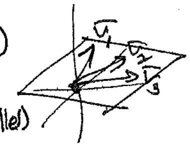
The parallel issue controling span $(\bar v_1, \bar v_2)$

gets worse for $\text{span}(\bar v_1, \bar v_2, \bar v_3) = \begin{cases} \{\bar 0\} & \text{if all } \bar v_i = \bar 0 \\ \text{line} & \text{if all } \bar v_i \text{ parallel (not all } \bar 0) \\ \text{plane} & \text{if } \bar v_i \text{ are coplanar (not all parallel)} \\ \text{3-dimensional subspace} & \text{otherwise} \end{cases}$



DEFIN 2.4.5. Say $\bar v_1, \dots, \bar v_k$ in $\mathbb{R}^n$ are linearly independent

if $\displaystyle\sum_{i=1}^{k} a_i \bar v_i = \sum_{i=1}^{k} b_i \bar v_i \overset{(*)}{\Rightarrow} a_1 = b_1, \dots, a_k = b_k$

(and linearly dependent otherwise).

Alternate phrasing DEFIN 2.4.10:

$\rightarrow \bar v_1, \dots, \bar v_k$ are lin. indep. if $\displaystyle\sum_{i=1}^{k} c_i \bar v_i = \bar 0 \overset{(**)}{\Rightarrow} c_1 = \dots = c_k = 0$

REMARK: Easily testable via solving $\begin{bmatrix} | & & | \\ \bar v_1 & \cdots & \bar v_k \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Why are (*), (**) equivalent?

Assuming (*), if $\displaystyle\sum_{i=1}^{k} a_i \bar v_i = \sum_{i=1}^{k} b_i \bar v_i$ then $\displaystyle\sum_{i=1}^{k} (a_i - b_i)\bar v = \bar 0 \overset{(*)}{\Rightarrow} a_1 - b_1 = \dots = a_k - b_k = 0$

i.e. $a_1 = b_1, \dots, a_k = b_k$

Assuming (**), if $\displaystyle\underbrace{\sum_{i=1}^{k} c_i \bar v_i}_{\text{call this } a_i} = \bar 0 = \underbrace{\sum_{i=1}^{k} 0 \cdot \bar v_i}_{\text{call this } b_i}$ then by (**), $c_1 = 0, c_2 = 0, \dots, c_k = 0$

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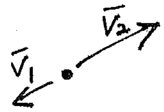
EXAMPLES:

① A single vector $\bar v$ is lin. indep. $\Leftrightarrow \bar v \neq \bar 0$ (if $\bar v = \bar 0$ then $c \cdot \bar v = c \cdot \bar 0 = \bar 0$ with $c \neq 0$)

② Two nonzero vectors $\underset{\bar 0}{\bar v_1} \neq, \underset{\bar 0}{\bar v_2} \neq$ are lin. indep. $\Leftrightarrow$ they're not parallel

i.e. lin. dependent $\Leftrightarrow$ parallel



The nontrivial dependence implies their parallelness:

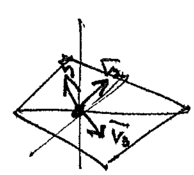
$$c_1 \bar v_1 + c_2 \bar v_2 = \bar 0 \quad \text{with } c_1 \neq 0$$

$$\Rightarrow \bar v_1 = \frac{-c_2}{c_1} \bar v_2$$

(or if $c_2 \neq 0$ then $\bar v_2 = -\frac{c_1}{c_2} \bar v_1$)

③ Three pairwise non-parallel $\bar v_1, \bar v_2, \bar v_3$ are lin. dependent

$\Leftrightarrow$ they are coplanar:



$c_1 \bar v_1 + c_2 \bar v_2 + c_3 \bar v_3 = \bar 0$ with $c_1 \neq 0$

implies $\bar v_1 = -\frac{c_2}{c_1} \bar v_2 - \frac{c_3}{c_1} \bar v_3 \in \text{span}(\bar v_2, \bar v_3)$, the plane spanned by $\bar v_2, \bar v_3$

(75)

The following result should not be too surprising...

