(39)
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§1.6: 4 big theorems

Once one introduces this notion...

DEF'N: A subset $C \subset \mathbb{R}^n$ is compact if it is closed and bounded (i.e. $\exists R>0$ with $B_R(0) \supset C$)

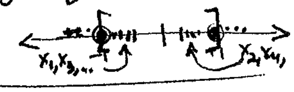
e.g.

one can use what we've learned to prove...

Not so exciting on their own; feel more like lemmas

THM 1.6.3 (Bolzano-Weierstrass)
Every sequence $\bar{x}_1, \bar{x}_2, \ldots \subset C$ a compact set in $\mathbb{R}^n$ has a convergent subsequence $\bar{x}_{i(1)}, \bar{x}_{i(2)}, \ldots$ whose limit is in C.

e.g. $x_n = (-1)^n\left(1-\frac{1}{n}\right) \in \mathbb{R}$, $\subset [-1,+1]$



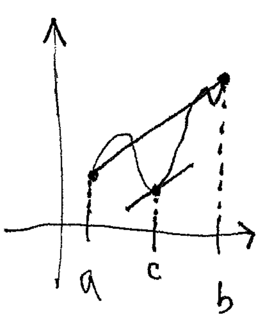
$\Downarrow$

THM 1.6.9 (Extreme Value thm)
If $C \subset \mathbb{R}^n$ is compact, and $f: C \to \mathbb{R}$ continuous, then f achieves a minimum and maximum value on C,
i.e. $\exists \bar{a}, \bar{b} \in C$ with $f(\bar{a}) \geq f(\bar{x})$ $\forall \bar{x} \in C$.
$f(\bar{b}) \leq f(\bar{x})$

important in fundamental thms of Calculus

THM 1.6.12 (Mean value thm)
$f: [a,b] \to \mathbb{R}$ continuous and f differentiable on (a,b)
$\Rightarrow \exists\, c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



THM 1.6.13 (Fundamental thm of algebra)
Every polynomial
$p(z) = z^k + a_{k-1}z^{k-1} + \ldots + a_2 z^2 + a_1 z + a_0$
with $k \geq 1$ has at least one root $z_0 \in \mathbb{C}$
i.e. $p(z_0) = 0$.

Not at all obvious, and very important!

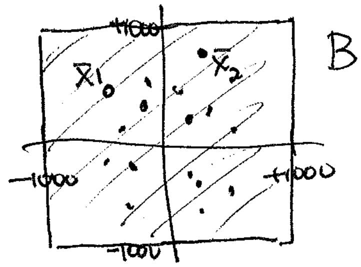
THM (Bolzano-Weierstrass) $(\bar{x}_i)_{i=1}^{\infty} \subset C \subset \mathbb{R}^n$ compact (= closed + bounded) $\Rightarrow \exists$ a convergent subsequence $(\bar{x}_{i(j)})_{j=1}^{\infty}$ with limit in C.

Try $C = (0,1) \subset \mathbb{R}^1$, $x_i = 1/i$ as counterexample

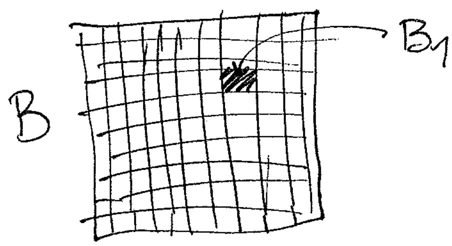
Try $C = \mathbb{R}^1$, $x_i = i$ as counterexample

proof: Since C is bounded, every $\bar{x}_i$ has all coordinates in $[-10^m, +10^m]$ for some m, so $(\bar{x}_i)_{i=1}^{\infty} \subset B$ for some large n-cubical box B,

e.g. $n=2$
$m=3$



Dividing each coordinate interval $[-10^m, +10^m]$ into 20 equal subintervals divides B into 20^n subboxes, at least one of which, call it B_1, has $\bar{x}_i \in B$ for infinitely many i:



Pick any i with $\bar{x}_i \in B_1$ and call this $i(1) := i$

Repeat this procedure, replacing • B with B_1

• $(\bar{x}_i)_{i=1}^{\infty}$ with $(\bar{x}_i)_{i=i(1)}^{\infty} \cap B_1$

producing a subbox $B_2 \subset B_1$ and $i(2) > i(1)$ with $\bar{x}_{i(2)} \in B_2$,

$B_3 \subset B_2$ $\quad i(3) > i(2) \quad \bar{x}_{i(3)} \in B_3$,

$\vdots$

We claim $(\bar{x}_{i(1)}, \bar{x}_{i(2)}, \ldots)$ is a convergent subsequence:

