But now we've seen that this matrix corresponds to (once we fixed $x_1, \dots, x_{d+1}$) a linear transformation $T_d : \underbrace{P_d}_{\substack{=: \{\text{polynomials } p(x) = a_0 + a_1 x + \dots + a_d x^d \\ \text{of degree} \le d\}}} \longrightarrow \mathbb{R}^{d+1}$

$$p(x) \longmapsto \begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{bmatrix}$$

(Note that P_d really is a copy of $\mathbb{R}^{d+1}$ if we identify $p(x) = \sum_{i=0}^{d} a_i x^i$ with $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix}$,

and the above map T_d really is linear since $\begin{bmatrix} a_0 \\ \vdots \\ a_d \end{bmatrix} + \begin{bmatrix} b_0 \\ \vdots \\ b_d \end{bmatrix}$ is identified with $\sum_{i=0}^{d} (a_i + b_i) x^i$)

So to show the matrix and T_d are invertible (injective, surjective), by Cor 2.5.10 we only need to show $\ker(T_d) = \{\vec{0}\}$, that is, if

$$\begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{then } p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d = 0$$

$$\Updownarrow$$

$$\left.\begin{matrix} p(x_1) = 0 \\ p(x_2) = 0 \\ \vdots \\ p(x_{d+1}) = 0 \end{matrix}\right\} \Rightarrow p(x) \text{ has } d+1 \text{ roots } x_1, \dots, x_{d+1} \text{ (at least)}$$

$\Rightarrow p(x) = 0$ since degree of $p(x)$ is $\le d$

think about this using division algorithm; each root x_i factors out $(x - x_i)$ from $p(x)$, so $p(x) = (x-x_1)(x-x_2)\cdots(x-x_d)\underbrace{q(x)}_{\substack{\text{degree } 0, \\ \text{i.e. constant} \\ \text{but having } x_{d+1} \text{ as a root,} \\ \text{so } 0.}}$

$$\begin{array}{r|l} & \tilde{p}(x) \leftarrow \text{degree one smaller than } p(x) \\ \hline x - x_i & p(x) \\ & \vdots \\ & \vdots \\ & \vdots \\ & 0 \end{array}$$

Thus T_d is invertible. ∎

RMK: Exercise 2.5.18 gives the exact Lagrange interpolation formula for $p(x)$ in terms of $\{(x_i, y_i)\}_{i=1,\dots,d+1}$

§2.6 Abstract vector spaces

By now we've run into a couple of "spaces" that we have identified with $\mathbb{R}^N$ for some N,

such as $P_d := \{\text{polynomials } p(x) = a_0 + a_1 x + \dots + a_d x^d \text{ of degree } \leq d\} = \mathbb{R}^{d+1}$

$$p(x) \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix}$$

or $\mathrm{Mat}(m,n) := \{m \times n \text{ matrices } A = (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}\} \leftrightarrow \mathbb{R}^{mn}$

$$A \longleftrightarrow \left.\begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{mn} \end{bmatrix}\right\} mn$$

We've also needed linear maps between them.

So let's ~~abstract~~ formalize this.

DEF'N 2.6.1: A (real) vector space V is a set with two operations defined, addition

$$V \times V \to V$$
$$(\bar{v}, \bar{w}) \mapsto \bar{v} + \bar{w}$$

and scalar multiplication

$$\mathbb{R} \times V \to V$$
$$(c, \bar{v}) \mapsto c\bar{v}$$

satisfying some axioms:

1. $\exists$ a zero vector $\bar{0} \in V$ with $\bar{0} + \bar{v} = \bar{v} + \bar{0} = \bar{v}\ \ \forall \bar{v} \in V$
2. $\forall \bar{v} \in V\ \exists$ a vector $-\bar{v} \in V$ with $\bar{v} + (-\bar{v}) = (-\bar{v}) + \bar{v} = \bar{0}$
3. $+$ is commutative: $\bar{v} + \bar{w} = \bar{w} + \bar{v}\ \ \forall \bar{v}, \bar{w} \in V$
4. $+$ is associative: $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$
5. $1 \in \mathbb{R}$ has $1 \cdot \bar{v} = \bar{v}\ \ \forall \bar{v} \in V$
6. $a, b \in \mathbb{R}$, $\bar{v} \in V$ have $a(b\bar{v}) = (ab)\bar{v}$
7. $(a+b)\bar{v} = a\bar{v} + b\bar{v}$
8. $a(\bar{v} + \bar{w}) = a\bar{v} + a\bar{w}$