THM 2.4.11: In $\mathbb{R}^n$, (a) every set of $n+1$ vectors $\bar{v}_1, \ldots, \bar{v}_{n+1}$ is lin. dependent and (b) no set of $n-1$ vectors $\bar{v}_1, \ldots, \bar{v}_{n-1}$ can span, (i.e. have $\mathrm{span}(\bar{v}_1, \ldots, \bar{v}_{n-1}) = \mathbb{R}^n$)

Proof. (a): Given $\bar{v}_1, \ldots, \bar{v}_{n+1} \in \mathbb{R}^n$, find a nontrivial dependence $c_1\bar{v}_1 + \ldots + c_{n+1}\bar{v}_{n+1} = \bar{0}$ (with c_i not all 0)

by solving
$$c_1\begin{bmatrix} | \\ \bar{v}_1 \\ | \end{bmatrix} + \ldots + c_{n+1}\begin{bmatrix} | \\ \bar{v}_{n+1} \\ | \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_{n+1} \\ | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

augmented matrix $\rightsquigarrow$
$$n\left\{\underbrace{\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_{n+1} \\ | & & | \end{bmatrix}}_{n+1}\left|\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right]\right. \overset{\text{row reduce}}{\rightsquigarrow} n\left\{\underbrace{\begin{bmatrix} & & \\ & \tilde{A} & \\ & & \end{bmatrix}}_{\substack{n+1 \text{ columns in } \tilde{A} \\ > n}}\left|\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right]\right.$$

$n+1$ columns in $\tilde{A}$ $>n$, so at least one nonpivotal column in $\tilde{A}$

i.e. at least one of the c_i can be chosen arbitrarily, so nonzero.

(b): Given $\bar{v}_1, \ldots, \bar{v}_{n-1} \in \mathbb{R}^n$, find a $\bar{b} \in \mathbb{R}^n$ with $\bar{b} \notin \mathrm{span}(\bar{v}_1, \ldots, \bar{v}_{n-1})$

by solving
$$\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_{n-1} \\ | & & | \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

augmented matrix $\rightsquigarrow$
$$n\left\{\underbrace{\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_{n-1} \\ | & & | \end{bmatrix}}_{n-1}\left|\begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix}\right]\right. \overset{\text{row reduce}}{\rightsquigarrow} n\left\{\begin{bmatrix} & * & \\ \hline \underbrace{0 \cdots\cdots 0}_{n-1} \end{bmatrix}\left|\begin{matrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_{n-1} \\ \tilde{b}_n \end{matrix}\right]\right.$$

$<n$, so at least one row contains no pivotal 1, so is all zeroes

Picking $\tilde{b}_n = 1$ one has no solutions $\begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$, so doing the inverse row operations to get the corresponding $\bar{b}$, one has no solutions to the original system ∎

(76) Combining spanning & lin. independence gives an important concept.

DEF'N: Given a subspace $E \subset \mathbb{R}^n$, a basis for E is a subset $\{\bar{v}_1, \ldots, \bar{v}_k\} \subset E$ that spans E and is lin. indep.

Equivalently, $\{\bar{v}_1, \ldots, \bar{v}_k\}$ is a basis for E $\Leftrightarrow$ every $v \in E$ has a $!$ expression $v = c_1\bar{v}_1 + \cdots + c_k\bar{v}_k$.

EXAMPLES: ① for an $m \times n$ matrix A, the solutions to $A\bar{x} = 0$, $\bar{x} \in \mathbb{R}^n$ always form a subspace of $\mathbb{R}^n$, and we can use row-reduction to find a basis.

$A\bar{x} = \bar{0} \Rightarrow A(c\bar{x}) = cA\bar{x} = \bar{0}$

$A\bar{x}_i = \bar{0} \Rightarrow A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{0} + \bar{0} = \bar{0}$

e.g. $A = [1 \ -1 \ -1]$

$$E := \{\bar{x} \in \mathbb{R}^3 : A\bar{x} = 0\} = \left\{\text{sol'ns to } [1 \ -1 \ -1]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow E$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =: \bar{v}_1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =: \bar{v}_2 \right\}$ (Why?)
- spans E?
- lin. indep.?

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② Every basis for $\mathbb{R}^n$ has exactly n elements (e.g. $\{\bar{e}_1, \ldots, \bar{e}_n\}$)

PROP: For a subspace $E \subset \mathbb{R}^n$, and $\{\bar{v}_1, \ldots, \bar{v}_k\} \subset E$, TFAE

(a) $\{\bar{v}_i\}_{i=1,\ldots,k}$ are a basis for E

(b) they are a minimal spanning set for E, i.e. removing any $\bar{v}_i$ no longer spans E

(c) they are a maximal lin. indep. set in E, i.e. adding any $\bar{v} \in E$ ruins their lin. independence.

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