10/28/2016 > **Elementary matrices** – an alternative view of row operations

Each of the 3 elementary row operations is the same as <u>multiplication on the left</u> by one of 3 types of <u>elementary matrices</u>: (also useful later in discussing determinants)

1. <u>Scaling</u> row i by c is achieved by multiplying on left by

$$E_1(i,c) := \begin{array}{r} \\ \\ \text{row } i \rightarrow \\ \\ \\ \end{array} \begin{array}{c} \text{col } i \\ \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & \boxed{c} & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

e.g.
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2(1/4)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 & -2 \\ 0 & 0 & 4 & 6 & -3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 4 & 6 & -3 \end{array}\right]$$

Note that $E_1(i,c)$ is invertible (for $c \neq 0$): $E_1(i,c)^{-1} = E_1(i,\frac{1}{c}) = \begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1/c & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}$

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2. adding a multiple c of row j to row i is achieved by mult. on left by

$$E_2(i,j,c) := \begin{array}{r} \\ \text{row } i \rightarrow \\ \\ \end{array} \begin{array}{c} \text{col } j \\ \begin{bmatrix} 1 & & \\ \cdots & \ddots & c \\ 0 & & 1 \end{bmatrix} \end{array}$$

e.g.
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2(2,1,-2)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 & -2 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}_{E_2(3,1,-4)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 0 & 0 & 4 & 6 & -3 \end{array}\right]$$

Note that $E_2(i,j,c)$ is invertible: $E_2(i,j,c)^{-1} = E_2(i,j,-c) = \begin{bmatrix} 1 & & -c \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

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3. exchanging rows i and j is multiplying on left by

$$E_3(i,j) = \begin{array}{r} \\ \text{row } i \rightarrow \\ \text{row } j \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$
$\quad\quad\quad\quad$ col i col j

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_3(1,3)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 4 & 4 & 8 & 10 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Note $E_3(i,j)^{-1} = E_3(i,j)$

PROP: $A, \tilde{A}$ are equivalent by a sequence of row operations

$$\iff A = E^{(1)}E^{(2)}\cdots E^{(r)}\tilde{A} \quad \text{for some elem. matrices } E^{(j)}$$

$$\left(\text{in which case } \tilde{A} = (E^{(r)})^{-1}(E^{(r-1)})^{-1}\cdots(E^{(2)})^{-1}(E^{(1)})^{-1}A\right)$$

proof: Should be clear from the above discussion.

COR: A is invertible $\iff A = E^{(1)}E^{(2)}\cdots E^{(r)}$ for some elem. matrices $E^{(j)}$
(and then $A^{-1} = (E^{(r)})^{-1}\cdots(E^{(2)})^{-1}(E^{(1)})^{-1}$).

proof: We've seen

$$A \text{ invertible} \iff A \text{ row reduces to } I \ (=\tilde{A})$$

$$\underset{\text{previous PROP}}{\iff} A = E^{(1)}E^{(2)}\cdots E^{(r)}$$

$$\left(\text{and then } I = \underbrace{(E^{(r)})^{-1}\cdots(E^{(2)})^{-1}(E^{(1)})^{-1}}_{\text{must be } A^{-1}}A\right) \blacksquare$$

REMARK: The book also points out a method to compute A^{-1} from A an $n\times n$ matrix:

THM 2.3.3: Row reducing $n\{[\underbrace{A}_{n} \mid I_n]$ either produces $\begin{cases} [I_n \mid A^{-1}] & \text{if } A \text{ is invertible} \\ \left[\begin{smallmatrix}\text{not}\\ I_n\end{smallmatrix} \middle| \text{something}\right] & \text{if } A \text{ is not invertible}\end{cases}$

proof: Think about it! $\blacksquare$

EXAMPLE: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has both $a \neq 0$,
and $\delta := \det A = ad - bc \neq 0$,

then $(A \mid I_2) = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$

$$d - \frac{cb}{a} = \frac{\det A}{a} = \frac{\delta}{a}$$

$$[I_2 \mid A^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{\delta} & -\frac{b}{\delta} \\ 0 & 1 & -\frac{c}{\delta} & \frac{a}{\delta} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{bc}{a\delta} & -\frac{b}{\delta} \\ 0 & 1 & -\frac{c}{\delta} & \frac{a}{\delta} \end{array}\right] \leftsquigarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{\delta} & \frac{a}{\delta} \end{array}\right]$$

Q: What happens if $a = 0$ (but $\det A \neq 0$)? try it!

