i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$) then $\bar{v}_1, \dots, \bar{v}_k$ are lin. independent.

In particular, if $\chi_A(t) = (t-\lambda_1)\cdots(t-\lambda_n)$ has no repeated roots (i.e. $\lambda_i \neq \lambda_j \ \forall\, 1 \leq i < j \leq n$)

then A is diagonalizable, since the λ_i-eigenvectors $\bar{v}_i$ give an eigenbasis for A.

proof: If $\bar{v}_1, \dots, \bar{v}_k$ are lin. dependent,

write down a nontrivial lin. dependence $c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_k\bar{v}_k \overset{(*)}{=} \bar{0}$

that has the fewest nonzero coefficients, and assume $c_1 \neq 0$ (by re-indexing, if needed). We'll get a contradiction by manufacturing one with fewer nonzero coefficients:

Apply A to $(*)$ to get $c_1 A\bar{v}_1 + c_2 A\bar{v}_2 + \dots + c_k A\bar{v}_k = \bar{0}$

i.e. $c_1\lambda_1\bar{v}_1 + c_2\lambda_2\bar{v}_2 + \dots + c_k\lambda_k\bar{v}_k = \bar{0}$

and subtract $\lambda_1(*)$ i.e. $c_1\lambda_1\bar{v}_1 + c_2\lambda_1\bar{v}_2 + \dots + c_k\lambda_1\bar{v}_k = \bar{0}$

giving $0 + \underbrace{c_2(\lambda_2-\lambda_1)}_{\neq 0}\bar{v}_2 + \dots + \underbrace{c_k(\lambda_k-\lambda_1)}_{\neq 0}\bar{v}_k = \bar{0}$

these new coefficients $c_i(\lambda_i-\lambda_1)$ are nonzero $\Leftrightarrow$ the old c_i was nonzero

Fewer total! ∎

11/28/2016 >

REMARK: This already shows most matrices A are diagonalizable, since $\chi_A(t)$ usually has distinct roots. In fact, with a little more work, one can write down a polynomial in the entries (a_{ij}) of A that must vanish to get repeated roots in $\chi_A(t)$.

e.g. for $n=2$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has repeated roots for $\chi_A(t) = \det\begin{bmatrix} t-a & -b \\ -c & t-d \end{bmatrix} = t^2 - \underbrace{(a+d)}_{B}t + \underbrace{ad-bc}_{C}$

$$\Leftrightarrow 0 = B^2 - 4C = (a+d)^2 - 4(ad-bc)$$
$$= a^2 + 2ad + d^2 - 4ad + 4bc$$
$$= (a-d)^2 + 4bc$$