A subspace $W \subset V$ is just a subset containing $\bar{0}$ and closed under $+$ and scalar multiplication

i.e. $\bar{w} \in W,\ c \in \mathbb{R} \Rightarrow c\bar{w} \in W$

$\bar{w}_1, \bar{w}_2 \in W \Rightarrow \bar{w}_1 + \bar{w}_2 \in W$

DEF'N 2.6.4: A linear transformation $T: V \to W$ is a map satisfying $T(c\bar{v}) = cT(\bar{v})$

$$T(\bar{v}_1 + \bar{v}_2) = T(\bar{v}_1) + T(\bar{v}_2)$$

EXAMPLES: ① $P_d = \{\text{degree} \le d \text{ polynomials}\} = \{p(x) = a_0 + a_1x + \dots + a_dx^d\}$

$\cup$

$P_{d-1} = \{\text{degree} \le d-1 \text{ polynomials}\} = \{\text{those with } a_d = 0\}$ is a subspace,

and $P_d \xrightarrow{\frac{d}{dx}} P_{d-1}$ is a linear transformation. (Why?)

$$p(x) \longmapsto p'(x)$$

$$\sum_{i=0}^{d} a_i x^i = p(x) = a_0 + a_1x + \dots + a_dx^d$$

$$p'(x) = \sum_{i=1}^{d-1} i a_i x^{i-1} = a_1 + 2a_2x + 3a_3x^2 + \dots + da_dx^{d-1}$$

② $\mathrm{Mat}(n,n) \xrightarrow{tr} \mathbb{R}$

$$A \longmapsto \underbrace{tr}_{\text{trace}}(A) = \sum_{i=1}^{n} a_{i,i}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

is a linear transformation (Why?)

⓪ We already saw that if we ~~picked~~ fixed $x_1, x_2, \dots, x_{d+1}$ in $\mathbb{R}$ then $P_d \longrightarrow \mathbb{R}^{d+1}$ was a linear transformation.

$$p(x) \longmapsto \begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_{d+1}) \end{bmatrix}$$

DEF'N 2.6.10–2.6.14: As before, we define

- Lin. independence of $\{\bar{v}_i\}_{i=1,\dots,k}$ in V, i.e. $\sum_{i=1}^{k} c_i \bar{v}_i = \bar{0} \Rightarrow c_1 = \dots = c_k = 0$
- $\operatorname{span}(\bar{v}_1, \dots, \bar{v}_k) := \{c_1 \bar{v}_1 + \dots + c_k \bar{v}_k : c_i \in \mathbb{R}\}$
- $\{\bar{v}_i\}_{i=1,\dots,k}$ are a basis for V if they are lin. indep. & $\operatorname{span}\{\bar{v}_i\}_{i=1,\dots,k} = V$

~~or~~ equivalently, every $\bar{v} \in V$ can be written uniquely as $\bar{v} = \sum_{i=1}^{k} x_i v_i$

(in which case, you call $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$ the coordinates of $\bar{v}$ with respect to the (ordered) basis $(\bar{v}_1, \dots, \bar{v}_k)$)

11/11/2016 Once you have picked (ordered) bases $(\bar{v}_1, \dots, \bar{v}_n)$ for V
$(\bar{w}_1, \dots, \bar{w}_m)$ for W

one can express any linear transformation $T: V \to W$ uniquely in these bases via an $m \times n$ matrix $A = (a_{ij})$

$$\text{where} \quad T(\bar{v}_j) = \sum_{i=1}^{m} a_{ij} \bar{w}_i$$

that is, $A = \begin{bmatrix} | & | & & | \\ T(\bar{v}_1) & T(\bar{v}_2) & \cdots & T(\bar{v}_n) \\ | & | & & | \end{bmatrix}$ where $T(\bar{v}_j)$ is expressed in coordinates with respect to $(\bar{w}_1, \dots, \bar{w}_m)$

As before, composition of linear maps corr. to multiplying matrices

EXAMPLES:

① $P_d = \{a_0 + a_1 x + \dots + a_d x^d\}$ polynomials has basis $(1, x, x^2, \dots, x^d)$

P_{d-1} has basis $(1, x, x^2, \dots, x^{d-1})$

The linear transformation $P_d \xrightarrow{d/dx} P_{d-1}$ expressed with respect to these choices of bases has matrix

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ \\ x^{d-1} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x & x^2 & x^3 \cdots & x^{d-1} & x^d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 2 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 3 & & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ & & & & & d-1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & d \end{bmatrix} \end{array}$$

e.g. $d = 3$

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \end{array}$$