Every $\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in B_1 has the same 10^m decimal digits $\begin{bmatrix} d_1^{(1)} \\ \vdots \\ d_n^{(1)} \end{bmatrix}$ for their entries

in B_2 —— " —— 10^{m-1} —— " —— $\begin{bmatrix} d_1^{(2)} \\ \vdots \\ d_n^{(2)} \end{bmatrix}$ —— " ——

$\vdots$

so if one defines $\bar{a} := \begin{bmatrix} d_1^{(1)} d_1^{(2)} \cdots \\ \vdots \\ d_n^{(1)} d_n^{(2)} \cdots \end{bmatrix}$ by the decimal expansion of its coordinates,

then it's easy to check $\lim_{j \to \infty} \bar{x}_{i(j)} = \bar{a}$, since $|\bar{x}_{i(j)} - \bar{a}| \leq \sqrt{\underbrace{(10^{m-j})^2 + \cdots + (10^{m-j})^2}_{n \text{ times}}}$

$\leq \sqrt{n} \cdot 10^{m-j} \to 0$ as $j \to \infty$.

Also $\bar{a} \in C$ since C is closed ∎

REMARK: This proof is highly non-constructive: even if we specify a concrete sequence $(\underset{\sin(10^m)}{\underset{\|}{x_m}})_{m=1}^{\infty} \subset C = [-1,+1]$ (in $\mathbb{R}^1$), see (EXAMPLE 1.6.4)

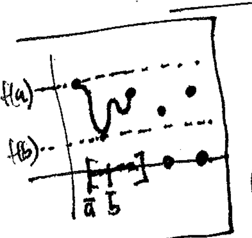
we have no idea what ~~the~~ sequence of subboxes $[-1,+1] \supset B_1 \supset B_2 \supset \dots$ will look like, and how to describe explicitly a convergent subsequence!

10/3/2016

(Extreme Value Thm)

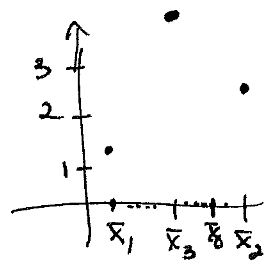
THM 1.6.9: For $f: C \to \mathbb{R}$ continuous with C compact, ($C \subset \mathbb{R}^n$)

$\exists\, \bar{a}, \bar{b} \in C$ with $f(\bar{a}) \geq f(\bar{x})\ \forall \bar{x} \in C$, $f(\bar{b}) \leq f(\bar{x})$



(i.e. f achieves a minimum $f(\bar{b})$, maximum $f(\bar{a})$ value on C)

proof: let's do max; then applying it to $-f(\bar{x})$ gives the min.



First show the values $f(\bar{x})$ are bounded. If not, then $\forall N = 1, 2, \dots\ \exists\, \bar{x}_N \in C$ with $f(\bar{x}_N) > N$.

Use Bolzano-~~Weierstrass to~~ find a convergent subsequence $(\bar{x}_{N(j)})_{j=1}^{\infty} \subset C$ with $\lim_{j\to\infty} \bar{x}_{N(j)} = \bar{x}_0 \in C$

Continuity implies $\lim_{j\to\infty} f(\bar{x}_{N(j)}) = f(\bar{x}_0)$.

This leads to a contradiction: for $j > f(\bar{x}_0)+1$, one has $f(\bar{x}_{N(j)}) > N(j) \geq j \geq f(\bar{x}_0)+1$,

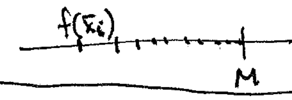
but if we pick ϵ with $1 > \epsilon > 0$ then $\exists J$ such that $|f(\bar{x}_{N(j)}) - f(\bar{x}_0)| < \epsilon < 1$

$\Rightarrow f(\bar{x}_{N(j)}) \leq f(\bar{x}_0) + \epsilon \leq f(\bar{x}_0)+1$.

When $j > \max\{f(\bar{x}_0)+1, J\}$, these are in conflict.

Once the values of $f(\bar{x})$ are bounded, we know they have a supremum M (sup) in $\mathbb{R}$

least upper bound, i.e. $M \geq f(\bar{x})\ \forall \bar{x} \in C$ but no $M' < M$ has this property.



But then $\exists\, \bar{x}_1, \bar{x}_2, \dots \in C$ with

$\lim_{i\to\infty} f(\bar{x}_i) = M$ (possibly $\bar{x}_1 = \bar{x}_2 = \dots \in C$ and $f(\bar{x}_1) = M$),

so $\exists$ a convergent subsequence $(\bar{x}_{i(j)})_{j=1}^{\infty}$ with $\lim_{j\to\infty} \bar{x}_{i(j)} = \bar{a} \in C$

and continuity gives $M = \lim_{j\to\infty} f(\bar{x}_{i(j)}) = f(\bar{a})$. ■