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§2.4 Span & linear dependence

Recall that a linear combination of $(\bar v_1, \ldots, \bar v_k)$ just means $a_1\bar v_1 + \ldots + a_k \bar v_k$ for $a_j \in \mathbb{R}$.

DEF'N 2.4.2

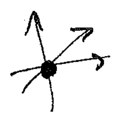
The span of $(\bar v_1, \ldots, \bar v_k)$ for $\bar v_j \in \mathbb{R}^n$ is this subspace:

$$\mathrm{span}(\bar v_1, \ldots, \bar v_k) := \{ a_1 \bar v_1 + \ldots + a_k \bar v_k \ : a_j \in \mathbb{R} \}$$

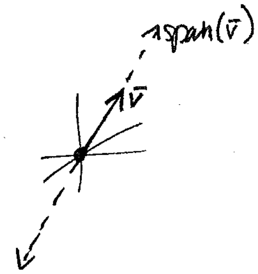
i.e. all lin. combs. of $(\bar v_1, \ldots, \bar v_k)$.

EXAMPLES:

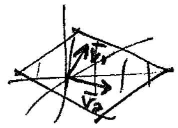
① $\mathrm{span}(\bar 0) = \{\bar 0\}$

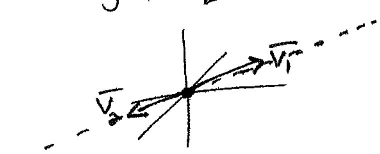


② $\mathrm{span}(\underset{\bar 0}{\underset{\neq}{\bar v}})$ = line through $\bar v$ (and $\bar 0$)



③ $\mathrm{span}(\underset{\bar 0}{\underset{\neq}{\bar v_1}}, \underset{\bar 0}{\underset{\neq}{\bar v_2}}) = \begin{cases} \text{plane containing } \bar v_1, \bar v_2 \text{ if } \bar v_1, \bar v_2 \text{ not parallel} \\ \text{line containing } \bar v_1 \text{ or } \bar v_2 \text{ if they are parallel} \end{cases}$





④ Does $\mathrm{span}\left(\begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right)$ contain $\begin{bmatrix}1\\0\\0\end{bmatrix}$? What about $\begin{bmatrix}1\\1\\1\end{bmatrix}$?

Can be rephrased as a linear system, solved as usual via row-reduction:

$\exists$? sol'ns to $a_1\begin{bmatrix}-1\\0\\2\end{bmatrix} + a_2\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

i.e. $\begin{bmatrix}-1 & 0\\0 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c}-1 & 0 & 1\\0 & 1 & 0\\2 & 3 & 0\end{array}\right] \overset{\text{row-reduce}}{\rightsquigarrow} \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 0\\2 & 3 & 0\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 0\\0 & 3 & 2\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 2\end{array}\right]$

no sol'ns

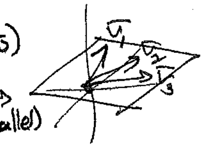
However

$\left[\begin{array}{cc|c}-1 & 0 & 1\\0 & 1 & 0\\2 & 3 & 1\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\\2 & 3 & 1\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\\0 & 3 & 3\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{array}\right] \Rightarrow \begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$(-1)\begin{bmatrix}-1\\0\\2\end{bmatrix} + (1)\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ ✓

The parallel issue controling $\operatorname{span}(\bar{v}_1,\bar{v}_2)$

gets worse for $\operatorname{span}(\bar{v}_1,\bar{v}_2,\bar{v}_3) = \begin{cases} \{\bar{0}\} & \text{if all } \bar{v}_i=\bar{0} \\ \text{line} & \text{if all } \bar{v}_i \text{ parallel (not all } \bar{0}) \\ \text{plane} & \text{if } \bar{v}_i \text{ are coplanar (not all parallel)} \\ \text{3-dimensional subspace} & \text{otherwise} \end{cases}$



DEF'N 2.4.5. Say $\bar{v}_1,\dots,\bar{v}_k$ in $\mathbb{R}^n$ are linearly independent

$$\text{if} \quad \sum_{i=1}^{k} a_i\bar{v}_i = \sum_{i=1}^{k} b_i\bar{v}_i \overset{(*)}{\Longrightarrow} a_1=b_1,\dots,a_k=b_k$$

(and linearly dependent otherwise).

Alternate phrasing DEF'N 2.4.10:

$\bar{v}_1,\dots,\bar{v}_k$ are lin. indep. if $\displaystyle\sum_{i=1}^{k} c_i\bar{v}_i=\bar{0} \overset{(**)}{\Longrightarrow} c_1=\dots=c_k=0$

[Why are $(*)$, $(**)$ equivalent?

Assuming $(*)$, if $\displaystyle\sum_{i=1}^{k} a_i\bar{v}_i = \sum_{i=1}^{k} b_i\bar{v}_i$ then $\displaystyle\sum_{i=1}^{k}(a_i-b_i)\bar{v}=\bar{0} \overset{(*)}{\Rightarrow} a_1-b_1=\dots=a_k-b_k=0$ i.e. $a_1=b_1,\dots,a_k=b_k$

Assuming $(**)$, if $\displaystyle\sum_{i=1}^{k} \underbrace{c_i}_{\text{call this } a_i}\bar{v}_i=\bar{0}=\sum_{i=1}^{k} \underbrace{0}_{\text{call this } b_i}\cdot\bar{v}_i$ then by $(**)$, $c_1=0, c_2=0,\dots,c_k=0$]

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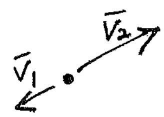
EXAMPLES:

① A single vector $\bar{v}$ is lin. indep. $\Leftrightarrow \bar{v}\neq\bar{0}$ (if $\bar{v}=\bar{0}$ then $c\cdot\bar{v}=c\cdot\bar{0}=\bar{0}$ with $c\neq 0$)

② Two nonzero vectors $\underset{\bar{0}}{\bar{v}_1 \neq}, \underset{\bar{0}}{\bar{v}_2 \neq}$ are lin. indep. $\Leftrightarrow$ they're not parallel

i.e. lin. dependent $\Leftrightarrow$ parallel



The nontrivial dependence implies their parallelness:

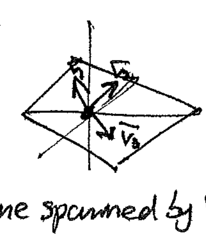
$$c_1\bar{v}_1+c_2\bar{v}_2=\bar{0} \quad \text{with } c_1\neq 0$$

$$\Rightarrow \bar{v}_1 = \frac{-c_2}{c_1}\bar{v}_2$$

$$\left(\text{or if } c_2\neq 0 \text{ then } \bar{v}_2=-\frac{c_1}{c_2}\bar{v}_1\right)$$

③ Three pairwise non-parallel $\bar{v}_1,\bar{v}_2,\bar{v}_3$ are lin. dependent

$\Leftrightarrow$ they are coplanar:



$c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3=\bar{0}$ with $c_1\neq 0$

implies $\bar{v}_1 = -\frac{c_2}{c_1}\bar{v}_2 - \frac{c_3}{c_1}\bar{v}_3 \in \operatorname{span}(\bar{v}_2,\bar{v}_3)$, the plane spanned by $\bar{v}_2,\bar{v}_3$